OPTIMAL INSURANCE AND REINSURANCE

CONTRACTS

Markus Schicht

82 Pages   August 2003

The paper presents an optimal insurance contract, describes reinsurance

fundamentals and optimal reinsurance contracts under given assumptions, and finally

discusses the problem of terrorism insurance coverage.

APPROVED:

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Date   Krzysztof Ostaszewski, Chair

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Date   Kulathavaranee Thiagarajah

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In the United States and many other countries, the insurance market has a big
importance for society and economy. Thus, it is very important to investigate this market
or parts of it, as the reinsurance market, for example. The question concerning an optimal
contract is of large interest and a topic of research.

This thesis presents optimal insurance and reinsurance contracts under given
assumptions. The first chapter should give an introduction to the purpose and properties of
insurance. The second chapter introduces probability spaces and random variables,
discusses utility functions and presents an optimal insurance contract. The third chapter
then gives an overview for the purpose of reinsurance, shows several types of reinsurance
contracts, and presents some information about large reinsurance companies. In the fourth
chapter, we show several principles of premium calculations and introduce a condition
under which a reinsurance contract is optimal, using a utility function approach. Then we
show the optimality of the change loss reinsurance contract. Chapter five finally discusses
the situation in the insurance market after the terror attacks on the United States on
September 11, 2001. We present fundamentals of the federal program of claim adjustment

in the case of future terrorism attacks until the end of the year 2005. Finally, we discuss

risk spreading methods for terrorism coverage after the year 2005. These methods are the

ceding of risk to the capital market or the creating of a reinsurance pool, as done in the

United Kingdom in 1993.

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OPTIMAL INSURANCE AND REINSURANCE

CONTRACTS

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A Thesis Submitted in Partial
Fulfillment of the Requirements
for the Degree of

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CHAPTER I

INTRODUCTION

Everybody has certain plans and visions for the future life. Unfortunately, these plans will not unfold with certainty.

Of course, we have to face this problem also with regard to our future financial situation. One approach to solve this problem is to buy insurance. To understand why, let us consider the purpose and general goal of insurance, as given by Bowers et al. (1997):

Definition 1 *Insurance is designed to protect against serious financial exposures that result from random events intruding on the plans of individuals.*

Today insurance has achieved great importance in commerce and society in general. The purchase of insurance is a central element in the financial security for the future for many human beings or companies, and insurance companies are an important part of the whole economy.

To give some examples, let us begin with the fact that every individual is going to die with certainty. The problem is that nobody knows the date of his or her death. Thus, many people purchase a life insurance in order to protect the financial security of the survivors in the family.

Another example is automobile insurance: People who drive a car have to face the possibility of an accident. In virtually all countries in the world, the purchase of automobile liability insurance is mandatory.

These are only two examples of insurance coverage, and individuals or institutions have the opportunity to purchase insurance for many different fields such as health, annuity, property and casualty and many more.

It is interesting to observe that typically customers have almost no influence on the details of a contract. Once we decide to buy a certain insurance, we accept all the details of the contract as given by the insurance company. On the other hand, the structure of a reinsurance contract (we will return to that issue later) is often quite flexible and negotiated between parties.

CHAPTER II

THEORY OF AN OPTIMAL INSURANCE CONTRACT

2.1 Probability Spaces and Random Variables

We begin our analysis with some definitions, as given by Billingsley (1995):

Definition 2 *Let Σ be a collection of subsets of a nonempty set Ω. Then Σ is called a σ-algebra, if*

(i) $\Omega \in \Sigma$,

(ii) $A \in \Sigma \Rightarrow A^c \in \Sigma$,

(iii) $A_i \in \Sigma, i \in \mathbb{N} \Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \Sigma$.

Definition 3 *A mapping $Pr \colon \Sigma \to [0,1]$ on a σ-algebra Σ is called a probability measure, if it satiesfies the following conditions:*

(i) $Pr(\emptyset) = 0, Pr(\Omega) = 1$,

(ii) If $(A_i)_{i \in \mathbb{N}}$ is a disjoint sequence of sets in Σ and $\bigcup_{i \in \mathbb{N}} A_i \in \Sigma$, then

$$Pr\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} Pr(A_i).$$

The triple (Ω, Σ, Pr) is called a probability measure space.

Definition 4

(i) *The Borel algebra $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by all open intervals $(a, b) \in \mathbb{R}$.*

(ii) *A function $f \colon \Omega \to \mathbb{R}$ is called Borel measurable on a probability space (Ω, Σ, Pr), if*

$$f^{-1}(E) \in \Sigma \;\; for \;\; all \;\; E \in \mathcal{B}(\mathbb{R}).$$

Definition 5 *A random variable X on a probability space (Ω, Σ, Pr) is a Borel measurable function*

$$X \colon \Omega \to \mathbb{R}.$$

Definition 6 *A property is called to hold almost surely (a.s.) in a probability space (Ω, Σ, Pr), if there exists a set $B \in \Sigma$ with $Pr(B) = 0$ such that the property holds on $\Omega \setminus B$.*

2.2 Utility Functions

We consider a situation as presented by Gwartney and Stroup (1982):

When a decision maker has to choose among alternatives of equal costs, he will select the option that creates the greatest benefit for him. In order to measure this benefit, we determine his utility.

Gwartney and Stroup (1982) define utility as a number that represents the level of subjective satisfaction that a decision maker derives from the choice of a specific alternative. In order to attach utility to alternatives, we define a utility function as follows:

Definition 7 *A function $u \colon \mathbb{R}_+ \to \mathbb{R}_+$, with existing second derivative u'' is called a utility function, if the following properties hold:*

(i) *$u(w)$ is the utility of the decision maker at a wealth $w \in \mathbb{R}_+$,*

(ii) $u'(w) > 0$ for all $w \in \mathbb{R}_+$,

(iii) $u''(w) < 0$ for all $w \in \mathbb{R}_+$.

Remark 1

(a) *Property (ii) means that the decision maker prefers more wealth to less wealth.*

(b) *Property (iii) means that the decision maker is risk averse: He will always prefer a certainty of a given amount of money δ to a situation where the expected return in δ.*

In order to derive the decision maker's best utility under conditions of uncertainty, we use the expected utility approach, as presented by Panjer (1998):

We denote the random loss of a decision maker in a time period by a nonnegative random variable X over a probability space (Ω, Σ, Pr). The decision maker has a wealth of w. We can observe that $0 \leq X \leq w$.

The decision maker has the choice of buying insurance for a premium P to cover the loss, or not buying it. Without buying insurance, his expected utility is

$$E[u(w - X)].$$

If the decision maker buys insurance, then his wealth at the end of the time period will be

$$w - P,$$

if we neglect the time value of money. We observe that the decision maker decides to

purchase insurance as long as

$$u(w - P) > E[u(w - X)].$$

Lemma 1 *If $u''(w) < 0$ for all $w \in \mathbb{R}_+$, then*

$$u(w) - u(z) \le (w - z)u'(z) \quad for \ \ all \ \ w, z \in \mathbb{R}_+.$$

Proof:

Let $C_{w,z} : [0, 1] \to \mathbb{R}$ be defined by

$$C_{w,z}(\lambda) = u(\lambda w + (1 - \lambda)z), \lambda \in [0, 1].$$

First we want to prove that $C_{w,z}$ is a concave function: We know that if $u''(w) < 0$ for all $w \in \mathbb{R}_+$, then u is concave on $\mathbb{R}_+$. Let $w, z \in \mathbb{R}_+, \lambda_1, \lambda_2 \in [0, 1], \alpha \in (0, 1)$. Then:

$C_{w,z}(\alpha\lambda_1 + (1 - \alpha)\lambda_2)$

$$
\begin{aligned}
&= \ u[(\alpha\lambda_1 + (1 - \alpha)\lambda_2)w + (1 - \alpha\lambda_1 - (1 - \alpha)\lambda_2)z] \\
&= \ u[\alpha\lambda_1 w + (1 - \alpha)\lambda_2 w + z - \alpha\lambda_1 z - (1 - \alpha)\lambda_2 z] \\
&= \ u[\alpha(\lambda_1 w - \lambda_1 z) + (1 - \alpha)(\lambda_2 w - \lambda_2 z) + z] \\
&= \ u[\alpha(\lambda_1 w + (1 - \lambda_1)z) + (1 - \alpha)(\lambda_2 w + (1 - \lambda_2)z)] \\
&\ge \ \alpha u[\lambda_1 w + (1 - \lambda_1)z] + (1 - \alpha)u[\lambda_2 w + (1 - \lambda_2)z] \\
&= \ \alpha C_{w,z}(\lambda_1) + (1 - \alpha)C_{w,z}(\lambda_2)
\end{aligned}
$$

Since we know now that $C_{w,z}$ is a concave function,

$$C_{w,z}(t) = C_{w,z}(t \cdot 1 + (1-t)0)$$

$$\geq tC_{w,z}(1) + (1-t)C_{w,z}(0) \quad for \ all \ t \in (0,1]$$

Therefore,

$$C_{w,z}(t) - C_{w,z}(0) \geq t[C_{w,z}(1) - C_{w,z}(0)]$$

$$= t[u(w) - u(z)] \quad for \ all \ t \in (0,1]$$

$$\Rightarrow \frac{C_{w,z}(t) - C_{w,z}(0)}{t} \geq u(w) - u(z) \text{ for all } t \in (0,1]$$

$$\Rightarrow \lim_{t\to 0} \frac{C_{w,z}(t) - C_{w,z}(0)}{t} = \frac{d}{dt} C_{w,z}(t)\Big|_{t=0} \geq u(w) - u(z)$$

Let us calculate the derivative of $C_{w,z}$ with respect to t:

$$\frac{d}{dt} C_{w,z}(t) = (w - z)\left\{\frac{d}{dt} u(tw + (1-t)z)\right\}$$

Let us consider $t = 0$, and we obtain:

$$\frac{d}{dt} C_{w,z}(t)\Big|_{t=0} = (w - z)u'(z)$$

$$\geq u(w) - u(z)$$

$\square$

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Theorem 1 *(Jensen's Inequality) Let X be a random variable with $E(X) < \infty$, and let $u''(w) < 0$ for all $w \in \mathbb{R}$. Then,*

$$u(E[X]) \geq E[u(X)].$$

Proof:

Since $E(X) < \infty$, we can conclude from Lemma 1 that

$$u(w) \leq u(E[X]) + u'(E[X])(w - E[X]) \quad for \ \ all \ \ w \in \mathbb{R}.$$

If we replace w by X and take the expected value on both sides, we obtain:

$$
\begin{aligned}
E[u(X)] &\leq u(E[X]) + u'(E[X])(E[X] - E[X]) \\
&= u(E[X])
\end{aligned}
$$

$\square$

Jensen's Inequality says that

$$u(w - E[X]) \geq E[u(w - X)].$$

This indicated that a risk-averse decision maker is willing to pay a higher premium for insurance than the expected value of his loss.

Panjer (1998) provides a list of commonly used utility functions. We present them together with their first and second derivatives:

(i) Quadratic Utility:

$$u(x) = x - \frac{x^2}{2b}, \text{ for } x < b,$$

$$u'(x) = 1 - \frac{x}{b},$$

$$u''(x) = -\frac{1}{b}.$$

(ii) Exponential Utility:

$$u(x) = 1 - e^{-ax}, \text{ for } x > 0 \text{ and } a > 0,$$

$$u'(x) = ae^{-ax},$$

$$u''(x) = -a^2 e^{-ax}.$$

(iii) Power Utility:

$$u(x) = \frac{1}{\alpha}(x^\alpha - 1), \text{ for } x > 0 \text{ and } \alpha \in (0, 1),$$

$$u'(x) = x^{\alpha-1},$$

$$u''(x) = (\alpha - 1)x^{\alpha-2}.$$

(iv) Logarithmic Utility:

$$u(x) = a \cdot lnx + b, \text{ for } x > 0 \text{ and } a > 0,$$

$$u'(x) = \frac{a}{x},$$

$$u''(x) = -\frac{a}{x^2}.$$

2.3 Mathematical Investigation of an Optimal Insurance Contract

In the following section, we present a part of the work of Arrow (1963). The basics of that part are described by Bowers et al. (1997).

As we have already stated, a decision maker has a given amount of wealth, denoted by w and encounters a random loss in the next time period, denoted by a nonnegative random variable X over a probability space (Ω, Σ, Pr).

Now we want to answer the question what opportunities the decision maker has. The decision maker has many opportunities, but we want to investigate this basic choice: The decision maker can purchase an insurance contract that will pay a portion of his loss (including the possibility that this portion can be his entire loss). This portion can be expressed by a function

$$I \colon \mathbb{R}_+ \to \mathbb{R}_+.$$

Such an insurance contract should not pay more than the loss, meaning it should not be an incentive to incur the loss. In addition, we assume that the decision maker does not have to make a payment to the insurance company when the loss occurs. Hence, if x is the loss of the decision maker in the next time period, we assume that

$$0 \leq I(x) \leq x \ \ for \ \ all \ \ x \in \mathbb{R}_+. \tag{2.1}$$

To simplify the situation, let us assume that all feasible insurance contracts with expected claims

$$E[I(X)] = \gamma, \ \ \gamma \in \mathbb{R}_+, \tag{2.2}$$

can be purchased for the same price. The decision maker has to decide on that price, meaning he has to decide on the amount of money to be paid for insurance. That amount is denoted by P.

We will now investigate the following key question: Which one of the insurance contracts that fulfill (2.1) and (2.2) and require a premium P should be purchased by the decision maker in order to maximize his expected utility?

We begin this investigation with a definition of a basic form of insurance and reinsurance contract.

Definition 8 *Let $M \in \mathbb{R}_+$. Then the function $I_M \colon \mathbb{R}_+ \to \mathbb{R}_+$, defined by*

$$I_M(x) = \begin{cases} 0 & \text{if } x < M, \\ x - M & \text{otherwise,} \end{cases}$$

is called a stop-loss or excess-of-loss insurance.

Remark 2

(i) *M is called deductible amount or deductible.*

(ii) *Benefit payments do not begin until the decision maker's loss is greater than the deductible. If the loss exceeds the deductible, the insurer only pays the difference of the loss and the deductible.*

We assume that the decision maker's loss X is a continuous random variable. Let f denote the probability density function (*p.d.f.*), and F denote the cumulative distribution function (*c.d.f.*) of X.

