Quantitative Risk Management of Variable Annuity Guaranteed Benefits: Opportunities and Challenges

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Variable annuity guaranteed benefits

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- Brief introduction
- Guaranteed minimum maturity benefit – Individual model
- Guaranteed minimum maturity benefit - Average model
- Guaranteed minimum withdrawal benefit
- Guaranteed minimum accumulation benefit
- Guaranteed minimum income benefit

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Guaranteed minimum maturity benefit – Individual model

Guaranteed minimum maturity benefit - Average model

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Quantitative risk management of variable annuities

- Variable annuity is in essence an “individual retirement plan”.
- Individual Retirement Account (IRA) and Employer-sponsored 401(k)/403(b) plan
- First variable annuities in the US was introduced by the Teachers Insurance and Annuities Association - College Retirement Equity Fund (TIAA-CREF) in 1952.
- Nowadays they refer mostly to the products in the invididual annuity markets.

Product design

- Arguably the most complex equity-based guarantee available to individual investors;
- Policyholders make contributions (called **purchase payments**) into subaccounts; typically single purchase payment at the inception;
- The term of the product can be broken down into two parts:
  - Accumulation phase; (resembles mutual funds)
  - Income phrase. (various types of guaranteed benefits)

Product design

- The value of each account varies with the performance of the particular fund in which it invests;

$$F_t = F_0 \frac{S_t}{S_0} e^{-mt}, \qquad 0 \leq t \leq T,$$

  where m is the annualized rate of total fees and charges.

  - F_t – total value of subaccounts at time t;
  - S_t – value of equity-index at time t;

- Without any guarantee, equity participation involves no risk to the variable annuity writer, who merely acts as a steward of the policyholders' funds.

- In order to compete with mutual funds, nearly all major variable annuity writers start to offer various types of investment guarantees, which transfer certain financial risks to the insurers. (Liabilities)

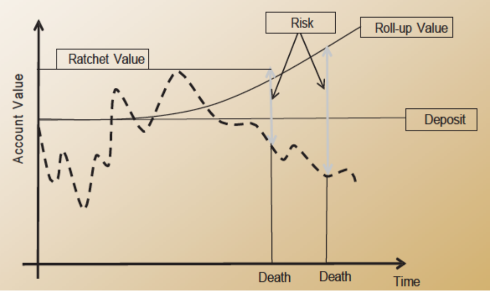
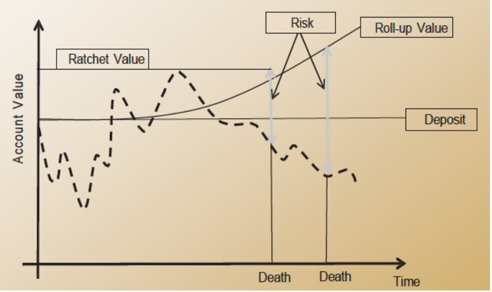


Abbildung : Guaranteed minimum benefits (GMxB)

- GMMB, GMDB, GMWB, GMAB, GMIB, GMSB

- Opportunities: Most current literature focuses on pricing and hedging. The market practice of risk management is based on Monte Carlo simulations. Growing needs for efficient computing techniques.



- Challenges: Most guarantee products involve complicated embedded options (more than standard options in finance). Difficult to assess interacting risks and long-term risks.

Research Goals

- Develop models that capture the essence of product features; (think about Black-Scholes)

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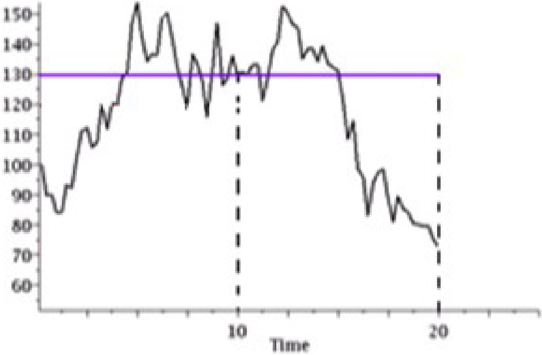
- Enjoy the journey, not just the destination!

