ASSET-LIABILITY MANAGEMENT WITH MULTIPLE SHORTFALL AND

TAIL CONDITIONAL EXPECTATION CONSTRAINTS

Wolfgang Karcher

140 Pages

August 2006

This thesis provides an asset-liability management methodology for defined benefit plans. The methodology is based on the concept of shortfall and tail conditional expectation constraints which are used to present one way of how to manage the pension fund of such a defined benefit plan.

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The downturn of the stock markets between the years 2000 and 2002 was a revealing experience for most investors. These years of falling stock markets made them realize how risky stock markets are. With this development of the stock markets in mind, it is desirable to have tools which can help control the risk of an investment and which provide strategies of how to come to investment decisions.

This thesis is devoted to the specific challenges that a pension fund manager faces when managing the fund of a certain type of pension plans - defined benefit plans. It provides an asset-liability management methodology based on the concept of shortfall constraints and tail conditional expectation constraints. These constraints are used to present a way of how to restrict the risk of the pension fund of a defined benefit plan. Based on the idea of shortfall and tail conditional expectation constraints, a strategy is developed that can support the pension fund manager in his/her investment decisions.

ASSET-LIABILITY MANAGEMENT WITH MULTIPLE SHORTFALL AND

TAIL CONDITIONAL EXPECTATION CONSTRAINTS

WOLFGANG KARCHER

A Thesis Submitted in Partial
Fulfillment of the Requirements
for the Degree of

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CHAPTER I

INTRODUCTION

The downturn of the stock markets between 2000 and 2002 turned out to be a great challenge for most investors. Over that period, stock market indices declined steadily. The Dow Jones Industrial Average, for example, lost 24.21% of its value and dropped from $11,357.51$ at the start of 2000 to $8,607.52$ at the start of 2003 (Dow Jones & Company, 2006). In the same period, the NASDAQ declined by 66.48% of its value (NASDAQ, 2006). This development made investors realize how risky stock markets really are.

Risk can be defined as the uncertainty that exists as to the occurrence of some event (Greene, 1968, p. 2). The risk that an investor faces is then the uncertainty that exists as to what return he/she will earn. In this thesis, we will develop tools that can support investors in their investment decisions and that can help control the investment risk. In particular, we will discuss how a pension fund manager can handle the challenges which he/she faces when managing a pension fund. For this purpose, we need to introduce some terms and notions.

At first, we provide a brief overview of pension plans. Usually, pension plans are set up by an employer in order to deliver retirement benefits. There also exist other kinds of pension plans which are not so common: multi-employer plans, and plans

organized by various levels of government. In this thesis, we will focus on pension plans organized by an employer.

One criterion to classify pension plans is "the asset base for the liabilities for benefits promised to plan participants" (Gajek and Ostaszewski, 2004, p. 5). Then we can distinguish between the following types of pension plans:

- pension plans without a fund and

- pension plans with a fund.

Pension plans without a fund are also called pay-as-you-go plans. This means that there is no fund where assets are accumulated for the purpose of paying benefits (cf. Gajek and Ostaszewski, 2004, p. 5). By contrast, pension plans with a fund - so-called funded pension plans - do accumulate assets for the purpose of delivering retirement benefits. In the following, we will not further consider pay-as-you-go plans, but concentrate on funded pension plans. A detailed discussion about pay-as-you-go plans and funded pension plans is provided by Gajek and Ostaszewski (2004).

Another criterion to classify pension plans is "the method of correction of an imbalance between assets and liabilities" (Gajek and Ostaszewski, 2004, p. 6). Then we can distinguish between the following types of pension plans:

- defined contribution plans and

- defined benefit plans.

A defined contribution plan is defined as a pension plan for which only the

contributions are prescribed in advance (and benefits are determined by the performance of the assets of the plan), whereas a defined benefit plan is defined as a plan for which benefits are prescribed in advance, and asset performance affects contribution levels needed to fund benefits (Gajek and Ostaszewski, 2004, p. 6). A combination of the features of both types of pension plans is also possible: these pension plans are often referred to as hybrid plans and have become more and more popular in the US since the 1990s. Examples of typical hybrid designs include pension equity plans and cash balance plans which are discussed in detail by Green (2003) and Elliott and Moore (2000), respectively.

Since for a defined contribution plan, the benefits are not fixed in advance, but only depend on the asset performance, the employees bear the investment risk. For a defined benefit plan, the employer is responsible to the plan participants for paying the predeterminate retirement benefits at retirement. Thus, in this case, the investment risk is borne by the employer.

The characteristic in which we are mainly interested in this thesis is the level of investment risk that the employer has to bear. Since the employer bears the investment risk for a defined benefit plan, we will focus on this type of pension plans henceforth.

