Heterogeneous effect on the expected shortfall and conditional tail expectation

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Outline

1. Introduction
2. CTE in Statistics
3. Linear combinations of independent random variables
4. Heavy-tailed distribution

Expected shortfall

Value-at-risk: VaR

$$\mathrm{VaR}_p(X) = F_X^{-1}(p), \quad 0 < p < 1.$$

This concept was introduced to answer the following question: how much can we expect to lose in one day, week, year, with a given probability? In today's financial world, VaR has become the benchmark risk measure: its importance is unquestion since regulators accept this model as the basis for setting capital requirements for market risk exposure.

Expected shortfall

As the VaR at a fixed level only gives local information about the underlying distribution, a promising way to escape from this shortcoming is to consider the so-called expected shortfall over some quantile. Expected shortfall at probability level p is the stop-loss premium with retention VaR. Specifically,

$$\mathrm{ES}_p(X) = \mathrm{E}[(X - \mathrm{VaR}_p(X))_+],$$

where $x_+ = \max\{x, 0\}$.

Conditional tail expectation

The conditional tail expectation (CTE) represents the conditional expected loss given that the loss exceeds its VaR:

$$\mathrm{CTE}_p(X) = \mathrm{E}(X|X > VaR_p(X))$$

Thus the CTE is nothing but the mathematical transcription of the concept of 'average loss in the worst $100(1-p)\%$ case'. Defining by $c = \mathrm{VaR}_p(X)$ a critical loss threshold corresponding to some confidence level p, $\mathrm{CTE}_p(X)$ provides a cushion against the mean value of losses exceeding the critical threshold c.

Example

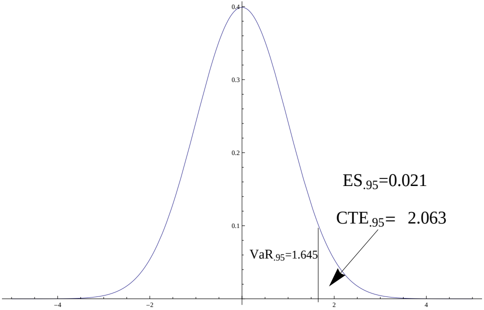

Figure: Plot of standard Normal distribution.

Relationship

For $0 < p < 1$, then

$$\mathrm{CTE}_p(X) = \mathrm{VaR}_p(X) + \frac{1}{\bar{F}_X(\mathrm{VaR}_p(X))}\mathrm{ES}_p(X),$$

where $\bar{F}_X = 1 - F_X$.

Nonparametric estimation

Many statisticians are interested in the nonparametric estimation of CTE.

- Chen (2008)-Journal of Financial Econometrics
  a) Sample average estimator
  b) Kernel estimator
- Cai and Wang (2008) -Journal of Econometrics
  Weighted double kernel local linear estimator of the conditional density

Individual risk model

Let X_i be the payment on policy i for $i = 1, \ldots, n$. We are interested in the distribution of the total claims on a number of policies with

$$S = X_1 + X_2 + \ldots + X_n,$$

where $X_1, \ldots, X_n$ are independent random variables.

Heterogenous effect

Let $X_1, \cdots, X_n$ be independent and identical exponential random variables, then, we are interested in

$$S = \lambda_1 X_1 + \lambda_2 X_2 + \ldots + \lambda_n X_n.$$

How the heterogeneity affects the expected shortfall?

Measuring the heterogeneity

Marshall and Olkin (1979)

Let $\{x_{(1)}, x_{(2)}, \cdots, x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. The vector $\mathbf{x}$ in $\mathbb{R}^{+n}$ is said to

- majorize the vector $\mathbf{y}$ in $\mathbb{R}^{+n}$ (denoted by $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$$

for $j = 1, \cdots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

Example:

$$(0.2, 1, 9) \overset{m}{\succeq} (0.2, 4, 6).$$

Measuring the heterogeneity

- weakly submajorize the vector **y** in $\mathbb{R}^{+n}$ (denoted by $\mathbf{x} \succeq_w \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{[i]} \geq \sum_{i=1}^{j} y_{[i]}$$

for $j = 1, \cdots, n$, where $\{x_{[1]}, x_{[2]}, \cdots, x_{[n]}\}$ denotes the decreasing arrangement of the components of the vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$.

Example:

$$(9, 1, 0.1) \succeq_w (5, 3, 2).$$

Convolutions

Kochar and Xu, Journal of Multivariate Analysis, 2010, 165-176

Kochar and Xu (2010) proved the following result.