Lemma 2 *If $E(X)$ exists, then $\lim_{x \to \infty} x(1 - F(x)) = 0$.*

Proof:

We present the proof as suggested by Fisz (1963):

$$\lim_{x \to \infty} x(1 - F(x)) = \lim_{x \to \infty} x \cdot Pr(X > x)$$

$$= \lim_{x \to \infty} \left(x \cdot \int_x^\infty f(y)\, dy \right)$$

$$= \lim_{x \to \infty} \left(\int_x^\infty x \cdot f(y) \, dy \right)$$

$$\leq \lim_{x \to \infty} \left(\int_x^\infty y \cdot f(y) \, dy \right)$$

$$= 0$$

$\square$

As we have stated above, we denote the expected amount that the insurer has to pay by γ. Then:

$$E[I_M(X)] = \int_M^\infty (x - M) f(x) \, dx \qquad (2.3)$$

$$= \int_M^\infty (x - M) \, dF(x)$$

$$= \int_M^\infty -(x - M) \, d(1 - F(x))$$

$$= -(x - M)(1 - F(x)) \Big|_{x=M}^{x \to \infty} + \int_M^\infty (1 - F(x)) \, d(x - M)$$

$$= -\lim_{x \to \infty} (x - M)(1 - F(x)) + \int_M^\infty (1 - F(x)) \, dx$$

$$= \int_M^\infty (1 - F(x)) \, dx \qquad \text{(using Lemma 2)} \qquad (2.4)$$

$$= \gamma$$

For a given γ, we are able to show that a unique deductible, denoted by M^*, exists:

Lemma 3 *Let $\gamma \in (0, E[X])$. Then there exists exactly one $M^* \in \mathbb{R}_+$ such that*

$$\gamma = \int_{M^*}^{\infty} (1 - F(x)) \, dx.$$

Proof:

Let us consider $\int_M^{\infty} (1 - F(x)) \, dx$ as a function of M. Then,

$$\frac{d}{dM} \int_M^{\infty} (1 - F(x)) \, dx = -1 + F(M),$$

by the Fundamental Theorem of Calculus.

We want to show that $\int_M^{\infty} (1 - F(x)) \, dx$ is a strictly monotone decreasing function in M, in order to imply that there exists exactly one M which solves the equation $\gamma = \int_M^{\infty} (1 - F(x)) \, dx$. First, we have to show that $F(M) < 1$.

Suppose that $F(M) = 1$. Then $Pr(X \geq M) = 0$. But this is a contradiction.

Thus we have shown that $F(M) < 1$. This implies that

$$\frac{d}{dM} \int_M^{\infty} (1 - F(x)) \, dx = -1 + F(M) < 0.$$

This means that $\int_M^{\infty} (1 - F(x)) \, dx$ is a strictly monotone decreasing function in M. Hence we can conclude that for every $\gamma \in (0, E[X])$ there exists exactly one $M^* \in \mathbb{R}_+$ such that

$$\gamma = \int_{M^*}^{\infty} (1 - F(x)) \, dx.$$

$\square$

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Now let us assume that the decision maker has defined a utility function u, which we have defined in Definition 7.

The following theorem is a central element of the work of Arrow (1963) and describes an optimal insurance contract:

Theorem 2 *(Bowers et al., 1997) If a decision maker*

- *has wealth of amount w,*

- *has defined a utility function u such that $u''(w) < 0$ for all $w \in \mathbb{R}_+$,*

- *faces a random loss X,*

- *will spend an amount of P on an insurance contract,*

and the insurance market offers for a payment of P all feasible insurance contracts I of the form $0 \leq I(x) \leq x$, with $E(I(X)) = \gamma$, then the expected utility of the decision maker will be maximized by purchasing an insurance contract of the form

$$
I_{M^*}(x) = \begin{cases} 0 & if \ \ x < M^*, \\ x - M^* & if \ \ x \geq M^*, \end{cases}
$$

where M^* is the unique solution of the equation

$$
\gamma = \int_M^\infty (x - M) f(x) \, dx.
$$

Proof:

Let $0 \le I(x) \le x$ for all $x \in \Omega$. Then, by applying Lemma 1, we get:

$$u(w - x + I(x) - P) - u(w - x + I_{M^*}(x) - P)$$

$$\le [I(x) - I_{M^*}(x)]u'(w - x + I_{M^*}(x) - P)$$

We want to prove that

$$[I(x) - I_{M^*}(x)]u'(w - x + I_{M^*}(x) - P) \le [I(x) - I_{M^*}(x)]u'(w - M^* - P)$$

- Case 1: $I(x) = I_{M^*}(x)$

  $\Rightarrow 0 \le 0$

- Case 2: $I(x) < I_{M^*}(x)$

  We know that $0 \le I(x) \le x$

  $\Rightarrow I_{M^*}(x) > 0$

  $\Rightarrow I_{M^*}(x) = x - M^*$

  $\Rightarrow M^* = x - I_{M^*}(x)$

  Hence, we obtain the following:

  $$[I(x) - I_{M^*}(x)]u'(w - x + I_{M^*}(x) - P) \le [I(x) - I_{M^*}(x)]u'(w - M^* - P)$$

  $$\Leftrightarrow [I(x) - I_{M^*}(x)]u'(w - M^* - P) \le [I(x) - I_{M^*}(x)]u'(w - M^* - P)$$

- Case 3: $I(x) > I_{M^*}(x)$

  Let $x \ge M^* \Rightarrow I_{M^*}(x) = x - M^*$

  Let $x < M^* \Rightarrow I_{M^*}(x) = 0 > x - M^*$

  Thus, $I_{M^*}(x) \ge x - M^*$ for all $x \in \Omega$

  $\Rightarrow I_{M^*}(x) - x - P \ge -M^* - P$

  $\Rightarrow u'(w - x + I_{M^*}(x) - P) \le u'(w - M^* - P)$, since u' is a decreasing function

  $\Rightarrow [I(x) - I_{M^*}(x)]u'(w - x + I_{M^*}(x) - P) \le [I(x) - I_{M^*}(x)]u'(w - M^* - P)$

Finally, we obtain:

$$[I(x) - I_{M^*}(x)]u'(w - x + I_{M^*}(x) - P) \le [I(x) - I_{M^*}(x)]u'(w - M^* - P)$$

Hence,

$$E[u(w - X + I(X) - P)] - E[u(w - X + I_{M^*}(X) - P]$$

$$
\begin{aligned}
&\le\ E[(I(X) - I_{M^*}(X))u'(w - X + I_{M^*}(X) - P)] \\
&\le\ E[(I(X) - I_{M^*}(X))u'(w - M^* - P] \\
&=\ u'(w - M^* - P)E[I(X) - I_{M^*}(X)] \\
&=\ u'(w - M^* - P)(\beta - \beta) \\
&=\ 0
\end{aligned}
$$

$$\Rightarrow E[u(w - X + I(X) - P)] \le E[u(w - X + I_{M^*}(X) - P]$$

The decision maker has a wealth w and faces a random loss of X. He receives benefits of $I(X)$, but has to pay a premium of P for the insurance contract. Therefore, $I_{M^*}(X)$ maximizes his expected utility.

$\square$

CHAPTER III

REINSURANCE FUNDAMENTALS

3.1 Preface

Reinsurance began in the 19th century. As described by the General Re Corporation (2003), the first reinsurance contract was written in 1852. But reinsurance did not become an important part of the insurance industry until the 20th century (Gastel, 1995). Indeed, during that century, the insurance industry prospered. Hence reinsurance became more and more important. New kinds of reinsurance concepts were conceived, and new applications were devised.

As Gastel (1995) describes, reinsurance attained more visible public profile in mid 1980s during the so called liability insurance crisis. Before that time, the general public did not know reinsurance except those people who were participating in that business.

Today reinsurance is a topic of increasing interest in many parts outside the insurance business as banks, governments, or the press. Yet many people who know basics of insurance do not know that reinsurance exists or what reinsurance is.

3.2 Introduction to Reinsurance

Definition 9 *Reinsurance is an insurance contract between two insurance companies. The insured company is called the primary insurer or ceding company, and the insurer of this ceding company is called the reinsurer or reinsurance company.*

One of the reasons for the necessity of such a form of insurance is very similar to a reason for a general insurance contract. The goal of the ceding company is to delete certain financial incertitude by transferring, or *ceding*, a part of the risk to a reinsurer. The reinsurer assumes that risk, but receives a premium for that and agrees to make a payment in case of an underlying loss, according to the contract.

An important characteristic of reinsurance is that it does not change the insurer's responsibility to the original policyholder. With or without that contract, the insurer has to meet the obligation of the issued policy. Thus, in most cases the original policyholder knows nothing about the existence of a reinsurance contract.

3.3 Main Functions of Reinsurance

As presented by Gastel (1995), reinsurance achieves four major functions which are all related closely:

1. Protection against Catastrophic Events

   Even if a ceding company has a very good geographical distribution of insurance holders, it still has to face the case that a catastrophic event (such as hurricanes, earthquakes, riots or chemical accidents) can take place and create a big loss, though the loss arisen from each single policyholder could be relatively small. With reinsurance, the primary insurer is more secure with regard to its solvency. Without reinsurance, the consequences of catastrophic losses could be disastrous for a ceding company.

2. Increase of Capacity

   In general, state solvency regulations proscribe a primary insurer from underwriting an insurance amount that is greater than 10 percent of its policyholder

surplus[1]. Therefore, many insurers would be heavily bounded in writing policies for higher limits. But with reinsurance, a primary insurer can write policies with higher capacities and cede a part of the liabilities to the reinsurer. Without reinsurance, there exists a large gap between bigger and smaller insurers, because smaller companies might be endangered by big liabilities that would not affect the financial welfare of bigger insurers. Reinsurance enables also smaller ceding companies to take part in the market competition due to the possibility of writing bigger amounts of insurance.

3. Liability Stabilization

A special characteristic of insurance is that losses are not known when the product is sold, and can fluctuate a lot. With reinsurance an insurer can control these losses: It can determine how much liability it wants to assume for each single policy or how big the maximum amount of accumulated losses should be. The amount above that so called net retention liability can be ceded to a reinsurer.

4. Financing

A reinsurance contract also provides a form of financing.

- By writing business, a primary insurer has to pay for the agent's commission immediately. That means that the company's policyholder surplus declines (due to high acquisition expenses). Thus, the company faces financial problems as the number of written policies increases since the ratio of premiums to policyholder surplus becomes to large. This problem is called *surplus strain*. To give an idea about the value of this ratio, Gastel

---

[1]difference of assets and liabilities

(1995) says a company is considered to be overextended if that ratio is larger than 3.

By purchasing reinsurance, the policyholder surplus will increase, since a part of the primary insurer's liabilities is ceded to the reinsurer. If a primary and a reinsurance company enter into a pro rata reinsurance contract (this form of reinsurance is presented later in this chapter on pages 21 until 24), the reinsurer pays a commission to the primary insurer for its expenses.

- Another form of financing can be achieved if a primary insurer and a reinsurer enter into a contract after the primary insurer suffered a loss. With the reinsurance contract, the reinsurer assumes a part of the loss, but in a form of a loan, repaid with future high premiums. Thus, the ceding insurer uses reinsurance as a tool of risk management obtaining capital in a situation of financial stress, and the reinsurer is receiving higher premiums for the reinsurance contract than usually.

3.4 Forms of Reinsurance

3.4.1 Treaty and Facultative Reinsurance

As discussed in Gastel (1995), Mehr and Cammack (1980), and Brown (2001), one classification of reinsurance contracts is based on the fact what insurance policies are covered after the ceding process. With regard to this consideration, there are two different types of reinsurance:

1. Treaty Reinsurance

   Treaty reinsurance covers an expanded group of policies, for example all of a specific type.

When treaty reinsurance is purchased, all policies that fall in the terms of that contract are covered, not only those policies that already existed but also new ones. Thus, treaty reinsurance is also called automatic reinsurance.

2. Facultative Reinsurance

Facultative reinsurance covers only one policy and is used if that policy is uncommon or covers a property of higher amount. The ceding company and the reinsurer have to negotiate about the details of that contract. This can be a very inconvenient and time-intensive process if a ceding company has many of those policies.

That kind of reinsurance contract is called facultative because the reinsurer keeps the faculty either to reject or to accept to reinsure the policy, contrary to treaty reinsurance.

The ceding company has also some power. It can choose whether to buy a reinsurance contract or not; it can choose the reinsurer, the amount of ceded risk and the time of purchase.

3.4.2 Proportional and Nonproportional Reinsurance

Another classification refers to the coverage of a reinsurance contract, as investigated in Gastel (1995), Mehr and Cammack (1980) and Brown (2001):

1. Proportional Reinsurance

By purchasing proportional or pro rata reinsurance, the insurer and the reinsurer split both the premiums and the losses on a proportional basis.

Let us consider an example:

Example 1 If we consider a 25 percent pro rata reinsurance contract, the reinsurer receives 25 percent of the premiums in return to accept to pay 25 percent

of all losses.

The primary insurer receives a commission from the reinsurer to cover expenses for underwriting and agent commissions (see also the financing function of reinsurance as described on page 20).

Pro rata reinsurance is often used by insurers that are new in the market (Mehr and Cammack, 1980).

There exist two different kinds of pro rata reinsurance contracts which are of value to consider:

a. Quota-Share Reinsurance

Under quota-share reinsurance, the part of the liability that is ceded by the primary insurer is a fixed percentage of each policy of a specific line. The ceded percentage is arranged once, and afterwards the splitting of losses and premiums follows that pattern. This type is reinsurance is the easiest and most convenient form of reinsurance: The percentage is known, and thus the payments are easy to determine for each party.

Example 2 In a 60 percent quota-share contract, the reinsurer has to pay 60 percent of all losses and receives 60 percent of the premiums. The insurer has to pay for 40 percent of all losses and keeps 40 percent of the premiums.

b. Surplus-Share Reinsurance

Under surplus-share reinsurance, the part of the liability that is retained in the ceding company is a fixed amount of money. Thus, the amount of insurance implies whether there is any obligation of the reinsurer or not: In the case that the retained amount is bigger than the insured amount,

no amount is ceded to the reinsurance company; otherwise, only the excess is reinsured. The premium split is calculated by the ratio of ceded to retained liability. This ratio is also used to share the losses.