When offering a funded defined benefit plan, the company should find a way to manage the additional investment risk. A pension fund manager is often hired for this purpose. The pension fund manager's task is then to decide how to invest the contributions that are made to the pension plan. Furthermore, the manager must make

sure that the employer can deliver the retirement benefits. To do so, the fund manager needs to manage the assets of the pension fund against its liabilities. This is often referred to as asset-liability management. The asset-liability management methodology which we will discuss in this thesis is based on concepts presented in Leibowitz et al. (1996).

In chapter II and chapter III, we will develop the concept of shortfall constraints and tail conditional expectation constraints. We will see how these concepts can help the pension fund manager control the overall risk of the pension fund.

In chapter IV, the results from chapter II and III will be used to develop a strategy on which the fund manager can base his/her investment decisions.

Throughout the thesis, examples will be given for several results. These examples are all based on the same values of the underlying variables such that the graphs can be compared to each other. The figures have been produced with the computer algebra system Maple.

CHAPTER II

SHORTFALL CONSTRAINTS

In this chapter, we want to analyze how pension fund managers can control the overall risk of the pension fund. For this purpose, we need to know how pension fund managers can measure the risk of the pension fund.

A comprehensive overview of risk measures is provided by Rachev et al. (2005, pp. 187). In the following, we will give a brief summary of this overview. For this purpose, we will focus on the risk of an investment and denote the random return of the investment by R. At first, however, we need a formal definition of a risk measure.

Definition 1 (Risk Measure)

A risk measure ρ is a mapping from a set of random variables S to the set of real numbers $\mathbb{R}$:

$$\rho : S \to \mathbb{R}.$$

Basically, there are two different groups of risk measures: dispersion measures and safety-first measures. Dispersion measures quantify the dispersion of the random variable R. This means that a large dispersion represents a high risk, whereas a small dispersion indicates less risky investments. The most common dispersion measures are (cf. Rachev et al., 2005):

1. Mean Standard Deviation (σ)

   The dispersion measure is the standard deviation σ of the random variable R:

   $$\sigma(R) = \sqrt{E\left(R - \mu\right)^2},$$

   where $\mu = E(R)$ is the expected return of R.

2. Mean Absolute Deviation (MAD)

   The dispersion measure is the absolute deviation of R from the mean:

   $$MAD(R) = E\left(|R - \mu|\right).$$

3. Mean Absolute Moment (MAM)

   The mean absolute moment is defined as

   $$MAM(R, q) = (E(|R - \mu|^q))^{\frac{1}{q}}, \quad q \geq 1.$$

4. Gini Index of Dissimilarity (GM)

   This dispersion measure is defined as

   $$GM(R, B) = \min\{E(|R - B|)\},$$

   where B is the random return of a benchmark, for example the return of a market index. The minimum is taken over all joint distributions of (R, B).

5. Mean Entropy (EE)

   The dispersion measure is the exponential entropy which is defined as

   $$EE(R) = e^{-E(\ln(f(R)))},$$

where $f(\cdot)$ is the probability density function of R. This dispersion measure is only defined for continuous return distributions.

Safety-first risk measures follow a different approach. They use different concepts of probability theory in order to quantify the risk of falling below a certain return. Examples of these risk measures include (cf. Rachev et al., 2005):

1. Shortfall Probability (SP)

   The safety-first risk measure is the shortfall probability which is defined as

   $$SP(R, m) = P(R < m),$$

   where m is a predetermined minimum acceptable return and $P(\cdot)$ is used to denote probabilities. In order to control the risk, this probability shouldn't be too large.

2. Value-at-Risk (VaR)

   The Value-at-Risk is defined as

   $$VaR_\alpha(R) = -\inf\{x | P(R \leq x) > \alpha\},$$

   where α is a predetermined probability. The larger the VaR is, the higher the risk is.

3. Tail Conditional Expectation/Expected Tail Loss (TCE)

   We define the tail conditional expectation as

   $$TCE_\alpha(R) = E\left(-R | -R \geq VaR_\alpha(R)\right),$$

where $VaR_\alpha(R)$ is the Value-at-Risk. This definition is slightly different from the definition given by Rachev et al. (2005). They define the TCE as $TCE_\alpha(R) = E\left(\max\{-R, 0\}|-R \geq VaR_\alpha(R)\right)$. This means that basically only negative values of R (i.e. positive losses) contribute to TCE. In our approach however, we allow for contributions of both negative and positive values of R. The TCE measures the expectation of the loss $-R$, given that the loss is greater than or equal to the Value-at-Risk. The larger the TCE is, the higher the risk is.

4. Lower Partial Moment (LPM)

This risk measure depends on two parameters: The first one is a so-called power index which specifies how risk averse the investor is, whereas with the second one - the target rate of return - the minimum acceptable return which shouldn't be fallen short is set (cf. Rachev et al., 2005, p. 193). The lower partial moment $LPM(\cdot)$ is then defined as

$$LPM(R, q, t) = \sqrt[q]{E(max\{t - R, 0\}^q)},$$

where q is the power index and t is the target rate of return. The larger the LPM is, the higher the risk is.