Guaranteed Minimum Maturity Benefit(GMMB)

- Policyholder receives the greater of a minimum guarantee and account value. The GMMB writer is liable for the difference between the guarantee and account value, should the former exceeds the latter.



- M&E charge, rider charge are made on a daily basis as a certain percentage of subaccount values. (Incomes)

$$M_t = m_x F_t, \qquad 0 \leq t \leq T,$$

where m_x is the charge allocated to fund GMxB rider.($m_x < m$)

Guaranteed Minimum Maturity Benefit(GMMB)

- Gross liability: (T is the maturity date)

$$e^{-rT}(G - F_T)_+ I(\tau_x > T),$$

  where τ_x is the future lifetime of policyholder aged x at issue. (put-option-like payoff)

- Net liability= Gross liability - Margin offset income

$$L_0 = e^{-rT}(G - F_T)_+ I(\tau_x > T) - \int_0^{T \wedge \tau_x} e^{-rs} m_e F_s ds.$$

  (exotic option?) Mostly negative and rarely positive.

F investment fund value; G guaranteed benefit; T maturity date; r risk-free interest rate; m_e GMMB fee rate.

Risk management of equity-linking insurance

- Actuarial risk management (under real-world measures)
  - Quantile risk measure (Value-at-Risk)

$$V_\alpha := \inf\{y : \mathbb{P}[L_0 \leq y] \geq \alpha\}.$$

Minimum capital required to ensure sufficient fund to cover liability with the prob. of α.

  - Conditional tail expectation (Expected Shortfall)

$$\mathrm{CTE}_\alpha := \mathbb{E}[L_0 | L_0 > V_\alpha].$$

Average capital required to cover the liabilities exceeding the quantile measure.

- Dynamic hedging (financial engineering method)

How do we model the equity-index process?

- The model depends on the assumptions on the dynamics of the equity-index.

$$F_t = F_0 \frac{S_t}{S_0} e^{-mt}, \qquad 0 \leq t < T.$$

- Net liability= Gross liability - Margin offset income

$$L_0 = e^{-rT}(G - \underline{F_T})_+ I(\tau_x > T) - \underline{\int_0^{T \wedge \tau_x} e^{-rs} m_e F_s ds}.$$

- There is standard life table model for the future lifetime τ_x; No consensus on equity-index/asset price process S_t.

F investment fund value; *G* guaranteed benefit; *T* maturity date; *r* risk-free interest rate; *M* margin offset (fees).

- In principle, insurance companies are allowed to use their own stochastic models for equity returns. However, the stochastic models should meet the calibration criteria set by the American Academy of Actuaries.

Tabelle : Calibration Standard for Total Return Wealth Ratios

Percentile	1 Year	5 Years	10 Years	20 Years
2.5%	0.78	0.72	0.79	n/a
5.0%	0.84	0.81	0.94	1.51
10.0%	0.90	0.94	1.16	2.10
90.0%	1.28	2.17	3.63	9.02
95.0%	1.35	2.45	4.36	11.70

Common equity-return models listed in the AAA guildeline

- Independent Lognormal (ILN) (Geometric Brownian motion)
- Monthly Regime-Switching Lognormal Model with 2 Regimes (RSLN2)
- Monthly Regime-Switching Lognormal Model with 3 Regimes (RSLN3-M)
- Daily Regime-Switching Lognormal Model with 3 Regimes (RSLN3-D)
- Stochastic Log Volatility with Varying Drift (SLV)

"Essentially, all models are wrong, but some are useful..." — *George E.P. Box*

- For simplicity, we assume the dynamics of equity prices is driven by geometric Brownian motion.

$$S_t = S_0 \exp(\mu t + \sigma B_t), \qquad t \geq 0.$$

  - Fixed make-up of funds;
  - Target volatility fund. (industry trends)