To illustrate that, let us give an example:

Example 3 A primary insurer has a policy with

- a policy limit of $1,000,000,

- and a premium of $12,000.

Let us assume a surplus-share reinsurance contract with a retention of $250,000. With respect to the liability,

- the primary insurer retains $250,000,

- and cedes $750,000 to the reinsurer.

Therefore, the ratio of ceded to retained liability is $\frac{\$750,000}{\$250,000} = \frac{3}{1}$, what implies that the reinsurer receives $9,000, and the primary insurer retains $3,000 of the premium.

If we assume a loss of $500,000, the reinsurer has to pay $375,000, and the primary insurer has to pay $125,000 in order to adjust the loss.

Obviously, the ratio of ceded to retained liability has to be calculated for each policy so that we do not have one ratio that is used for all surplus-share reinsurance contracts.

We want to round off the discussion of quota-share and surplus-share reinsurance with a comparison of both types of reinsurance with regard to the following topics:

(i) Retention

Under quota-share reinsurance, a fixed percentage of the liability of every policy is ceded, whereas a fixed amount of liability is ceded under a surplus-share contract (what implies that the percentage of liability beard by the reinsurer is increasing as the amount of insurance increases).

(ii) Ceding

Under surplus-share, the primary insurer has the possibility to choose the amount of liability that is to be ceded to the reinsurer.

(iii) Lines

In a surplus-share contract, the maximum amount of liability a reinsurance company is willing to assume is also expressed in "lines". For example, a "three-line surplus" describes a contract where the reinsurer is able to assume an amount of liability up to three times the ceding company's retention.

Sometimes, a surplus-share contract is layered: Many reinsurer assume a part of the liability that is ceded by the primary insurer. We present a corresponding example on page 26.

2. Nonproportional Reinsurance

Under nonproportional or excess of loss reinsurance, the primary insurer has to pay the full amount of losses up to a retention, whereas the reinsurer pays only those losses that exceed this retention.

The following types of excess of loss reinsurance are discussed by Mehr and Cammack (1980):

(a) Per Risk Excess Reinsurance

Per risk excess reinsurance protects a ceding company against individual losses above a retention amount. If a loss does not exceed the stated retention, the ceding company has to pay the entire amount, but otherwise the reinsurer pays the difference of the loss and the retention.

(b) Catastrophic Excess Reinsurance

As stated earlier, a primary insurer has to face the case of catastrophic events that could endanger its existence. The reason for that danger is the fact that not only one, but many policies can be subject to a loss.

We can easily understand that catastrophic excess reinsurance never covers one individual risk only, but always many of them. Hence, this form of reinsurance is always written as a treaty reinsurance. The reinsurer pays for losses due to catastrophic events in excess of a retention, but only up to a fixed amount.

This type of reinsurance is common for property insurance, and is in general the only reinsurance form for liability insurance.

(c) Stop Loss Reinsurance

Under stop loss reinsurance, the primary insurer and the reinsurer first determine a so called stop loss limit. This stop loss limit is stated as a percentage of the primary insurer's net premium income or a fixed amount of money (whichever is greater). If the primary insurer's losses exceed the stop loss limit, the reinsurer has to pay a - also pre-determined - percentage of the primary insurer's net premium income or a fixed sum of money (whichever is less).

Let us consider an example:

Example 4 A primary insurer and a reinsurer enter into a stop loss reinsurance contract with

- a stop loss limit of 70% of the primary insurer's net premium income or $300,000 (whichever is greater).

If losses incur that exceed the stop loss limit, the reinsurer has to pay

- up to 50% of the primary insurer's net premium income or $300,000 (whichever is less).

Let us assume that the primary insurer's net premium income is $500,000. Thus, the stop loss limit is $350,000. If the primary insurer's losses are not greater than $350,000, the reinsurer has no liability. But in the other case, the reinsurer has to pay the difference of the losses and $350,000, but not more than $250,000.

We can observe that the primary insurer is protected against any loss ratio that is greater than 70%, given the losses do not exceed $600,000.

As stated on page 24, a primary insurer can cede different parts of its liability to several reinsurer. We consider a situation, presented in Gastel (1995), where a policy is reinsured by three types of reinsurance contracts, provided by three different reinsurance companies:

Example 5 A primary insurer has issued a fire insurance policy with a limit of liability of $20,000,000. The company has three reinsurance contracts:

- A 90% quota-share treaty with a treaty limit of $500,000 with reinsurer A,

- an excess of loss treaty, covering losses between $500,000 and $5,000,000 with reinsurer B,

- and a facultative excess of loss, covering losses between $5,000,000 and $20,000,000 with reinsurer C.

The fire insurance policy is covered automatically under the two treaties, but the primary insurer has had to enter into the facultative contract with reinsurer C separately.

Let us consider the following different loss scenarios:

- At a loss of $150,000, the primary insurer has to pay $15,000, and reinsurer A has to pay $135,000.

- At a loss of $2,000,000, the primary insurer has to pay $50,000, whereas reinsurer A is responsible to pay $450,000, and reinsurer C reimburses $1,500,000.

- At a loss between $5,000,000 and $20,000,000, the primary insurer has to pay $50,000, reinsurer A has to pay $450,000, reinsurer B has to pay $4,500,000, and reinsurer C has to pay the difference of the loss and $5,000,000.

We can observe that the primary insurer has less diversification of its liability: In either case, the exposure will not be greater than $50,000, whereas the total liability can vary between $0 and $20,000,000. Especially the facultative excess of loss contract with reinsurer C is very important since the primary insurer was covered for liabilities up to $5,000,000, which is only $\frac{1}{4}$ or 25% of the maximum possible loss.

3.5 Global Players in the Reinsurance Business

As presented in Best's Review (2002), the top five global reinsurer in 2001, evaluated by *total gross premiums written in millions of dollars*, are

1. Munich Re - Segment Reinsurance

2. Swiss Re Group

3. Berkshire Halthaway Group

4. GE Global Insurance Holdings

5. Hannover Re

Munich Re and Swiss Re are by far the largest suppliers of reinsurance coverage. Thus, it is of interest to state some facts about these two companies:

	Munich Re	Swiss Re
Total Gross Premiums Written in 2001 (only on reinsurance)	$19,666 million	$18,569 million
Total Net Premiums Written in 2001 (only on reinsurance)	$16,614 million	$16,982 million
Total Assets in 2001	Euro 202,064 million	CHF 170,230 million

A comparison of companies with respect to *total gross premium written* is a very common approach.

On the other side, it is not very useful to compare companies with respect to *total assets*: For example, Munich Re owns several primary insurance companies, which affects the total asset, but does not affect Munich Re's reinsurance outcome.

The United States represent 40% of the entire worldwide reinsurance market and is therefore the largest market in the world (Munich Re, 2003). In the United States, Munich Re is represented by its subsidiary American Re, which had a market share of 10.7% in 2001 and is the second-largest reinsurer in the United States. The largest reinsurer in the United States is Berkshire Hathaway Insurance Group.

We can make an interesting observation in the reinsurance market by considering the London market place. As described is Best's Review (2002), until the beginning of the 1990s, the majority of reinsurance business was written in London. Due to natural catastrophic events and large losses, the London market declined, and a market consolidation took place. As a result, Bermudian reinsurer filled the gap that the London market crisis left. Today, more than 1,600 insurers have a business location in Bermuda, although the Bermuda market is no more than about 50 years old.

Reasons for that development are that no insurer, that is owned by non-Bermudian shareholders, is forced to pay taxes on income, capital gains, premiums or profits. Even if such taxes were to be introduced, international companies may apply to the Bermuda Government for an assurance against the imposition of any such taxes until 2016 (Bermuda Insurance Market, 2003).

CHAPTER IV

THEORY OF OPTIMAL REINSURANCE

4.1 Preface

The following chapter discusses several ideas of an optimal reinsurance contract. The fact that there are different results depends on certain assumption which are chosen in all the approaches. We present the work of Deprez and Gerber (1985), and Gajek and Zagrodny (2000), (2002). Our main focus will be on the earlier results of Gajek and Zagrodny (2000).

4.2 Premium Calculation Principles

We consider the following situation, as presented by Deprez and Gerber (1985):

- Let a random variable X over a probability space (Ω, Σ, Pr) denote a insurable risk.

- A premium calculation principle is a mapping $H \colon \mathcal{X} \to \mathbb{R}$, where $\mathcal{X}$ is the set of all insurable risks X.

Since we want to investigate a primary insurer's reinsurance strategy, we have to consider a nonnegative random variable Y over (Ω, Σ, Pr), which denotes the total claim of a primary insurer.

A reinsurer offers to cover a random amount R of Y, which is a measurable function of Y, for a premium P. Thus the total claim Y is divided into two parts: The part paid by the reinsurer, denoted by $R(Y)$, and the part paid by the insurer, de-

noted by $\tilde{R}(Y)$. If we think in terms of insurable risks, the reinsurer's insurable risk is $R(Y)$, and hence the premium for a reinsurance contract can be determined by $P = H(R(Y))$.

Deprez and Gerber (1985) present several premium calculation principles:

Definition 10 *Let P be the premium for a reinsurance contract.*

(i) Let $E(Y) < \infty$. Then the net premium principle is defined by

$$P = E(R(Y)).$$

(ii) Let $E(Y^2) < \infty$. Then the variance principle with parameter $\alpha > 0$ is defined by

$$P = E(R(Y)) + \alpha Var(R(Y)).$$

(iii) Let $E(Y^2) < \infty$. Then the standard deviation principle with safety loading parameter $\beta > 0$ is defined by

$$P = E(R(Y)) + \beta \sqrt{Var(R(Y))}.$$

An important property of such principles is convexity:

Definition 11 *A premium calculation principle is called convex, if it satisfies the following conditions:*

(i) $H(c) = c$ for all $c \in \mathbb{R}$,

(ii) $H(X + c) = H(X) + c$ for all $c \in \mathbb{R}$ and for all $X \in \mathcal{X}$,

(iii) $H(\alpha X + (1 - \alpha)Y) \le \alpha H(X) + (1 - \alpha)H(Y)$ *for all* $\alpha \in (0, 1)$ *and for all* $X, Y \in \mathcal{X}$.

Deprez and Gerber (1985) note that both the net premium principle and the variance principle are convex. A part of the proof of convexity of the standard deviation principle is also presented in their paper; we will show that part in Lemma 6.

4.3 Utility Function Approach

As introduced above, we denote the primary insurer's total claim by Y, and R is the part of Y that is paid by the reinsurer for a premium calculated by a given premium calculation principle.

In order to derive an optimal reinsurance contract, Deprez and Gerber (1985) use the idea of maximizing the primary insurer's utility function.

Let $\mathcal{H}$ be the set of all measurable functions $R\colon \mathbb{R}_+ \to \mathbb{R}$ such that $E(R^2(Y)) < \infty$.

We state the following definitions, as presented by Wouk (1979):

Definition 12 *A norm on a vector space V is a function*

$$\| \ \| \colon V \to \mathbb{R}$$

with the following properties: For all $v, w \in V$ and for all $\lambda \in \mathbb{R}$,

(i) $\|v\| \ge 0$ *and* $\|v\| = 0$ *if and only if $v = 0$,*

(ii) $\|\lambda v\| = |\lambda| \cdot \|v\|$,

(iii) $\|v + w\| = \|v\| + \|w\|$.

Definition 13 *Let $\mathcal{S}$ be a set. We call a function $\rho\colon \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ a metric, if the following properties hold for any elements $x, y, z \in \mathcal{S}$:*

(i) $\rho(x, y) = 0$ if and only if $x = y$,

(ii) $\rho(x, y) = \rho(y, x)$,

(iii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

The pair (S, ρ) is called a metric space.

Remark 3

(a) Property (ii) is called symmetry.

(b) Property (iii) is called triangle inequality.

Definition 14

(i) A sequence $\{x_n\}$ of numbers is called a Cauchy sequence, if for every $\epsilon > 0$ there exists a number $N \in \mathbb{N}$ such that if $m, n \in \mathbb{N}$ and $m, n \geq N$, then

$$|x_m - x_n| < \epsilon.$$

(ii) Let $(\mathcal{S}, \rho)$ be a metric space. A sequence $\{x_n\}$ in S is called a Cauchy sequence, if for every $\epsilon > 0$ there exists a number $N \in \mathbb{N}$ such that if $m, n \in \mathbb{N}$ and $m, n \geq N$, then

$$\rho(x_m, x_n) < \epsilon.$$

(iii) A metric space $(\mathcal{S}, \rho)$ is called complete, if every Cauchy sequence in $\mathcal{S}$ has a limit in $\mathcal{S}$.

Definition 15 *A space $\mathcal{S}$ is called an inner product space if it is a linear vector space over $\mathbb{R}$ and if there exists a mapping $< \cdot, \cdot >: \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ that satisfies the following conditions:*

(i) $< x, y > \ = \ < y, x >$ for all $x, y \in \mathcal{S}$,

(ii) $< x_1 + x_2, y > \ = \ < x_1, y > + < x_2, y >$ for all $x_1, x_2 \in \mathcal{S}$,

(iii) $< \lambda x, y > \ = \lambda < x, y >$ for all $\lambda \in \mathbb{R}$ and for all $x, y \in \mathcal{S}$,

(iv) $< x, x > \ > 0$ if $x \neq 0$.

Note that if $\mathcal{S}$ is an inner product space, then $\rho(x, y) = < x - y, x - y >^{\frac{1}{2}}$ is a metric. If an inner product space $\mathcal{S}$ is complete in the metric $\rho(x, y) = < x - y, x - y >^{\frac{1}{2}}$, then $\mathcal{S}$ is called a Hilbert space.