There is one major disadvantage when using dispersion measures as risk measures: they fail to distinguish between upward and downward return fluctuations. Thus, they do not reflect the common idea of risk as a potential worse. In chapter I, risk was defined as the uncertainty that exists as to the occurrence of some event

(Greene, 1968, p. 2). This definition does not differentiate between the upward and downward fluctuations of the return, either. For the rest of this thesis, however, we change this definition a little bit. In the following, we will always think of risk as a potential loss where $loss = -return$. This means that a positive loss is something bad, whereas a non-positive loss is a gain, i.e. something positive. Then, upward fluctuations differ from downward fluctuations. Therefore, we will not use dispersion measures any more to measure risk.

Following Leibowitz et al. (1996), we will use shortfall probability as risk measure in this chapter to develop the concept of shortfall constraints. We will see how shortfall constraints can help avoid unreasonably risky allocations in stocks and bonds/cash.

The Normal Distribution Assumption

Before we introduce the concept of shortfall constraints, we set up a model for the asset return. This makes it possible to derive formulas for the shortfall constraints.

It is quite common to assume normally distributed asset returns since the normal distribution is an acceptable approximation of the distribution of historical asset returns. We note that the probability density function of the normal distribution (cf. Patel and Read, 1982, p. 18) is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty,$$

where μ and σ are the expected return and the standard deviation, respectively. If X is

a normally distributed random variable with expected return μ and standard deviation σ, we briefly denote: $X \sim N(\mu, \sigma^2)$.

Besides it is very convenient to work with normally distributed random variables because they have the following desirable properties:

1. A linear combination of normally distributed random variables is also normally distributed (cf. Johnson and Wichern, 2002, p. 156).

2. If X is normally distributed with expected value μ and standard deviation σ, then $\frac{X-\mu}{\sigma}$ has a standard normal distribution (i.e. a normal distribution with expected value $\mu = 0$ and standard deviation $\sigma = 1$) (cf. Patel and Read, 1982, p. 19).

However, there are also some arguments against the normal distribution assumption. Fehr (2006) noted that according to Rachev, stock market crashes are more likely to happen in reality than they would happen if returns were modeled with a normal distribution:

A stock market crash like the one in October 1987 should occur only once in 10^{87} years when using the normal distribution, while historically one expects that such a crash happens every 38 years. This means that if one relies on the normal distribution, one greatly underestimates these risks. (original in German, author's own translation)

Another disadvantage of the normal distribution is that it can produce returns that are smaller than -1 since the density function is positive for $x \in \mathbb{R}$. This does not reflect

reality. That is why sometimes the log-normal distribution is preferred. The probability density function of the log-normal distribution with parameters μ and σ is given by (cf. Klugman et al., 2004, p. 59)

$$f(x) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

If we assume that $1 + return \sim LN(\mu, \sigma)$, where $LN(\cdot)$ denotes the log-normal distribution, then $1 + return \geq 0$ since the density function is positive only for $x > 0$. That is why it must hold: $return \geq -1$. Thus with this distribution, it is possible to model returns that do not assume values less than -1.

Nevertheless it is often possible to obtain useful results for practical applications when assuming normally distributed returns (cf. Fehr, 2006). That is why we will use it when we derive formulas for the shortfall constraints in the following sections. Although there are some disadvantages, it provides some valuable insights into the shortfall concept. The intention of our approach is not to reflect reality perfectly, but to get a better understanding of it:

> But we must also remember that whatever model we select it is only an approximation of reality. This is reflected in the following modeler's motto [...]:

> All models are wrong, but some models are useful. (Klugman et al., 2004)

The Asset Return Shortfall Constraint

In the introduction of this chapter, we decided to use shortfall probability to measure the risk of a pension fund. A small shortfall probability represents a low risk, whereas a high shortfall probability indicates a high risk. In order to control the risk of a pension fund, the pension fund manager can restrict the shortfall probability such that this probability will not exceed a value that the fund manager regards as critical. This restriction of the shortfall probability is referred to as shortfall constraint.

The asset return shortfall constraint allows for controlling the downside risk of the asset return. A fund manager, for example, may have the objective to meet a minimum acceptable asset return of 3% with a probability of 95%. In general, this constraint can be formulated as follows (cf. Leibowitz et al., 1996, p. 44):

There should be no more than a probability of α that the asset return will be less than a minimum acceptable asset return of m.

We assume that the fund manager can choose from two different types of investments: stocks and bonds/cash. The fund manager's task is then to decide what percentage to invest in stocks and what percentage to invest in bonds/cash. It is intuitive that the shortfall constraint will reduce the number of possible stock/bond/cash portfolio combinations. We will see how this constraint affects the fund manager's choice between stocks and bonds/cash after we find the formula for the asset return shortfall constraint.

In order to derive this formula, we need to introduce some notations:

- R_P: return of the pension fund portfolio,

- R_B: return of the bond portion in the portfolio, and

- R_E: return of the stock portion in the portfolio.