Difficulties with Monte Carlo simulations

- 2008 SOA Report on Economic Capital of Life Insurance Companies

> *"76% of the respondents in the survey whose companies have more than \$10 billion of annual revenue use a form of stochastic approach. In contrast, only 27% of the respondents whose companies have less than \$1 billion use a form of stochastic approach."*

> *"Even the most resourceful companies are often forced to find a balance between accuarcy, efficiency and timeliness of delivery."*

- Conditioning on that the policyholder's death occurs after maturity

$$
\begin{aligned}
L_0 &= e^{-rT}(G - F_T) - \int_0^T e^{-rs} m_e F_s \, \mathrm{d}s \\
&= e^{-rT} G - \left(e^{-rT} F_T + m_e \int_0^T e^{-rs} F_s \, \mathrm{d}s \right) \\
&\sim e^{-rT} G - X_T \qquad \text{(identity in distribution)}
\end{aligned}
$$

where X is determined by

$$
\mathrm{d}X_t = \left[(\mu - \frac{\sigma^2}{2} - m - r) X_t + m_e \right] \mathrm{d}t + \sigma X_t \, \mathrm{d}B_t .
$$

- Risk measures

$$\begin{aligned}\mathbb{P}(L_0 > V_\alpha) &= \mathbb{P}(L_0 > V_\alpha | \tau_x > T)\mathbb{P}(\tau_x > T) \\ &= \mathbb{P}(e^{-rT}G - X_T > V_\alpha)\mathbb{P}(\tau_x > T) \\ &= \mathbb{P}(X_T < K)\mathbb{P}(\tau_x > T)\end{aligned}$$

for some constant K.
Similarly, one can show that

$$\text{CTE}_\alpha = \frac{1}{\alpha}\mathbb{E}[(e^{-rT}G - X_T)I_{\{X_T<K\}}]\mathbb{P}(\tau_x > T).$$

- Using spectral expansion, we obtained

$$\mathbb{P}(L_0 > V_\alpha) = \frac{x_0}{2\pi^2} \exp\left(-\frac{1}{4wx_0}\right) w^{\frac{\nu+1}{2}} \exp\left(\frac{1}{4x_0}\right) \int_0^\infty e^{-(\nu^2+p^2)t/2}$$
$$\times W_{-\frac{\nu+1}{2},\frac{ip}{2}}\left(\frac{1}{2wx_0}\right) W_{\frac{1-\nu}{2},\frac{ip}{2}}\left(\frac{1}{2x_0}\right) \left|\Gamma\left(\frac{\nu+ip}{2}\right)\right|^2 \sinh(\pi p)p\,\mathrm{d}p$$

  where W is the Whitter function of the second kind and

$$t := \frac{\sigma^2 T}{4}, \nu = \frac{2(\mu - m - r)}{\sigma^2}, x_0 = \frac{\sigma^2}{4m_e}, K = x_0 w, w = \frac{e^{-rT}G - V_\alpha}{F_0}.$$

- We also obtained similar results for $\mathbb{E}(L_0 I_{\{L_0 > V_\alpha\}})$, which determines the conditional risk measure CTE_α.

Comparative study: GMMB (Feng and Volkmer(2012))

- 10-year GMMB with full fund of initial deposit $G/F_0 = 1$;
- Mean and standard deviation of log-returns per annum $\mu = 0.09, \sigma = 0.3$;
- Risk-free discount rate per annum $r = 0.04$;
- M&E charges and rider charges per annum $m = 0.01$;
- GMMB rider charge 35 basis points of account value $m_e = 0.0035$.

Methods	Direct integration	Inverse Laplace	Monte Carlo
$V_{95\%}/F_0$	28.935%	28.935%	29.111%
Initial value	33%	(28%, 33%)	-
Time (mins)	3.7916	3.54375	396.224
$\mathrm{CTE}_{95\%}/F_0$	40.041%	40.042%	40.029%
Time (mins)	1.9325	0.28775	-

Comparative study: GMMB (Feng and Volkmer(2014a))

The same valuation assumptions.

