Heterogeneity and the need for capital in the individual model

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Risk

Actuaries believe that the heterogeneity of the risks comprised in a given insurance portfolio tends to increase its dangerousness. This is turn leads to requiring more capital.

1. Life insurance-Spreeuw (1998).

   The author considered a portfolio consisting of insured individuals with different death probabilities in a given period. The insurer shares at maturity a fixed proportion of the profit or loss with all survivors.
   Assuming that the same premium is paid by all the insureds, Spreeuw (1998) showed that the single premium under the homogeneity assumption is higher than under the heterogeneity assumption.

1. Dahan et al- Insurance: Mathematics and Economics

   They studied endowment insurance contracts and whole life insurance contracts, where the insureds in the portfolios are exposed to a common life distribution, and studied the effect of the aging variability on the annual premium. They showed that the annual premium under the heterogeneity assumption is higher than under the homogeneous assumption.

Model assumptions

$$X_i = J_{q_i} C_i,$$

where

1. J_{q_i} is Bernoulli distributed with mean q_i;
2. J_{q_i} indicates whether at least one claim occurred for policy i ($J_{q_i} = 1$) if at least one claim has been reported by policyholder i;
3. C_i is then the total cost of all these claims.

Individual risk model

$$\sum_{i=1}^{n} X_i$$

Example

Assume that

$$\mathrm{E}C_i = \mu, \qquad \mathrm{Var}(C_i) = \sigma^2,$$

then

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = (\sigma^2 + \mu^2)\sum_{i=1}^{n} q_i - \mu^2 \sum_{i=1}^{n} q_i^2$$

One basic question is: how the heterogeneity (or dispersion) in

$$(q_1, \ldots, q_n)$$

affects the variance? Assume the other portfolio

$$Y_i = J_{p_i} C_i,$$

If $\sum p_i = \sum q_i$, then

$$\sum_{i=1}^{n} q_i^2 < \sum_{i=1}^{n} p_i^2 \Rightarrow \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) > \mathrm{Var}\left(\sum_{i=1}^{n} Y_i\right).$$

Measuring the heterogeneity
Marshall and Olkin (1979)

Let $\{x_{(1)}, x_{(2)}, \cdots, x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. The vector $\mathbf{x}$ in $\mathbb{R}^{+n}$ is said to

- majorize the vector $\mathbf{y}$ in $\mathbb{R}^{+n}$ (denoted by $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}$$

for $j = 1, \cdots, n-1$ and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

Example:

$$(0.2, 1, 9) \overset{m}{\succeq} (0.2, 4, 6).$$

Stochastic orders
a useful tool for risk management

1. Stochastic Dominance (denoted by $X \leq_{st} Y$)
$$P(X > t) \leq P(Y > t), \quad \forall t.$$
Note that
$$X \leq_{st} Y \Longleftrightarrow \mathrm{E}[g(X)] \leq \mathrm{E}[g(Y)],$$
for all the non-decreasing functions g.
2. Stop-loss order (denoted by $X \leq_{sl} Y$)
$$\mathrm{E}[(X-d)_+] \leq \mathrm{E}[(Y-d)_+], \quad d \in \mathbb{R}_+.$$
That is, if their corresponding stop-loss premiums are ordered for all possible levels d of the deductible. Note that
$$X \leq_{sl} Y \Longleftrightarrow \mathrm{E}[g(X)] \leq \mathrm{E}[g(Y)],$$
for all the non-decreasing and convex functions g.

1. Convex order (denoted by $X \leq_{cx} Y$)

$$X \leq_{sl} Y, \quad \mathrm{E}(X) = \mathrm{E}(Y).$$

2. Laplace transform order (denoted by $X \leq_{Lt} Y$)
Assume a utility function $1 - exp(-tx)$.

$$\mathrm{E}[exp(-tX)] \leq \mathrm{E}[exp(-tY)], \quad t \geq 0.$$

Relation with risk measures

1. Value-at-Risk

$$VaR_p(X) \leq VaR_p(Y) \Longleftrightarrow X \leq_{st} Y, \quad p \in [0,1].$$

2. Tail Value-at-Risk

$$TVaR_p(X) = \frac{1}{1-p} \int_p^1 VaR_q(X) dq.$$

Then,

$$TVaR_p(X) \leq TVaR_p(Y) \Longleftrightarrow X \leq_{sl} Y.$$

Heterogeneity in the claim occurrence probabilities

Theorem

(Ma, 1998) Consider two portfolios with individual claim amounts $X_i = J_{q_i} C_i$ and $Y_i = K_{p_i} C_i$ with $C_1, \ldots, C_n$ identically distributed. Then,

$$(p_1, \ldots, p_n) \overset{m}{\succeq} (q_1, \ldots, q_n) \Longrightarrow \sum_{i=1}^{n} Y_i \leq_{cx} \sum_{i=1}^{n} X_i.$$

This results shows that the heterogeneity decreases the dangerousness of the portfolio (as measured by the convex order between the corresponding total claim costs). It is due to the fact that the associated claim costs are identically distributed.