We assume that R_B and R_E are normally distributed:

- $R_B \sim N(\mu_B, \sigma_B^2)$, where μ_B and σ_B^2 are the expected return and the variance of R_B, respectively.

- $R_E \sim N(\mu_E, \sigma_E^2)$, where μ_E and σ_E^2 are the expected return and the variance of R_E, respectively.

Then the following relationships between the random variables R_P, R_B and R_E hold:

$$R_P = w \cdot R_E + (1-w) \cdot R_B, w \in [0,1], \tag{2.1}$$

$$\mu_P = w \cdot \mu_E + (1-w) \cdot \mu_B, \tag{2.2}$$

$$\sigma_P = \sqrt{w^2 \sigma_E^2 + (1-w)^2 \sigma_B^2 + 2w(1-w)\sigma_E \sigma_B \rho_{EB}}, \tag{2.3}$$

$$R_P = w \cdot R_E + (1-w) \cdot R_B \sim N(\mu_P, \sigma_P^2). \tag{2.4}$$

The condition $w \in [0,1]$ means that short positions are not possible (remark: an investor has a short position if he/she sells a stock or a bond without actually owning it (this would be the case for $w < 0$ or $w > 1$). For this purpose, the investor has to borrow the stock or the bond from someone else and promise to buy it back and return it after a predetermined time.).

Now we can translate the asset return shortfall constraint into a mathematical expression:

There should be no more than a probability of α that the asset return will be less than a minimum acceptable asset return of m.

$$\Leftrightarrow$$

$$P(R_P < m) \leq \alpha. \tag{2.5}$$

The following result gives the formula for the shortfall constraint of the asset return:

Result II.1 (The Asset Return Shortfall Constraint)

The asset return shortfall constraint $P(R_P < m) \leq \alpha$ is given by the inequality

$$\mu_P \geq m - z_\alpha \cdot \sigma_P. \tag{2.6}$$

Proof. According to (2.4), R_P has a normal distribution with expected value μ_P (2.2) and variance σ_P (2.3). Then $\frac{R_P - \mu_P}{\sigma_P} \sim N(0,1)$ and the following equivalences hold:

$$P(R_P < m) \leq \alpha$$
$$\Leftrightarrow \quad P\left(\frac{R_P - \mu_P}{\sigma_P} < \frac{m - \mu_P}{\sigma_P}\right) \leq \alpha$$
$$\Leftrightarrow \quad \Phi\left(\frac{m - \mu_P}{\sigma_P}\right) \leq \alpha$$
$$\Leftrightarrow \quad \frac{m - \mu_P}{\sigma_P} \leq z_\alpha$$
$$\Leftrightarrow \quad \mu_P \geq m - z_\alpha \cdot \sigma_P$$

Here, z_α denotes the α-percentile of the standard normal distribution and Φ is the

cumulative distribution function of the standard normal distribution. $\qquad\qquad\square$

Example 1

As mentioned in the chapter "Introduction", we use the same numerical values for the

underlying variables in all examples. These values are given in the pension fund

example in Leibowitz et al. (1996, p. 86). However, the asset return shortfall constraint

only depends on α and m so that we do not need any other values for this example.

We use $\alpha = 0.10$ and $m = -0.07$. Then $z_\alpha = -1.28$ and the asset return shortfall

constraint is given by

$$\mu_P \geq 1.28\sigma_P - 0.07.$$

In figure 1, p. 16, this asset return shortfall constraint is graphed in the

μ_P-σ_P-coordinate system. This graph concurs with the graph for the asset return

shortfall constraint given in Leibowitz et al. (1996).

The line that specifies the asset return shortfall constraint in figure 1, p. 16, is

referred to as the asset return shortfall line. It is given by the equation $\mu_P = m - z_\alpha\sigma_P$.

The minimum acceptable return m is the y-intercept of the asset return shortfall line.

The "shortfall constraint probability" α is represented by the α-percentile z_α of the

standard normal distribution. The negative value of z_α is the slope of the asset return

shortfall line. All portfolios that are represented by a point in the shaded area satisfy