Methods	Integration	Inverse Lap	Spectral	Green
$V_{90\%}$	12.55036%	12.55036%	12.55035%	12.55036%
Initial	10%	$(12\%, 14\%)$	$(12\%, 14\%)$	$(12\%, 14\%)$
Time	3.674(mins)	5.032(mins)	51.579(secs)	0.172(secs)
$\mathrm{CTE}_{90\%}$	30.29643%	30.29648%	30.29643%	30.29648%
Time	1.867(mins)	0.285(mins)	2.953(secs)	0(secs)

Tabelle : A comparison of computational methods for the GMMB rider

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- In practice, valuation actuaries use spreadsheets to run simulations based on traditional life insurance reserving methods. In each period, they compute

  Change in Surplus = Fee Income - Guaranteed Benifits
  - Expenses + Interest on Surplus

- The spreadsheet calculations are in essence the difference equation from a mathematical point of view. It is easy to show that the net liability used in practice for the GMMB is

$$L_0^* = e^{-rT}(G - F_T)_+ \, {}_T p_x - \int_0^T {}_s p_x e^{-rs} m_e F_s \, \mathrm{d}s,$$

  where ${}_s p_x$ is the prob. that a life-age x survives s years.

- The risk measures of L_0^* can be computed using PDEs (Feng (2014)) and by comonotonic approximation (Dhaene, Feng and Jing (2014)).

Connection between individual and average models

- Individual model:

$$L_0 = e^{-rT}(G - F_T)_+ I(\tau_x > T) - \int_0^{T\wedge\tau_x} e^{-rs} m_e F_s ds.$$

- Average model:

$$L_0^* = e^{-rT}(G - F_T)_+ \, {}_T p_x - \int_0^T {}_s p_x e^{-rs} m_e F_s \, \mathrm{d}s.$$

- Connection: Assuming all policies are **of equal size** and the future lifetimes of policyholders are mutually indepedent, we can show that

$$\frac{1}{N}\sum_{i=1}^{N} L_0^{(i)} \longrightarrow L_0^*, \qquad \text{almost surely.}$$

F investment fund value; G guaranteed benefit; T maturity date; r risk-free interest rate; m_d GMMB fee rate.

Differences between individual and average models

- Individual model: (Two sources of randomness)

$$L_0 = e^{-rT}(G - F_T)_+ I(\tau_x > T) - \int_0^{T \wedge \tau_x} e^{-rs} m_e F_s ds.$$

(The policyholder could die when the fund performs the worst and the guarantee benefit is highest.)

- Average model: (One source of randomness, the mortality risk is fully diversified.)

$$L_0^* = \mathbb{E}[L_0 | \mathcal{F}_T] = e^{-rT}(G - F_T)_+ \, {}_T p_x - \int_0^T {}_s p_x e^{-rs} m_e F_s \, \mathrm{d}s.$$

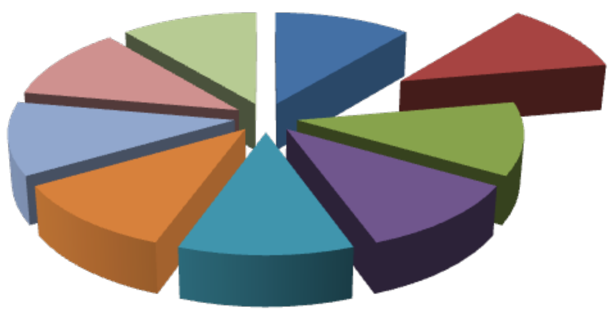
- They make no difference for pricing since $\mathbb{E}[L_0] = \mathbb{E}[L_0^*] = \mathbb{E}[\mathbb{E}[L | \mathcal{F}_T]]$, yet they are different in terms of tail probabilities.
- We can compute the tail probability $\mathbb{P}(L_0^* > V)$ using numerical PDE methods but it is much slower than the computation of $\mathbb{P}(L_0 > V)$ through Laplace transform.