Example

Consider a portfolio made of $n = 4$ policies, which are independent and identical exponential random variables with mean 1.

$$(.4, 0, 0, 0) \overset{m}{\succeq} (.2, .2, 0, 0) \overset{m}{\succeq} (.2, 0.1, 0.1, 0) \overset{m}{\succeq} (.1, 0.1, 0.1, 0.1)$$

Then, the corresponding variances are

$$.64 < .72 < .764 < .813.$$

Consider two portfolios with individual claim amounts $X_i = J_{q_i} C_i$ and $Y_i = K_{p_i} C_i$, if

1. $(p_1, \dots, p_n) \in \mathbb{D}$, $(q_1, \dots, q_n) \in \mathbb{D}$, where $\mathbb{D} = \{\mathbf{x} \in \mathbb{R}^n | 1 \geq x_1 \geq x_2 \geq \dots \geq x_n \geq 0\}$.
2. $$C_1 \geq_{st} C_2 \geq_{st} \dots \geq_{st} C_n$$
3. $$(p_1, \dots, p_n) \overset{m}{\succeq} (q_1, \dots, q_n)$$

Theorem

(Denuit and Frostig, 2006) for any decreasing convex function, provided the expectations exist,

$$\mathrm{E}\left(g\left(\sum_{i=1}^{n} X_i\right)\right) \geq \mathrm{E}\left(g\left(\sum_{i=1}^{n} Y_i\right)\right)$$

Example

Consider a portfolio made of $n = 4$ policies, which are independent exponential random variables with corresponding mean 4, 3, 2, 1. Note that

$$(.4, 0, 0, 0) \overset{m}{\succeq} (.2, .2, 0, 0) \overset{m}{\succeq} (.2, 0.1, 0.1, 0) \overset{m}{\succeq} (.1, 0.1, 0.1, 0.1).$$

Computing

$$E\left(\exp(-tX)\right)$$

gives the corresponding values,

$$.68 < .714 < .725 < .754.$$

Scale family

Denuit and Frostig (2006) Consider two portfolios with individual claim amounts $X_i = J_{q_i} a_i Z_i$ and $Y_i = J_{q_i} b_i Z_i$, where $Z_1, \ldots, Z_n$ are independent and identically distributed. If

1. $(q_1, \ldots, q_n) \in \mathbb{D}$
2. $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$ and $b_1 \geq b_2 \geq \ldots \geq b_n \geq 0$
3. $(b_1, \ldots, b_n) \stackrel{m}{\succeq} (a_1, \ldots, a_n)$ then,

$$\sum_{i=1}^{n} Y_i \geq_{sl} \sum_{i=1}^{n} X_i.$$

Example

Consider a portfolio with $n = 3$ policies. Their respective claim occurrence probabilities are $(q_1, q_2, q_3) = (.2, .1, .1)$. Assume that

$$\mathbf{a} = (10, 2, 2), \quad \mathbf{b} = (6, 6, 2)$$

$$\mathbf{c} = (6, 4, 4), \quad \mathbf{d} = (4, 4, 4).$$

Since

$$\mathbf{a} \overset{m}{\succeq} \mathbf{b} \overset{m}{\succeq} \mathbf{c} \overset{m}{\succeq} \mathbf{d},$$

it holds that

$$\sum_{i=1}^{n} J_{q_i} a_i Z_i \geq_{sl} \sum_{i=1}^{n} J_{q_i} b_i Z_i \geq_{sl} \sum_{i=1}^{n} J_{q_i} c_i Z_i \geq_{sl} \sum_{i=1}^{n} J_{q_i} d_i Z_i$$

Example

Assuming that the Z_i's are i.i.d. exponential random variables with unit mean, we compute stop-loss premiums for different retentions.

$$SL(t, \mathbf{a}) = \mathrm{E}\left[\left(\sum_{i=1}^{n} J_{q_i} a_i Z_i - t)_+\right)\right].$$

$$SL(1, \mathbf{a}) = 0.917, \quad SL(2, \mathbf{a}) = 0.726, \quad SL(3, \mathbf{a}) = 0.585;$$
$$SL(1, \mathbf{b}) = 0.710, \quad SL(2, \mathbf{b}) = 0.512, \quad SL(3, \mathbf{a}) = 0.372.$$

The effect of increasing the degree of heterogeneity is now clear: the amount of the stop-loss premium increases as the scale parameters become more dispersed, whatever the retention t .

General Semi-group family

The family $\{Z_\alpha\}$ is said to possess the semi-group property if

$$Z_\alpha + Z_\beta \sim Z_{\alpha+\beta}.$$

Theorem

Consider two portfolios with individual claim amounts $X_i = J_{q_i} Z_{\theta_i}$ and $Y_i = J_{q_i} Z_{\gamma_i}$, where $\{Z_\alpha\}$ is is a family of independent positive random variables with semi-group property. If

1. $(q_1, \ldots, q_n) \in \mathbb{D}$
2. $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n \geq 0$ *and* $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n \geq 0$ *then,*

$$(\gamma_1, \ldots, \gamma_n) \overset{m}{\succeq} (\theta_1, \ldots, \theta_n) \Rightarrow \sum_{i=1}^{n} Y_i \geq_{sl} \sum_{i=1}^{n} X_i.$$

General case

Theorem

Consider two portfolios with individual claim amounts $X_i = J_{p_i} Z_{\theta_i}$ and $Y_i = J_{q_i} Z_{\gamma_i}$, where $\{Z_\alpha\}$ is is a family of independent positive random variables with semi-group property. If

1. $(p_1, \ldots, p_n), (q_1, \ldots, q_n) \in \mathbb{D}$
2. $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n \geq 0$ *and* $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n \geq 0$
3. $(\ln(\mathbf{p}), \theta) \precsim (\ln(\mathbf{q}), \gamma)$ *then,*

$$\sum_{i=1}^{n} Y_i \geq_{sl} \sum_{i=1}^{n} X_i.$$

Thank you!