Figure 1
The Asset Return Shortfall Constraint
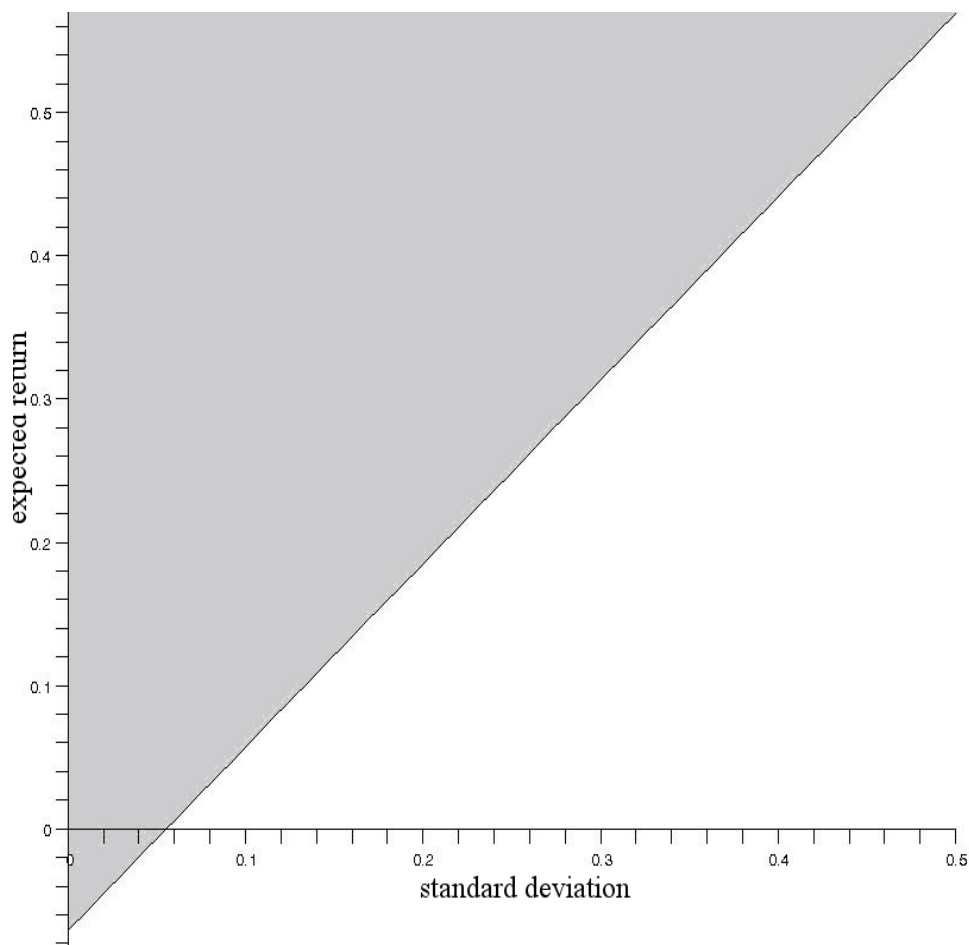
the asset return shortfall constraint. All other portfolios do not fulfill the asset return shortfall constraint.

We can see that this shortfall constraint reduces the number of portfolios from which the fund manager can choose. Before the asset return shortfall constraint was imposed, the fund manager could invest in all the portfolios that lie in the first quadrant of the μ_P-σ_P-coordinate system (we assume that $\mu_P > 0$; $\sigma_P > 0$ is always

true). After the asset return shortfall constraint is imposed, the choice is restricted to the portfolios that are represented by a point in the shaded area.

Sensitivity Analysis for the Asset Return Shortfall Constraint

As we have seen, the number of eligible portfolios is more limited when the asset return shortfall constraint is imposed, but the fund manager has some possibilities to adjust the extent of this limitation. Basically there are two "free" parameters in the asset return shortfall inequality (2.6) which can be changed by the manager: the shortfall constraint probability α and the minimum acceptable return m.

At first we assume that α is fixed. Figure 2, p. 18, shows how the asset return shortfall line reacts to changes in the predetermined minimum return m.

Since m represents the y-intercept of the asset return shortfall line, it is obvious that an increase in m makes the line moving upwards, whereas a decrease in m makes the line moving downwards. In this way, the fund manager can influence the number of portfolios from which he/she can choose. On the one hand, a smaller m increases the number of possible portfolio choices; on the other hand, however, this results in a higher risk since now, R_P in (2.5) is only required to be smaller than the less stringent value m with the same shortfall constraint probability α. Therefore, there is a trade-off between a large number of portfolio choices and a preferably small risk.

Now we assume that m is fixed. Figure 3, p. 19, shows how the asset return shortfall line reacts to changes in the shortfall constraint probability α.