Reliance on law of large numbers

- Law of large numbers may not apply
  - Unlike exchange-traded financial instruments, there is no standardized contract size for variable annuities. Purchase payments vary greatly by contract.

Policyholder A

$10,000



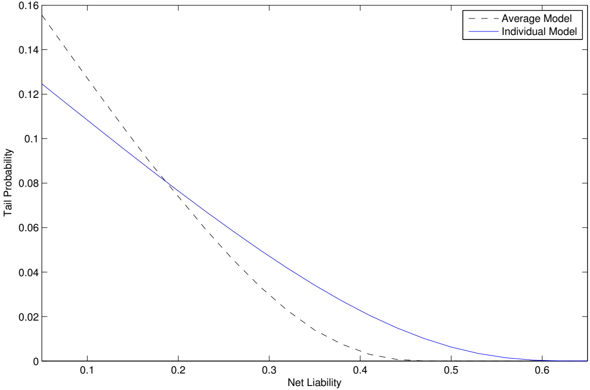
Policyholder B

$1,000

  - For example, if we use $1000 as the benefit base for calculation. The units from A's portfolio may be independent of B's unit. But the 10 units from A's portfolio are highly correlated as the contract lengths of these units all depend on A's future lifetime.

Comparison of Net Liabilities in Individual and Average Models

(Model parameters are the same as outlined in earlier slides)



(1) Individual serves good approximation of average (2) Financial risk is the dominating force in determining tail

Connection between individual and average models

- Convex order: (full diversification versus full concentration of mortality risk?)

$$L_0^* \longleftarrow \frac{1}{N}\sum_{i=1}^{N} L_0^{(i)} \leq_{\mathrm{cx}} L_0,$$

which implies that

$$\mathrm{CTE}(L_0^*) \leq \mathrm{CTE}(L_0).$$

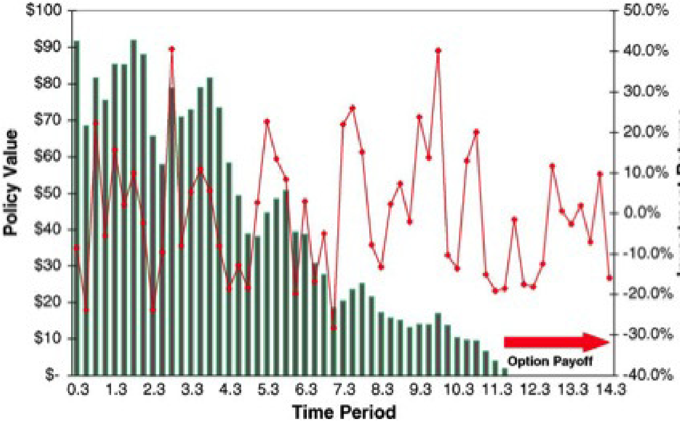
(NL_0 corresponds to the Fréchet upper bound of the joint contract lifetimes $(\tau_x^{(1)}, \tau_x^{(2)}, \cdots, \tau_x^{(n)})$).

- Question: Is $\mathrm{CTE}(L_0^*)$ the lower bound of the CTEs of liabilities with any dependency of $(\tau_x^{(1)}, \tau_x^{(2)}, \cdots, \tau_x^{(n)})$?

Guaranteed Minimum Withdrawal Benefit(GMWB)

- Contains no life insurance component.
- Provides a minimal payout based on the initial purchase payment.
- For example: A policyholder is guaranteed the ability to withdraw \$7 per annum per \$100 of initial investment until the original \$100 has been fully exhausted. (The benefit expires after $\$100/\$7 \approx 14.28$ years.)