Lemma 4 *(Cauchy-Schwarz Inequality) For any elements x, y in an inner product space $\mathcal{S}$,*

$$\mid < x, y > \mid^2 \ \leq \ < x, x > \cdot < y, y > .$$

Proof:

We present the proof as suggested by Wouk (1979):

Obviously, the statement is true if $y = 0$. Thus, we suppose that $y \neq 0$. Then we obtain that

$$< x + \alpha y, x + \alpha y > \ = \ < x, x > +2\alpha < x, y > +\alpha^2 < y, y > \ \geq 0 \ \ for \ all \ \ \alpha \in \mathbb{R}.$$

Now we set $\alpha = -\frac{<x,y>}{<y,y>}$. We conclude that

$$< x, x > -2\frac{< x, y >^2}{< y, y >} + \frac{< x, y >^2}{< y, y >^2} < y, y > \ = \ < x, x > -\frac{< x, y >^2}{< y, y >}$$

$$\geq \ 0$$

This inequality is equivalent to

$$< x, y >^2 \; \leq \; < x, x > \cdot < y, y > .$$

Finally, we conclude that

$$\mid < x, y > \mid^2 \; \leq \; < x, x > \cdot < y, y > .$$

$\square$

We use the two following theorems, usually stated for measure spaces (Dieudonné, 1975), only for probability spaces:

Theorem 3 *(Monotone Convergence Theorem) Let $\{f_n\}$ be a monotone increasing sequence of integrable functions. Then*

$$\lim_{n \to \infty} \int_\Omega f_n \, dPr = \int_\Omega \lim_{n \to \infty} f_n \, dPr.$$

Theorem 4 *(Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of integrable functions which converges a.s. to a real valued measurable function f. Suppose there exists an integrable function g such that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Then*

$$\lim_{n \to \infty} \int_\Omega f_n \, dPr = \int_\Omega \lim_{n \to \infty} f_n \, dPr.$$

Recall that $\mathcal{H}$ is the set of all measurable functions $R \colon \mathbb{R}_+ \to \mathbb{R}$ such that $E(R^2(Y)) < \infty$.

Lemma 5 $\mathcal{H}$ *is a Hilbert space with norm* $\|R\| = \sqrt{E(R^2(Y))}$.

Proof:

a) First let us show that $\mathcal{H}$ is a linear vector space over $\mathbb{R}$:

Since $\mathcal{H}$ is a space of integrable functions in $\mathbb{R}$, $\mathcal{H}$ is a linear vector space over $\mathbb{R}$.

b) Now let us show that the conditions (i) through (iv) in the definition of a Hilbert space are fulfilled:

We consider a mapping $< \cdot, \cdot >: \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$, defined by

$$< S, T >= E[S(Y)T(Y)] \ \ for \ \ S, T \in \mathcal{H}.$$

(i) $< S, T >= E[S(Y)T(Y)] = E[T(Y)S(Y)] =< T, S >$ for all $S, T \in \mathcal{H}$

(ii)

$$
\begin{aligned}
< S_1 + S_2, T > &= E[(S_1(Y) + S_2(Y))T(Y)] \\
&= E[S_1(Y)T(Y) + S_2(Y)T(Y)] \\
&= E[S_1(Y)T(Y)] + E[S_2(Y)T(Y)] \\
&= < S_1, T > + < S_2, T > \ \ \text{for all } S_1, S_2, T \in \mathcal{H}
\end{aligned}
$$

(iii)

$$
\begin{aligned}
< \lambda S, T > &= E[(\lambda S(Y))T(Y)] \\
&= \lambda E[S(Y)T(Y)] \\
&= \lambda < S, T > \ \ \text{for all } \lambda \in \mathbb{R} \text{ and for all } S, T \in \mathcal{H}
\end{aligned}
$$

(iv) Let $S \in \mathcal{H}, S \not\equiv 0$.

$$\Rightarrow \; < S, S > \; = \; E[S^2(Y)] > 0$$

c) Finally let us show that $\mathcal{H}$ is complete:

Let $\{S_k\}$ be a Cauchy sequence in $\mathcal{H}$. Then there exists a subsequence $\{S_{k_j}\}$ of $\{S_k\}$ such that

$$|S_k - S_{k_j}| < \left(\frac{1}{2}\right)^j \quad \text{whenever} \;\; k \geq k_j.$$

In particular, we obtain $|S_{k_{j+1}} - S_{k_j}| < \left(\frac{1}{2}\right)^j$.

We define a sequence $\{T_n\}$ in $\mathcal{H}$ by

$$T_n(x) = \sum_{j=1}^{n} |S_{k_{j+1}}(x) - S_{k_j}(x)|$$

and observe that $\{T_n\}$ is a monotone increasing sequence.

We conclude that

$$
\begin{aligned}
|T_n(x)| \; &= \; \left|\sum_{j=1}^{n} |S_{k_{j+1}}(x) - S_{k_j}(x)|\right| \\
&= \; \sum_{j=1}^{n} |S_{k_{j+1}}(x) - S_{k_j}(x)| \\
&< \; \sum_{j=1}^{n} \left(\frac{1}{2}\right)^j \\
&= \; \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}} - 1 \\
&= \; 2 - \left(\frac{1}{2}\right)^n - 1 \\
&= \; 1 - \left(\frac{1}{2}\right)^n
\end{aligned}
$$

This implies that $|T_n(x)| \leq 1$ for all $n \in \mathbb{N}$. Therefore, T_n is integrable for all $n \in \mathbb{N}$.

Since $\{T_n\}$ is a monotone increasing sequence of integrable functions such that $\int_\Omega T_n \, dPr$ is bounded, we can apply the Monotone Convergence Theorem and conclude that $\{T_n(x)\}$ converges a.s. and that T is integrable.

Equivalently, we conclude that

$$\sum_{j=1}^{\infty} |S_{k_{j+1}}(x) - S_{k_j}(x)|$$

converges a.s., and therefore,

$$\sum_{j=1}^{\infty} \left(S_{k_{j+1}}(x) - S_{k_j}(x) \right)$$

converges a.s.

Since

$$\sum_{j=1}^{n} \left(S_{k_{j+1}}(x) - S_{k_j}(x) \right) = S_{k_{n+1}}(x) - S_{k_1}(x),$$

we can conclude that $\{S_{k_j}(x)\}$ converges a.s. We define $S(x) = \lim_{j \to \infty} S_{k_j}(x)$.

With this result, we obtain

$$
\begin{aligned}
|S(x) - S_{k_j}(x)| &= \left| \sum_{i=j}^{\infty} \left(S_{k_{i+1}}(x) - S_{k_i}(x) \right) \right| \\
&\leq \sum_{i=j}^{\infty} |S_{k_{i+1}}(x) - S_{k_i}(x)| \\
&\leq \sum_{i=1}^{\infty} |S_{k_{i+1}}(x) - S_{k_i}(x)| \\
&= T(x) \ \ a.s. \ and \ for \ all \ k \in \mathbb{N}.
\end{aligned}
$$

Thus, we conclude that

$$|S(x) - S_{k_j}(x)|^2 < (T(x))^2 \ \ a.s. \ and \ for \ all \ j \in \mathbb{N}.$$

We know that T^2 is integrable since T is integrable. Therefore, $(S - S_{k_j})^2$ is integrable for all $j \in \mathbb{N}$ since it is bounded by T^2. Now we apply the Dominated Convergence Theorem and obtain:

$$\begin{aligned} \lim_{j \to \infty} E(S - S_{k_j})^2 &= E\big(\lim_{j \to \infty} (S - S_{k_j})^2 \big) \\ &= 0 \end{aligned}$$

Since $\{S_k\}$ is a Cauchy sequence with a convergent subsequence $\{S_{k_j}\}$, we can conclude that

$$\begin{aligned} \rho(S(x), S_k(x)) &\leq \rho(S(x), S_{k_j}(x)) + \rho(S_{k_j}(x), S_k(x)) \\ &\to 0 \ \ for \ \ n \to \infty, k \to \infty, \end{aligned}$$

where $S(x) := \lim_{j \to \infty} S_{k_j}(x)$. Finally, we have shown that if $\{S_n\}$ is a Cauchy sequence, then $\rho(S, S_n)$ is convergent. This implies that H is complete in the metric $\rho(U, V)$ for $U, V \in \mathcal{H}$.

$\square$

The Gâteaux derivative of a mapping $f \colon \mathcal{H} \to \mathbb{R}$ at $R \in \mathbb{R}$ is any linear continuous functional x^* from the dual of $\mathcal{H}$ such that

$$\lim_{t \to 0} t^{-1}(f(R + tH) - f(R)) =< x^*, H >$$

for any $H \in \mathcal{H}$. We write $\nabla f(R)(H)$ instead of $< x^*, H >$.

Definition 16 *Let u denote the primary insurer's utility function. A reinsurance contract is called optimal, if it maximizes*

$$E[u(-Y - P + R)],$$

which is the expected utility after purchasing reinsurance.

Theorem 5 *Let H be convex and Gâteaux differentiable. Then the reinsurance contract R^* is optimal, if and only if*

$$H'(R^*) = \frac{u'(-Y - H(R^*) + R^*)}{E[u'(-Y - H(R^*) + R^*)]}$$

Proof:

See Deprez and Gerber (1985).

4.4 Optimality of the Change Loss Reinsurance

Another approach of having an optimal reinsurance strategy was introduced by Gajek and Zagrodny (2000). They show that a special reinsurance contract is optimal under certain assumptions.

As we have stated above, we denote the total claim of a primary insurer in a given time period by a nonnegative random variable Y over a probability space (Ω, Σ, Pr). Earlier in this paper we have seen that the total claim Y is divided into two parts:

- The part paid by the reinsurer, denoted by R,

- and the part paid by the primary insurer, denoted by $\tilde{R}$.

The premium for reinsurance coverage R is denoted by P_r, and P_{max} denotes the amount which a primary insurer is able to pay for a reinsurance policy. Obviously,

$P_{max} \geq P_r$.

Let us assume that the reinsurance premium is calculated by the standard deviation principle, which we have introduced in Definition 10. This leads to the following relationship:

$$P_{max} \geq P_r = E(R(Y)) + \beta\sqrt{Var(R(Y))}. \tag{4.1}$$

We assume that $E(Y^2) < \infty$, otherwise the premium for reinsurance would be infinite. Observe that we obviously assume that $\beta > 0$, otherwise this premium would reduce to the net premium, which we do not consider here.

Obviously, a primary insurer is interested in minimizing the variance of the retained risk, which is calculated by $Var(\tilde{R}(Y))$. Also, it makes only sense for the insurer to accept those reinsurance contracts with

$$0 \leq \tilde{R}(Y) \leq Y,$$

what is equivalent to

$$0 \leq R(Y) \leq Y. \tag{4.2}$$

Thus, we have to minimize $Var(Y - R(Y)) = Var(\tilde{R}(Y))$ over the space

$$\{R \colon \mathbb{R}_+ \to \mathbb{R} \mid R \text{ is measurable}, E(R(Y)) + \beta\sqrt{Var(R(Y))} \leq P_{max},$$

$$0 \leq R(Y) \leq Y\}.$$

Definition 17 *Let $0 \leq r < 1$ and $M \in \mathbb{R}$. Then a contract of the form*

$$R_1(y) = \begin{cases} 0 & if \ \ y < M, \\ (1-r)(y-M) & otherwise, \end{cases}$$

is called change loss reinsurance. M depends on P_{max}, β and truncated moments of Y. M is called the change loss point.

Let F denote the distribution function of the total claim Y. Let $\mathcal{H}$ be the set of all measurable functions $R \colon \mathbb{R}_+ \to \mathbb{R}$ such that $E(R^2(Y)) < \infty$.

We begin with a basic definition (Wouk, 1979):

Definition 18 *Let $\mathcal{S}$ be a vector space over $\mathbb{R}$. Then a linear function from $\mathcal{S}$ into $\mathbb{R}$ is called a functional.*

Given a fixed premium $P_{max} > 0$ and a safety loading coefficient $\beta > 0$, let us define functionals $g \colon \mathcal{H} \to \mathbb{R}$ and $V \colon \mathcal{H} \to \mathbb{R}$ by

$$g(R) = E(R(Y)) + \beta \sqrt{Var(R(Y))} - P_{max},$$

$$V(R) = Var(Y - R(Y)).$$

Since we have shown in Lemma 5 that $\mathcal{H}$ is a Hilbert space, we can calculate the Gâteaux derivatives of g and V. For every $R, H \in \mathcal{H}$, we have

$$\nabla V(R)(H)$$

$$
\begin{aligned}
&= \lim_{t\to 0} t^{-1}\Big[Var\big(Y - R(Y) - tH(Y)\big) - Var\big(Y - R(Y)\big)\Big]\\
&= \lim_{t\to 0} t^{-1}\Big[E\big(Y - R(Y) - tH(Y)\big)^2 - \big(E(Y - R(Y) - tH(Y))\big)^2\\
&\quad - E\big(Y - R(Y)\big)^2 + \big(E(Y - R(Y))\big)^2\Big]\\
&= \lim_{t\to 0} t^{-1}\Big[E\{Y^2 - YR(Y) - tYH(Y) - YR(Y) + R^2(Y) + tH(Y)R(Y)\\
&\quad - tYH(Y) + tH(Y)R(Y) + t^2H^2(Y)\} - \{(E(Y))^2 - E(Y)E(R(Y))\\
&\quad - tE(Y)E(H(Y)) - E(Y)E(R(Y)) + (E(R(Y)))^2\\
&\quad + tE(H(Y))E(R(Y)) - tE(Y)E(H(Y)) + tE(H(Y))E(R(Y))\\
&\quad + t^2(E(H(Y)))^2\} - \{E(Y^2 - 2YR(Y) + R^2(Y)\}\\
&\quad + \{(E(Y))^2 - 2E(Y)E(R(Y)) + (E(R(Y)))^2\}\Big]\\
&= \lim_{t\to 0} t^{-1}\Big[-2tE(YH(Y)) + 2tE(H(Y)R(Y)) + t^2E(H^2(Y))\\
&\quad + 2E(Y)E(R(Y)) + 2tE(Y)E(H(Y)) - 2tE(H(Y))E(R(Y))\\
&\quad - t^2(E(H(Y)))^2 - 2E(Y)E(R(Y))\Big]\\
&= -2E[YH(Y)] + 2E[H(Y)R(Y)] + \lim_{t\to 0} tE[H^2(Y)]\\
&\quad + 2E(Y)E[H(Y)] - 2E[H(Y)]E[R(Y)] - \lim_{t\to 0} t(E[H(Y)])^2\\
&= -2E\big[H(Y)\big(Y - R(Y)\big)\big] + 2E\big[Y - R(Y)\big]E\big[H(Y)\big],
\end{aligned}
$$