Figure 2
**The Impact of the Minimum Acceptable Return m on the Asset Return
Shortfall line**
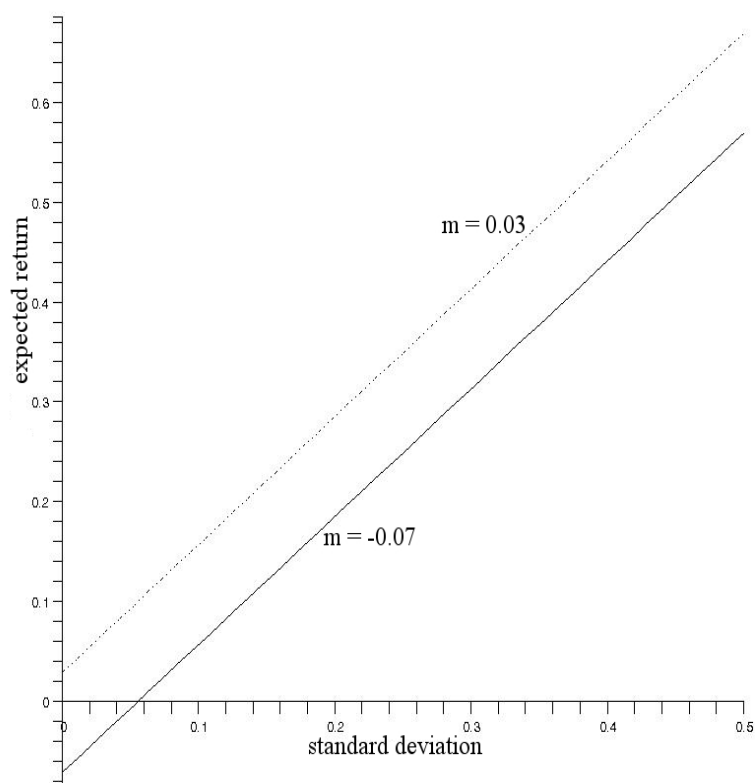
In order to analyze the impact of α on the asset return shortfall line, we assume

that $\alpha < 0.5$. This assumption simplifies the analysis without actually restricting the

choice of α in the shortfall constraint: $P(R_P < m) \leq \alpha$ is only reasonable for small

values of α, for example $\alpha = 5\%$. That is why values of α that are greater than or equal

to 0.5 will not be considered for the rest of the thesis.

If α is increased, the α-percentile z_α becomes larger. But z_α is less than 0 since we

Figure 3
The Impact of the Shortfall Constraint Probability α on the Asset Return Shortfall Line
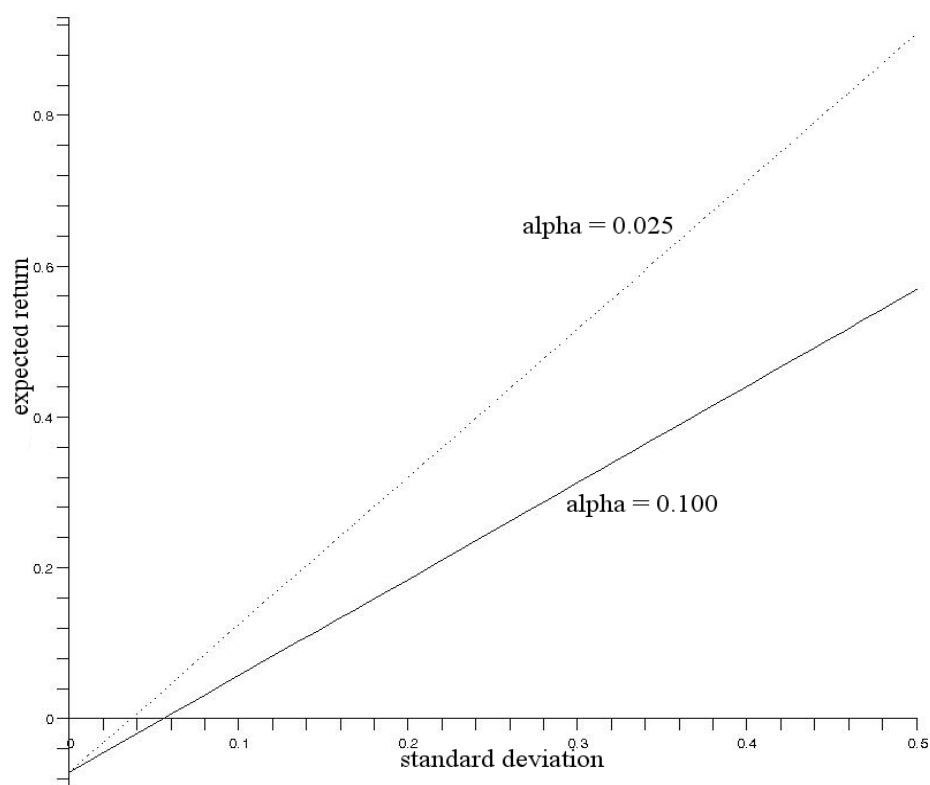
assume $\alpha < 0.5$. From the asset return shortfall constraint inequality (2.6),

$$\mu_P \geq m - z_\alpha \cdot \sigma_P,$$

we can see that with a larger α, the slope of the asset return shortfall line decreases, whereas with a smaller α, the slope increases. Again, the fund manager can influence the number of possible portfolio choices: the smaller the slope is, the larger the choice

of portfolios is. However, this will increase the risk as well since the shortfall constraint probability α has to be increased by the fund manager. We note that figure 2, p. 18, and figure 3, p. 19, concur with the graphs provided by Leibowitz et al. (1996) for this sensitivity analysis.