Guaranteed Minimum Withdrawal Benefit(GMWB)



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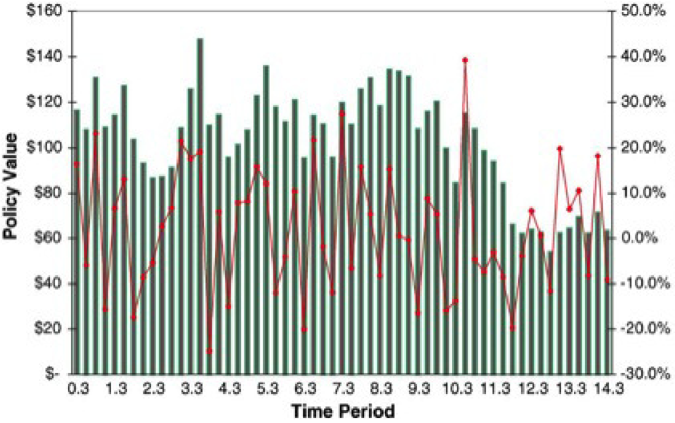

Mathematical Formulation

- Assume the dynamics of the equity-index is given by

$$\mathrm{d}S_t = \mu S_t \, \mathrm{d}t + \sigma S_t \, \mathrm{d}B_t.$$

- The accumulated value of VA subaccounts

$$F_t = F_0 \frac{S_t}{S_0} e^{-mt} - wt, \qquad t \geq 0.$$

- The dynamics of VA investment fund is driven by

$$\mathrm{d}F_t = [(\mu - m)F_t - w] \, \mathrm{d}t + \sigma F_t \, \mathrm{d}W_t, \qquad F_0 = G > 0.$$

Mathematical Formulation (Joint work with Hans W. Volkmer)

- Pricing from an investor's point of view (Milevsky and Salisbury (2006))
  - Present value of guaranteed income (Maturity: $T = G/w$)

$$w \int_0^T e^{-rt}\,\mathrm{d}t = \frac{w}{r}(1 - e^{-rT}).$$

  - Investment income

$$e^{-rT} F_T I(\tau_0 > T), \quad \tau_0 := \inf\{t : F_t < 0\}.$$

  - The fair price m_w is determined by

$$F_0 = \frac{w}{r}(1 - e^{-rT}) + \mathbb{E}^Q[e^{-rT} F_T I(\tau_0 > T)].$$

w withdrawal rate; r risk-free interest rate; T guaranteed period; F fund value; τ_0 first time fund is exhausted.

Mathematical Formulation (Joint work with Hans W. Volkmer)

- Pricing from an insurer's point of view
  - Present value of Income (Assets)

$$m_w \int_0^{\tau_0 \wedge T} e^{-rt} F_t \, \mathrm{d}t.$$

  - Present value of Outgo (Liabilities)

$$w \int_{\tau_0}^{T} e^{-rt} \, \mathrm{d}t \; I(\tau_0 < T) = \frac{w}{r}(e^{-r\tau_0} - e^{-rT}) I(\tau_0 < T).$$

  - The fair price m_w is determined by

$$\mathbb{E}^Q \left[m_w \int_0^{\tau_0 \wedge T} e^{-rt} F_t \, \mathrm{d}t \right] = \frac{w}{r} \mathbb{E}^Q \left[(e^{-r\tau_0} - e^{-rT}) I(\tau_0 < T) \right].$$

w withdrawal rate; r risk-free interest rate; T guaranteed period; F fund value; m_w fees to fund GMWB.

- Pricing from an investor's point of view

$$F_0 = \frac{w}{r}(1 - e^{-rT}) + \mathbb{E}^Q[e^{-rT} F_T I(\tau_0 > T)].$$

- Pricing from an insurer's point of view

$$\mathbb{E}^Q\left[m_w \int_0^{\tau_0 \wedge T} e^{-rt} F_t \,\mathrm{d}t\right] = \frac{w}{r}\mathbb{E}^Q\left[(e^{-r\tau_0} - e^{-rT}) I(\tau_0 < T)\right].$$

- Milevsky and Salisbury (2006) used numerical PDE methods to price the GMWB from an investor's perspective. We found closed-form solutions in both cases.
- One can prove using Dynkin's formula that the two methods of pricing are equivalent when $m_w = m$.