and

$$\nabla g(R)(H)$$

$$
\begin{aligned}
&= \lim_{t\to 0} t^{-1}\Big[g(R + tH) - g(R)\Big]\\
&= \lim_{t\to 0} t^{-1}\Big[E(R(Y)) + tE(H(Y)) + \beta\sqrt{Var\big(R(Y) + tH(Y)\big)} - P_{max}
\end{aligned}
$$

$$
-E(R(Y)) - \beta\sqrt{Var(R(Y))} + P_{max}\Big]
$$

$$
= E[H(Y)] + \beta \lim_{t\to 0} t^{-1}\left[\sqrt{Var\big(R(Y) + tH(Y)\big)} - \sqrt{Var\big(R(Y)\big)}\right]
$$

$$
= E[H(Y)] + \beta \lim_{t\to 0} t^{-1}\left[\frac{Var\big(R(Y) + tH(Y)\big) - Var\big(R(Y)\big)}{\sqrt{Var\big(R(Y) + tH(Y)\big)} + \sqrt{Var\big(R(Y)\big)}}\right]
$$

$$
= E[H(Y)] + \beta \lim_{t\to 0} t^{-1}\left[\frac{E\big(R(Y) + tH(Y)\big)^2 - \Big(E\big(R(Y) + tH(Y)\big)\Big)^2}{\sqrt{Var\big(R(Y) + tH(Y)\big)} + \sqrt{Var\big(R(Y)\big)}}\right.
$$

$$
\left. -\frac{E\big(R^2(Y)\big) - \Big(E\big(R(Y)\big)\Big)^2}{\sqrt{Var\big(R(Y) + tH(Y)\big)} + \sqrt{Var\big(R(Y)\big)}}\right]
$$

$$
= E[H(Y)] + \beta \lim_{t\to 0} t^{-1}\left[\frac{2tE\big(H(Y)R(Y)\big) + t^2 E\big(H^2(Y)\big)}{\sqrt{Var\big(R(Y) + tH(Y)\big)} + \sqrt{Var\big(R(Y)\big)}}\right.
$$

$$
\left. -\frac{2tE\big(H(Y)\big)E\big(R(Y)\big) - t^2\Big(E\big(H(Y)\big)\Big)^2}{\sqrt{Var\big(R(Y) + tH(Y)\big)} + \sqrt{Var\big(R(Y)\big)}}\right]
$$

$$
= E[H(Y)] + \beta\frac{E[R(Y)H(Y)] - E[R(Y)]E[H(Y)]}{\sqrt{Var[R(Y)]}},
$$

whenever $Var[R(Y)] > 0$ and V, g are as defined above.

Lemma 6 *Both g and V are convex.*

Proof:

Let us first show that g is a convex function:

For convenience, we denote $R_a(Y)$ by R_a and $R_b(Y)$ by R_b.

Let $0 < \alpha < 1$ and $R_a, R_b \in \mathcal{H}$. Then,

$$
g(\alpha R_a + (1-\alpha)R_b) \le \alpha g(R_a) + (1-\alpha)g(R_b)
$$

$$
\Leftrightarrow E[\alpha R_a + (1-\alpha)R_b] + \beta\sqrt{Var[\alpha R_a + (1-\alpha)R_b]} - P_{max}
$$

$$\leq \alpha\{E(R_a) + \beta\sqrt{Var(R_a)} - P\} + (1 - \alpha)\{E(R_b) + \beta\sqrt{Var(R_b)} - P_{max}\}$$

$$\Leftrightarrow \sqrt{Var[\alpha R_a + (1 - \alpha)R_b]} \leq \alpha\sqrt{Var(R_a)} + (1 - \alpha)\sqrt{Var(R_b)}$$

$$\Leftrightarrow Var[\alpha R_a + (1 - \alpha)R_b]$$

$$\leq \alpha^2 Var(R_a) + 2\alpha(1 - \alpha)\sqrt{Var(R_a)Var(R_b)} + (1 - \alpha)^2 Var(R_b)$$

$$\Leftrightarrow \frac{Var[\alpha R_a+(1-\alpha)R_b]-\alpha^2 Var(R_a)-(1-\alpha)^2 Var(R_b)}{2\alpha(1-\alpha)} \leq \sqrt{Var(R_a)Var(R_b)}$$

$$\Leftrightarrow \frac{Var(\alpha R_a)+Var((1-\alpha)R_b)+2Cov[\alpha R_a,(1-\alpha)R_b]-\alpha^2 Var(R_a)-(1-\alpha)^2 Var(R_b)}{2\alpha(1-\alpha)}$$

$$\leq \sqrt{Var(R_a)Var(R_b)}$$

$$\Leftrightarrow Cov[R_a, R_b] \leq \sqrt{Var(R_a)Var(R_b)}, \text{ which is obviously true.}$$

Now let us show that V is a convex function:

For convenience, we denote $R_a(Y)$ by R_a and $R_b(Y)$ by R_b.

Let $0 < \alpha < 1$ and $R_a, R_b \in \mathcal{H}$. Then,

$$V(\alpha R_a + (1 - \alpha)R_b) \leq \alpha V(R_a) + (1 - \alpha)V(R_b)$$

$$\Leftrightarrow Var\big(\alpha(Y - R_a)\big) + Var\big((1 - \alpha)(Y - R_b)\big) + 2Cov\big(\alpha(Y - R_a), (1 - \alpha)(Y - R_b)\big)$$

$$\leq \alpha Var(Y - R_a) + (1 - \alpha)Var(Y - R_b)$$

$$\Leftrightarrow \alpha^2 Var(Y - R_a) + (1 - \alpha)^2 Var(Y - R_b) + 2\alpha(1 - \alpha)Cov(Y - R_a, Y - R_b)$$

$$\leq \alpha Var(Y - R_a) + (1 - \alpha)Var(Y - R_b)$$

$$\Leftrightarrow \alpha(\alpha - 1)Var(Y - R_a) + (1 - \alpha)(-\alpha)Var(Y - R_b)$$

$$+ 2\alpha(1 - \alpha)Cov(Y - R_a, Y - R_b) \leq 0$$

$$\Leftrightarrow \alpha(\alpha - 1)Var(Y - R_a) + \alpha(\alpha - 1)Var(R_b - Y)$$

$$+ 2\alpha(\alpha - 1)Cov(Y - R_a, R_b - Y) \leq 0$$

$$\Leftrightarrow Var(Y - R_a) + Var(R_b - Y) + 2Cov(Y - R_a, R_b - Y) \geq 0$$

$$\Leftrightarrow Var\big((Y - R_a) + (R_b - Y)\big) \geq 0$$

$\square$

Let us define a subset $\mathcal{C} \subset \mathcal{H}$ by

$$\mathcal{C} = \{R \in \mathcal{H} \mid 0 \leq R(Y) \leq Y \ \text{ a.s. with respect to } \ F \ \text{ on } \ \mathbb{R}_+\}.$$

Lemma 7 $\mathcal{C}$ *is a convex set.*

Proof:

Let $R_a, R_b \in \mathcal{C}$, let $0 < \alpha < 1$.

$\Rightarrow 0 \leq \alpha R_a(y) + (1 - \alpha)R_b(y) \leq y$ a.s.

$\square$

Our goal is to minimize $V(\cdot)$ over sets

$$\mathcal{M}_\leq = \{R \in \mathcal{H} \mid g(R) \leq 0 \ \text{ and } \ R \in \mathcal{C}\}$$

and

$$\mathcal{M}_= = \{R \in \mathcal{H} \mid g(R) = 0 \ \text{ and } \ R \in \mathcal{C}\},$$

respectively. These are nonlinear constrained problems, which we can solve with the Lagrangian method. We define the Lagrangian function by

$$L_\lambda(R) = V(R) + \lambda g(R) + \psi_\mathcal{C}(R),$$

where λ is a Lagrange multiplier and $\psi_\mathcal{C} \colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\psi_\mathcal{C}(R) = \begin{cases} 0 & \textit{if } R \in \mathcal{C}, \\ +\infty & \textit{otherwise.} \end{cases}$$

Since $R_1 \in \mathcal{C}$ and $\lambda \geq 0$, L_λ is differentiable at R_1 in the direction of any point $R \in C$ with the directional derivative equal to

$$[\nabla V(R_1) + \lambda \nabla g(R_1)](R - R_1),$$

where $\nabla V(R_1)$ and $\nabla g(R_1)$ denote the Gâteaux derivatives of V and g, respectively.

Lemma 8 *Let $R_1 \in \mathcal{C}$ and $\lambda \geq 0$ be such that*

   (i) $\lambda g(R_1) = 0$,

   (ii) $L_\lambda(R) \geq L_\lambda(R_1)$ for all $R \in \mathcal{H}$.

Then R_1 minimizes V over the set $\mathcal{M}_\leq$. It also minimizes V over $\mathcal{M}_=$ whenever $g(R_1) = 0$ (since it minimizes V over $\mathcal{M}_\leq$).

Proof:

Let $R \in \mathcal{H}$. Then,

$$
\begin{aligned}
V(R_1) &= V(R_1) + \lambda g(R_1) + \psi_C(R_1) \\
&= L_\lambda(R_1) \\
&\leq L_\lambda(R)
\end{aligned}
$$

Also, for $R \in M_\leq$,

$$
\begin{aligned}
L_\lambda(R) &= V(R) + \lambda g(R) + \psi_C(R) \\
&= V(R) + \lambda g(R) \\
&\leq V(R)
\end{aligned}
$$

$\Rightarrow V(R_1) \leq L_\lambda(R) \leq V(R)$ for all $R \in \mathcal{M}_\leq$.

Since $\mathcal{M}_= \subset \mathcal{M}_\leq$, R_1 minimizes V over $M_=$ if $g(R_1) = 0$.

$\square$

The main difficulty now is to find $\lambda \geq 0$ for which (ii) holds. In order to preserve the existence of such lambdas we assume that the following conditions are satisfied:

$$E(Y) - M - \int_{[M,\infty)} (y - M)\, dF$$

$$+\frac{r}{\beta}\sqrt{\int_{[M,\infty)} (y - M)^2\, dF - \left(\int_{[M,\infty)} (y - M)\, dF\right)^2} = 0, \qquad (4.3)$$

$$\int_{[M,\infty)} (y - M)\, dF + \beta\sqrt{\int_{[M,\infty)} (y - M)^2\, dF - \left(\int_{[M,\infty)} (y - M)\, dF\right)^2}$$

$$= \frac{P_{max}}{1 - r}. \qquad (4.4)$$

Corollary 1 *If M and r are solutions to (4.3) and (4.4), then*

$$E(Y) + \beta\sqrt{Var(Y)} \geq P_{max}$$

Proof:

Assume that M and r are solutions to (4.3) and (4.4). Let us consider (4.4), and we observe that

$$\int_{[M,\infty)} (y - M)\, dF + \beta\sqrt{\int_{[M,\infty)} (y - M)^2\, dF - \left(\int_{[M,\infty)} (y - M)\, dF\right)^2} \geq P_{max}$$

$$\Rightarrow \int_{[0,\infty)} y\, dF + \beta\sqrt{\int_{[0,\infty)} y^2\, dF - \left(\int_{[0,\infty)} y\, dF\right)^2} \geq P_{max}, \text{ since } M \geq 0$$

$\square$

$E(Y) + \beta\sqrt{Var(Y)} \leq P_{max}$ means that the primary insurer is able to cede its whole portfolio a reinsurance company. It is not important to consider this situation from both practical and theoretical reasons.

Corollary 1 shows that $E(Y) + \beta\sqrt{Var(Y)} \geq P_{max}$ is a necessary condition for solvability of (4.3) and (4.4). The following lemma shows that $E(Y) + \beta\sqrt{Var(Y)} > P_{max}$ is a sufficient condition for solvability of (4.3) and (4.4):

Lemma 9 *Assume that $P_{max} > 0$, $\beta\sqrt{Var(Y)} > 0$ and*

$$E(Y) + \beta\sqrt{Var(Y)} > P_{max}.$$

Then there exist $M > 0$ and $r \in [0, 1)$ satisfying (4.3) and (4.4).

Proof:

Let us notice that

$$A = \left\{ M \geq 0 : \int_{[M,\infty)} (y - M)^2 \, dF > \left(\int_{[M,\infty)} (y - M) \, dF \right)^2 \right\} \neq \emptyset$$

since $0 \in A$. Hence $M_1 := \sup A$ is well defined though M_1 might be $+\infty$.

Furthermore, $M_1 > 0$:

Suppose that $M_1 = 0$.

$\Rightarrow \int_{[M,\infty)} (y - M)^2 \, dF - \left(\int_{[M,\infty)} (y - M) \, dF \right)^2 = 0$ for all $M > 0$

$\Rightarrow \int_{[M,\infty)} (y - M)^2 \, dF - \left(\int_{[M,\infty)} (y - M) \, dF \right)^2 = 0$ for $M = 0$, since

$\int_{[M,\infty)} (y - M)^2 \, dF - \left(\int_{[M,\infty)} (y - M) \, dF \right)^2$ is a continuous function in M.

But this is a contradiction to the assumption that $Var(Y) > 0$.

We derive from (4.3) that

$$r = \beta \frac{\int_{[0,M)} (M - y)\, dF}{\sqrt{\int_{[M,\infty)} (y - M)^2\, dF - \left(\int_{[M,\infty)} (y - M)\, dF \right)^2}}, \qquad (4.5)$$

which is well defined for every $M \in [0, M_1)$. Let us define a function $h \colon [0, M_1) \to \mathbb{R}$ by

$$h(M) = \left(1 - \beta \frac{\int_{[0,M)} (M-y)\, dF}{\sqrt{\int_{[M,\infty)} (y-M)^2\, dF - \left(\int_{[M,\infty)} (y-M)\, dF \right)^2}} \right)$$

$$\cdot \left(\int_{[M,\infty)} (y - M)\, dF + \beta \sqrt{\int_{[M,\infty)} (y - M)^2\, dF - \left(\int_{[M,\infty)} (y - M)\, dF \right)^2} \right).$$

Our goal is to show that every $M \in [0, M_1)$ such that $h(M) = P_{max}$ is a solution to (4.3) and (4.4), together with a corresponding r as given in (4.5).