The Curve of Possible Stock/Bond Combinations

In the next subsection, we analyze the interaction between the asset return shortfall constraint and the curve of possible stock/bond combinations. This curve represents all the portfolios which the pension fund manager can choose from. In particular, we will focus on the corresponding graph in the μ_P-σ_P-coordinate system. That is why we need a formula for the curve of possible stock/bond combinations in terms of μ_P and σ_P. The following result provides this formula:

Result II.2 (The Curve of Possible Stock/Bond Combinations)
We assume that $\mu_E \geq \mu_B$. Then the curve of possible stock/bond combinations (stock/bond-curve) in terms of μ_P and σ_P is given by

$$\mu_P = \pm\sqrt{\frac{1}{a}\sigma_P^2 - \frac{c}{a} + \left(\frac{b}{2a}\right)^2} - \frac{b}{2a}, \quad \mu_P \in [\mu_B, \mu_E], \tag{2.7}$$

where

$$a = \frac{\sigma_E^2 + \sigma_B^2 - 2\sigma_E\sigma_B\rho_{EB}}{(\mu_E - \mu_B)^2}, \tag{2.8}$$

$$b = \frac{2(\mu_E + \mu_B)\sigma_E\sigma_B\rho_{EB} - 2\mu_B\sigma_E^2 - 2\mu_E\sigma_B^2}{(\mu_E - \mu_B)^2}, \tag{2.9}$$

$$c = \frac{\mu_B^2\sigma_E^2 + \mu_E^2\sigma_B^2 - 2\mu_E\mu_B\sigma_E\sigma_B\rho_{EB}}{(\mu_E - \mu_B)^2}. \tag{2.10}$$

Proof. The formula can be derived by straight-forward calculations:

$$\mu_P = w\mu_E + (1-w)\mu_P$$

$$\Rightarrow \quad w = \frac{\mu_P - \mu_B}{\mu_E - \mu_B}, \ 1 - w = \frac{\mu_E - \mu_P}{\mu_E - \mu_B}$$

With (2.3), we get

$$\sigma_P^2 = w^2\sigma_E^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\sigma_E\sigma_B\rho_{EB}$$

$$= \frac{(\mu_P - \mu_B)^2}{(\mu_E - \mu_B)^2}\sigma_E^2 + \frac{(\mu_E - \mu_P)^2}{(\mu_E - \mu_B)^2}\sigma_B^2 + 2\frac{(\mu_P - \mu_B)(\mu_E - \mu_P)\sigma_E\sigma_B\rho_{EB}}{(\mu_E - \mu_B)^2}$$

$$= \overbrace{\frac{\sigma_E^2 + \sigma_B^2 - 2\sigma_E\sigma_B\rho_{EB}}{(\mu_E - \mu_B)^2}}^{=a}\mu_P^2$$

$$+ \overbrace{\frac{2(\mu_E + \mu_B)\sigma_E\sigma_B\rho_{EB} - 2\mu_B\sigma_E^2 - 2\mu_E\sigma_B^2}{(\mu_E - \mu_B)^2}}^{=b}\mu_P$$

$$+ \underbrace{\frac{\mu_B^2\sigma_E^2 + \mu_E^2\sigma_B^2 - 2\mu_E\mu_B\sigma_E\sigma_B\rho_{EB}}{(\mu_E - \mu_B)^2}}_{=c}$$

$$= a\mu_P^2 + b\mu_P + c$$

Thus

$$\mu_P^2 + \frac{b}{a}\mu_P = \frac{\sigma_P^2}{a} - \frac{c}{a}$$

$$\Leftrightarrow \left(\mu_P + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 = \frac{\sigma_P^2}{a} - \frac{c}{a}$$

$$\Leftrightarrow \mu_P = \pm\sqrt{\frac{1}{a}\sigma_P^2 - \frac{c}{a} + \left(\frac{b}{2a}\right)^2} - \frac{b}{2a}$$

It is obvious that $\mu_P \in [\mu_B, \mu_E]$ since

$$\mu_P = w\mu_E + (1-w)\mu_B, \ w \in [0,1] \text{ and } \mu_E \geq \mu_B.$$

$\square$

The Asset Return Shortfall Constraint for w

In chapter IV, we will see that it is useful to have a condition for the asset return shortfall constraint that restricts the weight w. This is provided by the following result.