Risk Measures of GMWB

- Net liability of the GMWB

$$L := w \int_{\tau_0 \wedge T}^{T} e^{-rt}\,\mathrm{d}t - m_w \int_0^{\tau_0 \wedge T} e^{-rt} F_t\,\mathrm{d}t.$$

- Approximations of $\mathbb{P}(L > V_\alpha)$ can be found by joint distribution of the GBM with affine drift and its integral.
  - Joint Laplace transform is a solution to the doubly confluent hypergeometric equation. (Feng and Volkmer (2014c))
- Numerical PDE method?
- Importance sampling?

w withdrawal rate; r risk-free interest rate; T guaranteed period; F fund value; m_w annual fee for GMWB.

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Guaranteed minimum accumulation benefit

- Under a GMAB policy there may be multiple renewal dates on which the guarantee for the next period is reset to the greater of the guarantee from previous period and the then-current account value at the renewal.
- If the guarantee is in-the-money then the insurer must pay out the difference and the guarantee for next period stays the same as before.
- If the guarantee is out-of-the-money, there is no guarantee payment but the guarantee level is automatically reset to the separate fund value at the time.

Mathematial formulation (Joint work with Bingji Yi and Hans Volkmer)

- Consider the payoff under GMAB with one rollover at T_1 and maturity at T_2. The present value of GMAB liability at T_1 is given by

$$e^{-rT_1} G_0 \left(1 - e^{-mT_1} \frac{F_0}{G_0} \frac{S_{T_1}}{S_0}\right)_+$$

The present value of GMAB liability at T_2 is given by

$$e^{-rT_2} G_0 \max\left(1, e^{-mT_1} \frac{F_0}{G_0} \frac{S_{T_1}}{S_0}\right) \left(1 - e^{-m(T_2-T_1)} \frac{S_{T_2}}{S_{T_1}}\right)_+$$

Variable annuity guaranteed benefits

Guaranteed Minimum Income Benefit

- The policyholder has the option to take the lump sum payment of account value or a life annuity with guaranteed minimum income.
- Insurer's liability at maturity

$$\max\{Xa_{65}(T) - F_T, 0\}$$

where T is the time until retirement age 65, $a_{65}(T)$ is the market price of a life annuity issued to a 65-year-old at time T and X is the guaranteed minimum monthly income.

Current literature: Using interest rate models

- First make assumptions on interest rate models. For example, Cox-Ingersoll-Ross model

$$\mathrm{d}R(t) = (a - bR(t))\,\mathrm{d}t + \sigma\sqrt{R(t)}\,\mathrm{d}B(t).$$

  Bond price does have closed-form solution.

$$B(t, T) = f(t, R(t)),$$

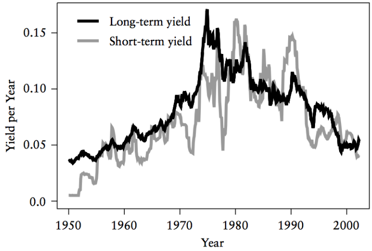
  where $f(t, r) = \exp\{-rC(t, T) - A(t, T)\}$ and A, C are all expressed in explicit forms.

- Then the life annuity price becomes a rather complex random variable

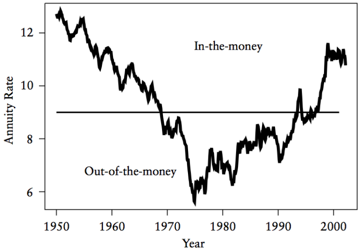
$$a_{65}(65 - x) = \sum_{k=1}^{12\times(65-x)} B(t, k).$$

Future work:

- Another approach is to introduce a reasonable stochastic model for the annuity prices directly.



(a) Interest rates



(b) Annuity prices

Thank you very much for your attention!