By using the definition of h and one of the assumptions, we see that

$$h(0) = E(Y) + \beta \sqrt{Var(Y)} > P_{max}.$$

We know that h is continuous on $[0, M_1)$. Thus, we have to show that

$$\liminf_{M \nearrow M_1} h(M) \leq 0,$$

and then we are done by applying the Intermediate-Value Theorem.

Let us consider $\liminf_{M \nearrow M_1} h(M)$.

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- If $M_1 = \infty$, then the first factor in the right-hand side of $h(M)$ tends to $-\infty$ as $M \nearrow M_1$, while the second one is always nonnegative.

- If $M_1 < \infty$, then

$$\left(\int_{[M,\infty)} (y - M) \, dF \right)^2 = \left(\int_{[M,\infty)} (y - M) \cdot 1 \, dF \right)^2$$

$$\leq \left(\int_{[M,\infty)} (y - M)^2 \, dF \right) \cdot \left(\int_{[M,\infty)} 1^2 \, dF \right)$$

$$= \left(\int_{[M,\infty)} (y - M)^2 \, dF \right) \cdot Pr(Y \geq M)$$

$$\leq \int_{[M,\infty)} (y - M)^2 \, dF,$$

by applying the Cauchy-Schwartz Inequality.

In fact, we have an equality for all $M > M_1$ by definition of M_1. In the case of an equality we have to consider two cases:

(i) Both sides of the equality are nonzero. Then,
   $Pr(Y \geq M_1) = 1$, which is a contradiction.

(ii) Both sides of the equality are equal to 0. Then,

   $\int_{[M,\infty)} (y - M) \, dF = 0$

   $\Rightarrow Pr([M,\infty)) = 0$

   $\Rightarrow Pr(Y > M_1) = 0$, since the equality is fulfilled for all $M > M_1$

   $\Rightarrow \lim_{M \nearrow M_1} \left(\int_{[M,\infty)} (y - M)^2 \, dF - \left(\int_{[M,\infty)} (y - M) \, dF \right)^2 \right) = 0$, and again
   the first factor in the right-hand side of $h(M)$ tends to $-\infty$ as $M \nearrow M_1$.

Now we apply the Intermediate-Value Theorem and conclude that there exists an $M \in (0, M_1)$ such that $h(M) = P_{max}$.

We still have to show that $r \in [0, 1)$:

Let us consider (4.5), and we observe that $r \geq 0$. In order to obtain that $r < 1$, we have to consider the definition of h and (4.5):

$$h(M) = P_{max}$$
$$\Rightarrow (1 - r)(E(Y) + \beta \sqrt{Var(Y)}) \geq P_{max} > 0$$
$$\Rightarrow 1 - r > 0$$

$\square$

Lemma 9 indicates that (4.3) and (4.4) have solutions under the reasonable assumption that the primary insurer does not cede its entire portfolio to the reinsurer. The existence of such solutions is needed in our final result:

Theorem 6 *Assume that $P_{max} > 0$ and $\beta \sqrt{Var(Y)} > 0$. Let $M \geq 0$ and $r \in [0, 1)$ such that*

$$E(Y) - M - \int_{[M,\infty)} (y - M) \, dF$$

$$+ \frac{r}{\beta} \sqrt{\int_{[M,\infty)} (y - M)^2 \, dF - \left(\int_{[M,\infty)} (y - M) \, dF \right)^2} = 0,$$

$$\int_{[M,\infty)} (y - M) \, dF + \beta \sqrt{\int_{[M,\infty)} (y - M)^2 \, dF - \left(\int_{[M,\infty)} (y - M) \, dF \right)^2}$$

$$= \frac{P_{max}}{1 - r}.$$

Then the function

$$R_1(y) = \begin{cases} 0 & \text{if } y \le M, \\ \\ (1-r)(y-M) & \text{otherwise,} \end{cases}$$

is the optimal reinsurance arrangement for the insurer under restrictions (4.1) and (4.2), i.e. it gives the minimum value of V over the set $M_\le$. The function V attains its minimum over $M_=$ at R_1, since $P_{max} = E(R_1) + \beta \sqrt{Var(R_1)}$, by (4.4).

Proof:

We can observe that $Pr(Y > M) > 0$:

We suppose that $Pr(Y > M) = 0$. This implies that the left side of equation (4.4) is equal to 0. But by assumption, $P_{max} > 0$ (which is the right side of (4.4)). Thus we have a contradiction.

So we know that $Pr(Y > M) > 0$.

We define

$$\lambda = \frac{2r}{\beta} \sqrt{ \int_{[M,\infty)} (y-M)^2 \, dF - \left(\int_{[M,\infty)} (y-M) \, dF \right)^2 }. \tag{4.6}$$

λ is well-defined since $\beta > 0$ and the radicand is nonnegative by the Cauchy-Schwarz inequality, and $\lambda \ge 0$.

We shall check if (i) of Lemma 8 holds. By (4.4),

$$g(R_1) = E(R_1(Y)) + \beta \sqrt{Var(R_1(Y))} - P_{max}$$

$$= \int_{[0,\infty)} R_1(y) \, dF + \beta \sqrt{ \int_{[0,\infty)} \left(R_1(y)\right)^2 dF - \left(\int_{[0,\infty)} R_1(y) \, dF \right)^2 }$$

$$- P_{max}$$

$$= (1-r)\left(\int_{[M,\infty)} (y-M)\, dF\right.$$

$$\left. +\beta\sqrt{\int_{[M,\infty)} (y-M)^2\, dF - \left(\int_{[M,\infty)} (y-M)\, dF\right)^2}\right) - P_{max}$$

$$= 0$$

Thus (i) of Lemma 8 holds.

In order to check if (ii) holds, let us consider $R \in \mathcal{C}$ (otherwise $\psi_{\mathcal{C}}(R) = +\infty$ and (ii) holds evidently).

$$\Rightarrow 0 \le R(y) \le y \text{ a.s. on } \mathbb{R}_+$$

$$\Rightarrow R(y) \ge R_1(y) \text{ a.s. on } [0, M), \text{ since } R_1(y) = 0 \text{ for all } y \in [0, M)$$

$$\Rightarrow R(y) - R_1(y) \ge 0 \text{ a.s. on } [0, M)$$

We have shown earlier that the Gâteaux derivative of L_λ at R_1 in the direction of any point $R \in C$ is equal to

$$[\nabla V(R_1) + \lambda \nabla g(R_1)](R - R_1),$$

where $\nabla V(R_1)$ and $\nabla g(R_1)$ denote the Gâteaux derivatives of V and g, respectively.

The Gâteaux derivatives of V and g have the following structure:

$$\nabla V(R_1)(H) =$$

$$-2\int_{[0,\infty)}[y - R_1(y)]H(y)\,dF + 2\int_{[0,\infty)}[y - R_1(y)]\,dF\int_{[0,\infty)}H(y)\,dF,$$

$$\nabla g(R_1)(H) =$$

$$\int_{[0,\infty)}H(y)\,dF + \beta\frac{\int_{[0,\infty)}R_1(y)H(y)\,dF - \int_{[0,\infty)}R_1(y)\,dF\int_{[0,\infty)}H(y)\,dF}{\sqrt{\int_{[0,\infty)}R_1^2(y)\,dF - (\int_{[0,\infty)}R_1(y)\,dF)^2}}.$$

In order to prove that $\nabla g(R_1)(H)$ is well-defined, we have to consider the radicand $\int_{[0,\infty)}(R_1^2(y)\,dF - (\int_{[0,\infty)}R_1(y)\,dF)^2$:

$$\int_{[0,\infty)}R_1^2(y)\,dF = (\int_{[0,\infty)}R_1(y)\,dF)^2$$

$$\Leftrightarrow \int_{[M,\infty)}R_1^2(y)\,dF = (\int_{[M,\infty)}R_1(y)\,dF)^2$$

$$\Leftrightarrow Pr(Y \geq M) = 1, \text{ which is a contradiction.}$$

As a sum of two convex functions, L_λ is convex in $\mathcal{C}$. Then, we conclude for all $R \in \mathcal{C}$ that

$$L_\lambda(R) - L_\lambda(R_1)$$

$$\geq \quad \nabla V(R_1)(R - R_1) + \lambda\nabla g(R_1)(R - R_1)$$

$$= 2\left\{ -\int_{[0,\infty)} [y - R_1(y)][R(y) - R_1(y)]\, dF \right.$$

$$\left. +\int_{[0,\infty)} [y - R_1(y)]\, dF \int_{[0,\infty)} [R(y) - R_1(y)]\, dF \right\}$$

$$+ \lambda \left\{ \int_{[0,\infty)} [R(y) - R_1(y)]\, dF + \beta \right.$$

$$\left. \cdot \frac{\int_{[0,\infty)} R_1(y)[R(y) - R_1(y)]\, dF - \int_{[0,\infty)} R_1(y)\, dF \int_{[0,\infty)} [R(y) - R_1(y)]\, dF}{\sqrt{\int_{[0,\infty)} R_1^2(y)\, dF - (\int_{[0,\infty)} R_1(y)\, dF)^2}} \right\}$$

$$= -2\int_{[0,M)} y[R(y) - R_1(y)]\, dF - 2\int_{[M,\infty)} [y - R_1(y)][R(y) - R_1(y)]\, dF$$

$$+2\left(\int_{[0,M)} y\, dF + \int_{[M,\infty)} [y - R_1(y)]\, dF \right) \int_{[0,\infty)} [R(y) - R_1(y)]\, dF$$

$$+\lambda \int_{[0,\infty)} [R(y) - R_1(y)]\, dF + \lambda\beta \left(\frac{\int_{[M,\infty)} R_1(y)[R(y) - R_1(y)]}{\sqrt{\int_{[M,\infty)} R_1^2(y)\, dF - (\int_{[M,\infty)} R_1(y)\, dF)^2}} \right.$$

$$\left. -\frac{\int_{[M,\infty)} R_1(y)\, dF \int_{[0,\infty)} [R(y) - R_1(y)]\, dF}{\sqrt{\int_{[M,\infty)} R_1^2(y)\, dF - (\int_{[M,\infty)} R_1(y)\, dF)^2}} \right)$$

$$= \int_{[0,\infty)} [R(y) - R_1(y)]\, dF \left(2\int_{[0,M)} y\, dF + 2\int_{[M,\infty)} [y - R_1(y)]\, dF + \lambda \right.$$

$$\left. -\lambda\beta \frac{\int_{[M,\infty)} R_1(y)\, dF}{\sqrt{\int_{[M,\infty)} R_1^2(y)\, dF - (\int_{[M,\infty)} R_1(y)\, dF)^2}} \right)$$

$$-2\int_{[0,M)} y[R(y) - R_1(y)]\, dF - 2\int_{[M,\infty)} [y - R_1(y)][R(y) - R_1(y)]\, dF$$

$$+\lambda\beta \frac{\int_{[M,\infty)} R_1(y)\, dF[R(y) - R_1(y)]\, dF}{\sqrt{\int_{[M,\infty)} R_1^2(y)\, dF - (\int_{[M,\infty)} R_1(y)\, dF)^2}} \right\}$$

$$= \int_{[0,\infty)} [R(y) - R_1(y)]\, dF \left(2\int_{[0,M)} y\, dF + 2\int_{[M,\infty)} [y - R_1(y)]\, dF + \lambda \right.$$

$$
\left.-\lambda\beta\frac{\int_{[M,\infty)} R_1(y)\,dF}{\sqrt{\int_{[M,\infty)} R_1^2(y)\,dF - (\int_{[M,\infty)} R_1(y)\,dF)^2}} - 2M\right)
$$

$$
+ \int_{[M,\infty)} [R(y) - R_1(y)]
$$

$$
\cdot\left(-2y + 2R_1(y) + \lambda\beta\frac{R_1(y)}{\sqrt{\int_{[M,\infty)} R_1^2(y)\,dF - (\int_{[M,\infty)} R_1(y)\,dF)^2}}\right) dF
$$

$$
-2\int_{[0,M)} y[R(y) - R_1(y)]\,dF + \int_{[0,\infty)} 2M[R(y) - R_1(y)]\,dF
$$

$$
= \int_{[0,\infty)} [R(y) - R_1(y)]\,dF\left(2\int_{[0,M)} y\,dF + 2\int_{[M,\infty)} [y - R_1(y)]\,dF + \lambda\right.
$$

$$
\left.-\lambda\beta\frac{\int_{[M,\infty)} R_1(y)\,dF}{\sqrt{\int_{[M,\infty)} R_1^2(y)\,dF - (\int_{[M,\infty)} R_1(y)\,dF)^2}} - 2M\right)
$$

$$
+ \int_{[M,\infty)} [R(y) - R_1(y)]
$$

$$
\cdot\left(2(-y + R_1(y)) + \lambda\beta\frac{R_1(y)}{\sqrt{\int_{[M,\infty)} R_1^2(y)\,dF - (\int_{[M,\infty)} R_1(y)\,dF)^2}}\right) dF
$$

$$
+2\int_{[0,M)} [M - y][R(y) - R_1(y)]\,dF + 2M\int_{[M,\infty)} [R(y) - R_1(y)]\,dF
$$

$$
= \int_{[0,\infty)} [R(y) - R_1(y)]\,dF\left(2\int_{[0,M)} y\,dF + 2\int_{[M,\infty)} [y - R_1(y)]\,dF + \lambda\right.
$$

$$
\left.-\lambda\beta\frac{\int_{[M,\infty)} R_1(y)\,dF}{\sqrt{\int_{[M,\infty)} R_1^2(y)\,dF - (\int_{[M,\infty)} R_1(y)\,dF)^2}} - 2M\right)
$$

$$
+ \int_{[M,\infty)} [R(y) - R_1(y)]
$$

$$
\cdot\left(2(M - y + R_1(y)) + \lambda\beta\frac{R_1(y)}{\sqrt{\int_{[M,\infty)} R_1^2(y)\,dF - (\int_{[M,\infty)} R_1(y)\,dF)^2}}\right) dF
$$

$$
+2\int_{[0,M)} [M - y][R(y) - R_1(y)]\,dF
$$

$$
= \int_{[0,\infty)} [R(y) - R_1(y)] \, dF \left(2E(Y) - 2M - 2 \int_{[M,\infty)} R_1(y) \, dF - 2r \cdot \right.
$$