Theorem

Let $X_1, \cdots, X_n$ be independent and identical exponential random variables, then

$$(\lambda_1, \cdots, \lambda_n) \succeq_w (\lambda_1', \cdots, \lambda_n') \Rightarrow \mathrm{ES}_p \left(\sum_{i=1}^{n} \lambda_i X_i \right) \geq \mathrm{ES}_p \left(\sum_{i=1}^{n} \lambda_i' X_i \right)$$

where $\succeq_w$ means weak submajorization.

Applications

Suppose that a total claim is composed of several subclaims which come from different exponential distributions. The actuary wants to know the properties of expected shortfall in order to make a good policy for the insurance company. Our result

1. reveals that greater the degree of heterogeneity among subclaims, the larger the expected shortfall is.
2. provides a sharp lower bound for the expected shortfall of subclaims at each probability level p based on the mean of heterogeneous subclaims.

Applications

Example: Suppose that the total claim is composed of 3 subclaims coming from exponential distributions with hazard rates λ_1, λ_2 and λ_3. Let us assume

$$(\lambda_1, \lambda_2, \lambda_3) = (1, 2, 3).$$

Then, the arithmetic mean and the harmonic mean are 2 and $18/11$, respectively. In the Figure below, we used Mathematica to plot the expected shortfalls of the total claim when the parameters are $(1, 2, 3)$, and their arithmetic mean and harmonic mean. It is seen that the harmonic mean provides the sharp bound for the expected shortfall as stated in our result.

Applications

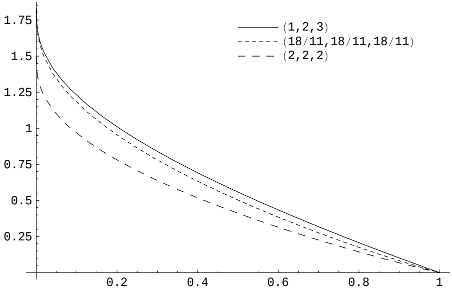

Figure: Plot of the expected shortfall of the total claim of three subclaims with exponential parameters $(1, 2, 3)$, the harmonic mean parameters $(18/11, 18/11, 18/11)$ and the arithmetic mean parameters $(2, 2, 2)$.

Gamma distribution

Let $X_1, \ldots, X_n$ be independent and identical gamma random variables. Diaconis and Perlman (1987, The Symposium on dependence in Statistics and Probability) studied the linear combinations of gamma random variables. They pointed out that if

$$(\lambda_1, \cdots, \lambda_n) \overset{m}{\succeq} (\lambda'_1, \cdots, \lambda'_n)$$

then

$$\mathrm{Var}\left(\sum_{i=1}^{n} \lambda_i X_i\right) \geq \mathrm{Var}\left(\sum_{i=1}^{n} \lambda'_i X_i\right).$$

Gamma distribution

Kochar and Xu, Journal of Statistical Planning and Inferences, 2011, 418-428

Kochar and Xu (2011) proved the following result.

Theorem

Let $X_1, \ldots, X_n$ be independent and identical gamma random variables. Then,

$$(\lambda_1, \cdots, \lambda_n) \succeq_w (\lambda'_1, \cdots, \lambda'_n) \Rightarrow \mathrm{ES}_p\left(\sum_{i=1}^{n} \lambda_i X_i\right) \geq \mathrm{ES}_p\left(\sum_{i=1}^{n} \lambda'_i X_i\right).$$

Heavy-tailed distribution

For heavy-tailed distribution or regularly varying right tail at ∞ with tail index $\alpha > 0$ if its survival function is of the following form,

$$\bar{F}(t) = t^{-\alpha} L(t), \quad t > 0, \alpha > 0,$$

where L is a slowly varying function; that is, L is a positive function on $(0, \infty)$ with property

$$\lim_{t \to \infty} \frac{L(ct)}{L(t)} = 1, \quad c > 0.$$

Well-known heavy-tailed distribution

- Pareto distribution
- Log-normal distribution
- Burr distribution
- Cauchy distribution
- Weibull distribution with shape parameter less than 1

Xu (2010) proved the following result.

Theorem

Let $X_1, \ldots, X_n$ are i.i.d. heavy-tailed random variables on $\Re_+$, and denote $Y = \sum_{i=1}^n w_i X_i$ and $Y = \sum_{i=1}^n w_i' X_i$, $w_i, w_i' \in \Re_+$, then, for $p \to 1$,

$$\sum_{i=1}^n w_i^\alpha \geq \sum_{i=1}^n w_i'^\alpha \sim CTE_p(Y) \geq CTE_p(Y').$$

Thank you!