$$
\frac{\sqrt{\int_{[M,\infty)} (y-M)^2 \, dF - (\int_{[M,\infty)} (y-M) \, dF)^2} \int_{[M,\infty)} (1-r)(y-M) \, dF}{\sqrt{\int_{[M,\infty)} (1-r)^2 (y-M)^2 \, dF - (\int_{[M,\infty)} (1-r)(y-M) \, dF)^2}}
$$

$$
\left. + \frac{2r}{\beta} \sqrt{\int_{[M,\infty)} (y-M)^2 \, dF - \left(\int_{[M,\infty)} (y-M) \, dF \right)^2} \right)
$$

$$
+ \int_{[M,\infty)} [R(y) - R_1(y)]
$$

$$
\cdot \left(2(M - y + R_1(y)) + \lambda \beta \frac{R_1(y)}{\sqrt{\int_{[M,\infty)} R_1^2(y) \, dF - (\int_{[M,\infty)} R_1(y) \, dF)^2}} \right) dF
$$

$$
+ 2 \int_{[0,M)} [M - y][R(y) - R_1(y)] \, dF \qquad \text{(using (4.6))}
$$

$$
= \int_{[0,\infty)} [R(y) - R_1(y)] \, dF \left(2E(Y) - 2M - 2 \int_{[M,\infty)} (1-r)(y-M) \, dF \right.
$$

$$
- 2r \int_{[M,\infty)} (y-M) \, dF
$$

$$
\left. + \frac{2r}{\beta} \sqrt{\int_{[M,\infty)} (y-M)^2 \, dF - \left(\int_{[M,\infty)} (y-M) \, dF \right)^2} \right)
$$

$$
+ \int_{[M,\infty)} [R(y) - R_1(y)] \Big(2(M - y + R_1(y))
$$

$$
+ \lambda \beta \frac{R_1(y)}{\sqrt{\int_{[M,\infty)} R_1^2(y) \, dF - (\int_{[M,\infty)} R_1(y) \, dF)^2}} \Big) dF
$$

$$
+ 2 \int_{[0,M)} [M - y][R(y) - R_1(y)] \, dF
$$

$$
= \int_{[0,\infty)} [R(y) - R_1(y)] \, dF \left(2E(Y) - 2M - 2 \int_{[M,\infty)} (y-M) \, dF \right.
$$

$$
\left. + \frac{2r}{\beta} \sqrt{\int_{[M,\infty)} (y-M)^2 \, dF - \left(\int_{[M,\infty)} (y-M) \, dF \right)^2} \right)
$$

$$
+ \int_{[M,\infty)} [R(y) - R_1(y)] \Big(2(M - y + R_1(y))
$$

$$+\lambda\beta\frac{R_1(y)}{\sqrt{\int_{[M,\infty)} R_1^2(y)\, dF - (\int_{[M,\infty)} R_1(y)\, dF)^2}}\Bigg)\, dF$$

$$+2\int_{[0,M)} [M-y][R(y)-R_1(y)]\, dF$$

$$= \int_{[0,\infty)} [R(y)-R_1(y)]\, dF \cdot 0$$

$$+\int_{[M,\infty)} [R(y)-R_1(y)]\Bigg(2[M-y+(1-r)(y-M)]$$

$$+2r\frac{(1-r)(y-M)\sqrt{\int_{[M,\infty)}(y-M)^2\, dF - (\int_{[M,\infty)}(y-M)\, dF)^2}}{\sqrt{\int_{[M,\infty)}(1-r)^2(y-M)^2\, dF - (\int_{[M,\infty)}(1-r)(y-M)\, dF)^2}}\Bigg)\, dF$$

$$+2\int_{[0,M)} [M-y][R(y)-R_1(y)]\, dF \qquad \text{(using (4.3) and (4.6))}$$

$$= \int_{[M,\infty)} \Big[R(y)-R_1(y)\Big]\Big[2\Big(M-y+(1-r)(y-M)+r(y-M)\Big)\Big]\, dF$$

$$+2\int_{[0,M)} [M-y][R(y)-R_1(y)]\, dF$$

$$= 2\int_{[0,M)} [M-y][R(y)-R_1(y)]\, dF$$

$$\geq 0 \;\forall\; R \in \mathcal{R}.$$

Hence, assumption (ii) of Lemma 8 is satisfied. Now we can conclude that R_1 minimizes V over $\mathcal{M}_\leq$ (and also minimizes V over $\mathcal{M}_=$ by using (4.4)). That implies that R_1 is an optimal reinsurance contract for an insurer under (4.1) and (4.2).

$\square$

Remark 4

(i) *The optimal reinsurance coverage is in fact a combination of proportional and excess of loss reinsurance. Up to a certain amount of claims M, the reinsurer pays nothing and the primary insurer has to cover the full liabilities. Above that amount M, the reinsurer pays a proportional amount of the difference of the*

total claim and M, whereas the primary insurer is responsible for the rest.

(ii) *If we consider (4.2), we see that $r \to 0$ as $\beta \to 0$, because otherwise the equation would not hold anymore. This indicates that R_1 tends to an excess of loss reinsurance when $\beta \to 0$. As Daykin et al. (1993) describe, it is well known that an excess of loss reinsurance coverage is an optimal coverage for a primary insurer under the net premium principle. That indicates that the optimality of the change loss reinsurance corresponds with this older result.*

Let P_{max} be fixed. Now we want to compare the optimal strategies described in Theorem 6 with a quota share reinsurance contract. To do so, Gajek and Zagrodny (2000) suggest the use of a reinsurance risk reduction function $\Phi \colon \mathcal{H} \to \mathbb{R}$, defined by

$$\Phi(R) = \sqrt{Var\big(Y\big)} - \sqrt{Var\big(Y - R(Y)\big)}.$$

Let R_1 denote the change loss reinsurance, and R_2 the quota share reinsurance. Determining $\frac{\Phi(R_1)}{\Phi(R_2)}$ gives us how many times $\Phi(R_1)$ is larger than $\Phi(R_2)$.

We know that $\tilde{R}_2(Y) = \alpha Y$ and $R_2(Y) = (1-\alpha)Y$ for $0 \le \alpha \le 1$.

$$\Rightarrow \sqrt{Var\big(\tilde{R}_2(Y)\big)} = \alpha \sqrt{Var(Y)}$$

$$\Rightarrow \Phi(R_2) = \sqrt{Var(Y)} - \alpha \sqrt{Var(Y)} = (1-\alpha)\sqrt{Var(Y)}$$

Let us calculate $1 - \alpha$ by assuming that all the money is spent on that type of reinsurance strategy:

$$P_{max} = E(R_2(Y)) + \beta\sqrt{Var\big(R_2(Y)\big)}$$
$$= (1-\alpha)E(Y) + \beta\sqrt{Var\big((1-\alpha)Y\big)}$$
$$= (1-\alpha)E(Y) + \beta(1-\alpha)\sqrt{Var(Y)}$$
$$= (1-\alpha)\Big(E(Y) + \beta\sqrt{Var(Y)}\Big)$$

$$\Leftrightarrow 1-\alpha = \frac{P_{max}}{E(Y)+\beta\sqrt{Var(Y)}}$$

This implies that $\Phi(R_2) = \frac{P_{max}\sqrt{Var(Y)}}{E(Y)+\beta\sqrt{Var(Y)}}$.

The question is now: What value of β is reasonable for both the primary insurer and the reinsurer? As described by Gajek and Zagrodny (2000), this is a difficult question. They assume that both parties accept a probability of 0.95 that $R(Y)$ is not greater than the premium for the reinsurance contract. That means:

$$Pr(R(Y) \le P_{max}) = 0.95.$$

We use the Central Limit Theorem and obtain

$$Pr(R(Y) \le P_{max})$$

$$= Pr\left(\frac{R(Y)-E(R(Y))}{\sqrt{Var(R(Y))}} \le \frac{P_{max}-E(R(Y))}{\sqrt{Var(R(Y))}}\right)$$

$$= Pr\left(\frac{R(Y)-E(R(Y))}{\sqrt{Var(R(Y))}} \le \frac{E(R(Y))+\beta\sqrt{Var(R(Y))}-E(R(Y))}{\sqrt{Var(R(Y))}}\right)$$

$$= Pr\left(\frac{R(Y) - E(R(Y))}{\sqrt{Var(R(Y))}} \leq \beta\right)$$

$$= 0.95$$

$$\Rightarrow \beta \approx 1.645$$

Example 6 Let us consider an example given by Gajek and Zagrodny (2000):

We assume that the total claim of a primary insurer follows a Normal distribution with mean 10^9 and variance 10^{16}, i.e.

$$Y \sim N(10^9, 10^{16}).$$

We calculate with a value for the safety loading parameter of

$$\beta = 1.645,$$

as we explained above, and the amount a primary insurer is able to spend on reinsurance coverage is

$$P_{max} = 2.91 \cdot 10^9.$$

Gajek and Zagrodny (2000) state solutions of (4.3) and (4.4) as

$$M = 8.506 \cdot 10^8 \ \ and \ \ r = 0.052,$$

that means the reinsurer pays 94.8% of that amount of the primary insurer's claim that exceeds $8.506 \cdot 10^{16}$.

We want to compare the reinsurance risk reductions for both change loss and quota share reinsurance coverage. This means that we have to calculate the variance of the total claims Y, and the variance of that part of Y that is paid by the primary insurer, denoted by $\tilde{R}(Y)$.

Let f denote the $p.d.f.$ of Y. Then we get the following results for the change loss reinsurance:

$$
\begin{aligned}
E(\tilde{R}_1(Y)) &= E[Y - R_1(Y)] \\
&= E(Y) - E[R_1(Y)] \\
&= 10^9 - \int_{8.506 \cdot 10^8}^{\infty} (1 - 0.052)(y - 8.506 \cdot 10^8) f(y)\, dy \\
&\approx 855,552,294.3
\end{aligned}
$$

and

$$
Var(\tilde{R}_1(Y))
$$

$$
\begin{aligned}
&= Var[Y - R_1(Y)] \\
&= E\Big[(Y - R_1(Y)) - E(Y - R_1(Y))\Big]^2 \\
&= E\Big[(Y - R_1(Y)) - 855,552,294.3\Big]^2 \\
&= \int_{-\infty}^{8.506 \cdot 10^8} (y - 855,552,294.3)^2 f(y)\, dy \\
&\quad + \int_{8.506 \cdot 10^8}^{\infty} (y - (1 - 0.052)(y - 8.506 \cdot 10^8) - 855,552,294.3)^2 f(y)\, dy \\
&\approx .2942527421 \cdot 10^{15}
\end{aligned}
$$

$$
\Rightarrow \sqrt{Var(\tilde{R}_1(Y))} \approx 0.1715379673 \cdot 10^8
$$

Notice that we calculated with rounded values for M and r, respectively, that are given by Gajek and Zagrodny (2000), and for β. For the calculation of the above integrals we used Maple® 6.

The value for $Var(\tilde{R}_1(Y))$ is not the one that Gajek and Zagrodny (2000) obtained, but we put this down to our calculation with rounded values, which are squared.

Nevertheless, we obtain an approximate result for the reinsurance risk retention of the change loss coverage of

$$\Phi(R_1) = \sqrt{Var(Y)} - \sqrt{Var(\tilde{R}_1(Y))} = 0.8284620327 \cdot 10^8.$$

For the same premium $P_{max} = 2.91 \cdot 10^8$, the primary insurer is able to purchase a quota share contract with parameter

$$\begin{aligned}
\alpha &= 1 - \frac{P_{max}}{E(Y) + \beta\sqrt{Var(Y)}} \\
&= 1 - \frac{2.91 \cdot 10^8}{10^9 + 1.645 \cdot 10^8} \\
&= 0.75
\end{aligned}$$

Then we obtain a reinsurance risk retention of the quota share coverage of

$$\begin{aligned}
\Phi(R_2) &= (1 - \alpha)\sqrt{Var(Y)} \\
&= 0.25 \cdot 10^8
\end{aligned}$$

Hence,

$$\frac{\Phi(R_1)}{\Phi(R_2)} = 3.313848131$$

indicates that the risk reduction with change loss reinsurance is 3.313848131 times greater than having quota share protection.

Gajek and Zagrodny (2000) obtained a value of

$$\frac{\Phi(R_1)}{\Phi(R_2)} = 3.5,$$

which indicates that the difference due to the rounding errors is not very large.

Example 7 Let us assume that a primary insurer has 3 policies written on home-owner fire insurance. The probability of having a loss of X for each policy is given by

$$Pr(X = x) = \begin{cases} 0.96 & if \ x = 0, \\ 0.03 & if \ x = 500, \\ 0.01 & if \ x = 200,000, \\ 0 & otherwise. \end{cases}$$

Let us assume that the random losses for each policy are independent from each other. The question is: What is an optimal reinsurance contract for the primary insurer? In order to answer this question, we have to determine the parameters of R_1, which are solutions M and r of

$$E(Y) - M - \int_{[M,\infty)} (y - M) \, dF$$

$$+ \frac{r}{\beta} \sqrt{\int_{[M,\infty)} (y - M)^2 \, dF - \left(\int_{[M,\infty)} (y - M) \, dF \right)^2} = 0, \qquad (4.3)$$

and

$$\int_{[M,\infty)} (y - M)\, dF + \beta \sqrt{\int_{[M,\infty)} (y - M)^2\, dF - \left(\int_{[M,\infty)} (y - M)\, dF\right)^2}$$

$$= \frac{P_{max}}{1 - r}, \qquad (4.4)$$

where Y denotes the total claim of the primary insurer.

We assume that the amount the primary insurer is able to spend on reinsurance coverage is $P_{max} = 15,000$, and that the safety loading parameter is $\beta = 1.645$.

First, we determine the density function of Y.

$$Pr(Y = y) = \begin{cases} 0.96^3 & if \ y = 0, \\ 3 \cdot 0.03 \cdot 0.96^2 & if \ y = 500, \\ 3 \cdot 0.03^2 \cdot 0.96 & if \ y = 1,000, \\ 0.03^3 & if \ y = 1,500, \\ 3 \cdot 0.01 \cdot 0.96^2 & if \ y = 200,000, \\ 6 \cdot 0.01 \cdot 0.03 \cdot 0.96 & if \ y = 200,500, \\ 3 \cdot 0.01 \cdot 0.03^2 & if \ y = 201,000, \\ 3 \cdot 0.01^2 \cdot 0.96 & if \ y = 400,000, \\ 3 \cdot 0.01^2 \cdot 0.03 & if \ y = 400,500, \\ 0.01^3 & if \ y = 600,000, \\ 0 & otherwise. \end{cases}$$

Now we are able to determine solutions M and r of (4.3) and (4.4). We use Maple® 6 for this task and obtain

$$M = 14,900.92$$

and

$$r = 0.7419230843.$$

Thus, the change loss reinsurance has the form

$$R_1(y) = \begin{cases} 0 & if \ \ y \le 14,900.92, \\ (1 - 0.7419230843)(y - 14,900.92) & otherwise, \end{cases}$$

This implies that the reinsurer pays about 25.81% of that amount of the primary insurer's total claim that exceeds $14,900.02$, given the premium for the reinsurance contract is

$$\begin{aligned} P_r = E\big(R_1(Y)\big) + \beta\sqrt{Var\big(R_1(Y)\big)} &= 1,434.47 + 1.645 \cdot \sqrt{8,370.99} \\ &= 1,584.97. \end{aligned}$$

We observe that the reinsurance premium is only about 26.22% of the primary insurer's expected total claim, which is $E(Y) = 6,045$.

4.5 Results using General Risk Measures

As presented by Gajek and Zagrodny (2002), we can develop the theory of an optimal reinsurance contract in a different manner.

Let us review that the primary insurer retains the part

$$\tilde{R}(Y) = Y - R(Y)$$

of its total claim Y. A dangerous situation emerges if the deviation of $\tilde{R}$ is large. Thus, let us state the following definition:

Definition 19 *A function*

$$\varphi \colon \mathbb{R} \to \mathbb{R}_+.$$

that measures the primary insurer's loss due to deviation of $\tilde{R}$ is called a harm function.

Gajek and Zagrodny (2002) mention several choices for such a harm function: A standard approach is to use

- $\varphi(t) = t^2$,

other choices can be

- $\varphi(t) = |t|$,

- $\varphi(t) = t^+$,

- or $\varphi(t) = (t^+)^2$,

where $t^+ := max\{0, t\}$.

Given a certain harm function, the risk measure can be defined by

$$\rho(R) = E\left[\varphi\big(Y - R(Y) - E(Y - R(Y))\big)\right].$$

Gajek and Zagrodny (2002) show that a change loss coverage is the optimal rein-surance strategy if we use the risk measure

$$\rho(R) = E\left[\left(Y - R(Y) - E\big(Y - R(Y)\big)\right)^{+}\right]^{2}.$$

On the other side, an optimal reinsurance contract has the form

$$R_3(y) = \begin{cases} 0 & if \ y \leq M_1, \\ y - M_1 & if \ M_1 < y \leq M_2, \\ M_1 - M_2 & if \ y > M_2, \end{cases}$$

if we are using the risk measures

$$\rho(R) = E\left|Y - R(Y) - E\big(Y - R(Y)\big)\right|,$$

or

$$\rho(R) = E\left(Y - R(Y) - E\big(Y - R(Y)\big)\right)^{+}.$$

Remark 5 *We can observe that R_2 is a special form of excess of loss reinsurance where the reinsurer's liability is bounded by $M_2 - M_1$. Hence we can see that this con-tract contains two retentions: The primary insurer's retention M_1, and the reinsurer's retention M_2.*

CHAPTER V

DISCUSSION OF TERRORISM INSURANCE

The terrorist attacks on the United States on September 11, 2001 have led to major changes in the whole insurance business. Though terrorism attacks took place before (e.g., the Oklahoma bombing on April 19, 1995), the claims had never have such a big impact as than on September 11, 2001.

5.1 Insured Losses from September 11, 2001

As Bouriaux and Scott (2002) present, the accumulated insured losses are ranging somewhere between 40 billion and 54 billion dollars.

One estimation by the Insurance Information Institute in 2002 is that the total insured losses were about 40.2 billion dollars. About 27 billion dollars or about 67% of these 40.2 billion dollars were covered by reinsurance companies.

The loss estimates from the attacks for both Munich Re and Swiss Re together are ranging between 4 billion and 5 billion dollars, as presented in Best's Review (2002). The consequences for these two reinsurer are a total of 3.4 billion dollars in pre-tax operating losses for 2001, as described in Bouriaux and Scott (2002).

5.2 "Insurance Terrorism Act"

After the attacks on September 11, 2001, the question concerning claim adjustment for future losses caused by terrorism was discussed. One reason for that was the fact that most property and casualty (P/C) insurers have begun to exclude terrorism

coverage from their policies after September 11, 2001, as described in Bouriaux and Scott (2002).

Bouriaux and Scott (2002) state that the insurance industry favored the creation of a reinsurance pool similar to the one in the United Kingdom. But the federal government rejected this concept (we will return to that issue later) and developed a new policy in the form of government provided reinsurance. On November 26, 2002, United States President George W. Bush signed a new law being called "Insurance Terrorism Act" (H.R. 3210), which created that reinsurance program. We want to summarize the main elements of that act as follows:

- Terrorism exclusions in P/C contracts that were in force on enactment date are void.

- The aggregate deductible for insurers is 10 billion dollars in 2004 and 15 billion dollars in 2005.

- The federal government provides financial help by paying 90% of the amount that is greater than the deductible, but not more than 100 billion dollars.

- The program will terminate on December 31, 2005.

We can observe that the federal government has the role of a reinsurer and applied the change loss reinsurance as defined in Definition 17 with $r = 0.9$, and $M = 10$ billion dollars (for 2004) and $M = 15$ billion dollars (for 2005), respectively. The maximum amount of coverage is 100 billion dollars.

5.3 Situation in the Insurance Industry

As Bouriaux and Scott (2002) describe, insurance against terrorist attacks is not available today for some properties at any premium amount. For most of those policies

where such coverage is available, the premiums are higher as than before September 11, 2001, the deductibles are larger, and chemical, biological or radiological attacks are excluded.

The consequences are that many property owners, who want to have coverage, have to bear the risk on their own or have to pay high prices for coverage. As presented by Barnes (2002), owners of commercial properties in the center of big cities or airport operators are most affected.

We can also point out these tendencies by a statement in a February 2002 report from the General Accounting Office, saying that the risks for losses of terrorism attacks are shifted from reinsurer to primary insurers and then to the insured.

But not only for the insured (or non-insured), but also for primary insurers and reinsurer, the times are difficult.

- Uncertainty

  As presented in an article in Best's Review (2002), primary insurers and reinsurer are waiting for a creation of certainty that should be set up by the federal government for the case of future terrorism attacks, particularly with regard to their maximum amount of losses. The article says, insurers were able and willing to pay for the claims from the attacks on September 11, 2001. But things would look totally different in the case of another attack with such extents.

- Losses and bankruptcy

  119 insurers (property and life) and reinsurer declared losses from the terrorism attacks on September 11, 2001.

  In addition, the number of business failures in the P/C insurance industry in 2002 was the highest for the last ten years, as presented in Best's Review (2003): In 2002, 38 P/C insurance companies (that represents 1.33 %) became insolvent.

In 2000 and 2001, only 30 companies declared insolvent in each year (1.02% and 1.03%, respectively).

In 2002, the whole reinsurance industry was focused on risk selection and rate adequacy (Swiss Re, 2003). The charges for reinsurance coverage were again, as in 2001, increasing, while many risks were excluded from coverage. In addition, the shift to nonproportional reinsurance products was continuing.

5.4 Emerging Solutions of the Problem

We want to investigate how solutions for these problems in the insurance industry could look like. First, let use observe that terrorism risk has its own nature with regard to some aspects:

- Missing historical data

  Though several terrorism attacks took place until now, there is still not enough data to predict terrorism losses. Instead, insurers have to make forecasts without the underlying mathematical justification, which is given by the law of large numbers.

- Losses can be catastrophic

  As intended on September 11, 2001, terrorism attacks can hit many properties at one time. If insurers would offer coverage for terrorism attacks, similar properties would be contained in one pool. Thus, the consequences for insurers could be that losses can have catastrophic dimensions.

  We can observe that pooling does not make any sense at all in this situation, since many pool members can be susceptible to large losses.

5.4.1 Catastrophe Insurance

Insurers argue that losses from terrorism events are difficult to predict. The same argument applies for catastrophe insurance, which is still available in the market. This leads to the question why catastrophe coverage is available, but terrorism coverage not.

Although terrorism events and natural disasters have something in common, there are aspects that show that there exist some important differences, as described by Bouriaux and Scott (2002):

- Inputs for simulations

  While it is possible to simulate catastrophic events using their characteristics like wind, for example, we do not know any inputs to simulate terrorism events.

- Changes in methods

  It is very difficult to forecast new methods or tools used by terrorists for future attacks.

- Payment for claims

  A basic assumption regarding to catastrophe insurance is that most claims are paid within one year after the event took place. But terror attacks can create claims which may take several years to develop.

5.4.2 Capital Market

After hurricane Andrew in 1992, a way of risk transfer was developed by the setting-up of insurance-linked capital market instruments such as catastrophe bonds, for example. Hence, we can use the same approach to cede terrorism risk to the capital market, as presented by Bouriaux and Scott (2002). This transfer includes some advantages:

- Expand capacities

  By offering terrorism coverage, capital markets would create additional capacity and help to decrease premiums.

- Better risk diversification

  By creating portfolios that include several forms of risk exposure, the risk of terrorism coverage could be decreased. Also, capital markets can connect insurance risk and financial risk in order to create a better diversification.

Bouriaux and Scott (2002) present a simulation of terrorism risk by using ten different loss distributions. They assume that the rate of return r can be calculated as

$$r = \frac{Future\ \ cash\ \ flow\ \ to\ \ the\ \ investor}{Amount\ \ invested} - 1,$$

where the future cash flow to the investor is equal to the amount of premium collected plus the interest on reserves and premiums minus the operating expenses minus the amount of claim payments.

Bouriaux and Scott (2002) assume that the interest on reserves and premiums is 6%, that the operating expenses are 10% of the aggregate insurance amount, and that a required return for the insurance risk is calculated using Ibbotson-Singuefield Data. Whit these assumptions, they obtain the following results:

- The premiums that the capital market must charge for coverage are relatively high: They vary between 12% and 44% of the insured amount.

- The expected values of losses are also very high: They vary between 4.59% and 34.45% of the insured amount.

Nevertheless, we do not have to forget that the market for insurance-linked instruments has stayed very small. This could indicate that a transfer of terrorism risk to the capital market would not have the wished attractiveness.

5.4.3 Pool Re

As Barnes (2002) describes, several bombings happened in the United Kingdom in the beginning of the 1990s. Some of them took place in London's financial area and created large losses; one bombing, the so called Bishopsgate bombing of April 1993, created a loss of about 1 billion Pound Sterling.

After one attack in 1992, insurers began to exclude terrorism coverage on policies for commercial properties. At the end of 1992, most insurers covered terrorism coverage only up to an amount of 100,000 Pound Sterling.

The Association of British Insurers began to work on a solution for this situation. A work group developed the idea of establishing a reinsurance pool, the *Pool Reinsurance Ltd*, or just Pool Re, which was then founded in the summer of 1993. Pool Re is a new form of reinsurance with the following properties:

- Pool Re is a mutual insurer.

- Every insurance company operating in the United Kingdom can become a voluntary member.

- Members can cede terrorism coverage that exceeds 100,000 Pound Sterling to Pool Re by paying a reinsurance premium.

- The reinsurance premium is exactly that amount that an insurer charges for additional terrorism coverage above 100,000 Pound Sterling.

- The United Kingdom government indemnifies Pool Re if claims exceed Pool Re's capital.

- At the end of 2001, Pool Re has had about 220 members.

Since its foundation, Pool Re has reimbursed members for ten large claims for a total amount of 600 million Pound Sterling, whereas it received premiums in an amount of 1.7 billion Pound Sterling.

Due to its simple structure with just a few full-time employees, Pool Re's operating costs are very low (about 1 million Pound Sterling each year), so that its surplus was 904 million Pound Sterling at the end of 2000.

If liabilities exceed Pool Re's resources in a certain year, the members are asked to administer up to 10% of their ceded premium amount for the appropriate year. After that, Pool Re can use its aggregate investment income in order to adjust claims. Finally, Pool Re can ask the United Kingdom government for compensation. So far, Pool Re has never asked its members or the government for any financial support in order to adjust claims.

In the case of an underwriting profit in a year, Pool Re is able to return premiums to its participating members, up to an accumulated amount of 10% of the total underwriting profit.

If Pool Re's surplus exceeds 1 billion Sterling Pounds, it has to pay a premium to the government, since the government indemnifies Pool Re. The premium is calculated by a specific formula. The premium payment may not decrease Pool Re's surplus to an amount below 1 billion Pound Sterling. In 2001, Pool Re made its first payment to the government, at an amount of 204 million Pound Sterling.

All this shows that some major changes took place in the United Kingdom insur-

ance market since the establishment of Pool Re:

- In general, terrorism coverage is available for commercial property owners.

- Insurers have certainty about their maximum loss in the case of a terrorism attack.

- Insurers can participate in that program on a voluntary basis.

- In the long term, the government has small or no expenses.

All together, the Pool Re model can be considered as a solution for the problems existing in the United States right now. With such a system, it would be more likely that terrorism coverage would be offered to commercial property owners, while we can observe a lack in that market right now. An appropriate model could help to decrease the number of business failures in the insurance industry to a modest number.

Overall, a concept like Pool Re could help the whole economy, which would be harmed in the case of a new terrorism attack due to missing spread of risk - or in other words: due to missing insurance for some properties.

As we have stated earlier, the insurance industry in the United States favored the creation of a reinsurance pool after the attacks on September 11, 2001. But the United States Treasury Department refused this idea since it did not want to be the final financier. When we consider the federal legislation now, we observe that the United States Treasury Department has the role of the final lender.

As Barnes (2002) describes, United States congress and the White House opposed the role of the federal government in an industry that is regulated by the states. But in the Pool Re model, the government has no big influence on the whole concept: It only decides whether an event is an act of terrorism or not.

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Finally, we can observe that the Pool Re model could be the model of choice for the United States, since it provides the needs for both the insurance industry and the consumers. It could be a field of future research whether the model is a desirable concept for the federal government or not.

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