Result II.3 (The Asset Return Shortfall Constraint for w)

Let

$$a = \mu_E^2 + \mu_B^2 - 2\mu_E\mu_B - z_\alpha^2 \left(\sigma_E^2 + \sigma_B^2 - 2\sigma_E\sigma_B\rho_{EB}\right), \tag{2.11}$$

$$b = 2\left(\mu_E\mu_B + \mu_B m - \mu_B^2 - \mu_E m + z_\alpha^2\sigma_B^2 - z_\alpha^2\sigma_E\sigma_B\rho_{EB}\right), \tag{2.12}$$

$$c = \mu_B^2 + m^2 - 2\mu_B m - z_\alpha^2\sigma_B^2. \tag{2.13}$$

Then the asset return shortfall constraint $P(R_P < m) \leq \alpha$ is equivalent to the following conditions if $\mu_P \geq m$:

- *If $a > 0$, then $w \in ((-\infty, w_1]\bigcap[0,1])\bigcup([w_2, +\infty)\bigcap[0,1])$, where $w_{1/2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ and $w_1 \leq w_2$. If the root $\sqrt{b^2-4ac}$ has no real solution, then $w \in [0,1]$.*

- *If $a < 0$, then $w \in [w_1, w_2]\bigcap[0,1]$, where $w_{1/2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ and $w_1 \leq w_2$. If the root $\sqrt{b^2-4ac}$ has no real solution, then there does not exist a w that satisfies the asset return shortfall constraint.*

- *If $a = 0$ and $b < 0$, then $w \in (-\infty, -\frac{c}{b}]\bigcap[0,1]$.*

- *If $a = 0$ and $b > 0$, then $w \in [-\frac{c}{b}, +\infty)\bigcap[0,1]$.*

- If $a = 0$, $b = 0$ and $c \geq 0$, then $w \in [0,1]$.

- If $a = 0$, $b = 0$ and $c < 0$, then there does not exist a w that satisfies the asset return shortfall constraint.

Proof.

$$P(R_P < m) \leq \alpha$$

$$\overset{Result\,II.1}{\Longleftrightarrow} \quad \mu_P \geq m - z_\alpha \cdot \sigma_P$$

$$\Leftrightarrow \quad w\mu_E + (1-w)\mu_B$$

$$\geq m - z_\alpha \sqrt{w^2\sigma_E^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\sigma_E\sigma_B\rho_{EB}}$$

It makes sense to assume $\mu_P \geq m$ (the minimum acceptable return m shouldn't be greater than the expected value μ_P of the portfolio). Then the last inequality is equivalent to

$$\left(w\mu_E + (1-w)\mu_B - m\right)^2$$

$$\geq \left(-z_\alpha\sqrt{w^2\sigma_E^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\sigma_E\sigma_B\rho_{EB}}\right)^2$$

$$\Leftrightarrow \quad \overbrace{\mu_E^2 + \mu_B^2 - 2\mu_E\mu_B - z_\alpha^2\left(\sigma_E^2 + \sigma_B^2 - 2\sigma_E\sigma_B\rho_{EB}\right)}^{=a}w^2$$

$$+ \overbrace{2\left(\mu_E\mu_B + \mu_B m - \mu_B^2 - \mu_E m + z_\alpha^2\sigma_B^2 - z_\alpha^2\sigma_E\sigma_B\rho_{EB}\right)}^{=b}w$$

$$+ \underbrace{\mu_B^2 + m^2 - 2\mu_B m - z_\alpha^2\sigma_B^2}_{=c} \geq 0$$

$$\Leftrightarrow \quad aw^2 + bw + c \geq 0$$

If $a > 0$, the function $f(w) = aw^2 + bw + c$ is a parabola which is opened to the top.

The nulls of this function are $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $w_1 < w_2$. That is why

$f(w) \geq 0 \Leftrightarrow w \leq w_1$ or $w \geq w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, the

inequality $aw^2 + bw + c \geq 0$ is true for all w. If $a < 0$, the function $f(w) = aw^2 + bw + c$

is a parabola which is opened to the bottom. The nulls of this function are

$w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $w_1 < w_2$. That is why $f(w) \geq 0 \Leftrightarrow w_1 \leq w \leq w_2$. If the root

$\sqrt{b^2 - 4ac}$ has no real solution, the inequality $aw^2 + bw + c \geq 0$ is false for all w. If

$a = 0$ and $b > 0$, this function reduces to a straight line, i.e. $f(w) \geq 0 \Leftrightarrow w \geq -\frac{c}{b}$. If

$a = 0$ and $b < 0$, we get $f(w) \geq 0 \Leftrightarrow w \leq -\frac{c}{b}$. If $a = 0$, $b = 0$ and $c \geq 0$, then $f(w) \geq 0$,

otherwise $f(w) < 0$. This completes the proof. $\qquad\square$

Example 2

For this example, we need the following values of the underlying variables:

Table 1
Values for Example 2

	Expected Return	Standard Deviation of Returns	Correlation with Bonds	Correlation with Stocks
Stocks	13.0%	17.00%	0.35	1.00
Bonds	8.0%	6.96%	1.00	0.35

Source: Leibowitz et al. (1996, p.86)

In order to distinguish between the variables in Result II.3 and Result II.2, we use

the subscripts asc (asset return shortfall constraint) and sbc (stock/bond-curve),

respectively. For α and m, we choose $\alpha = 0.10$ and $m = -0.07$. With the values in the

table and the formulas in the results, we get:

$$a_{asc} = -0.0392, \ b_{asc} = 0.0173, \ c_{asc} = 0.0146.$$

Thus,

$$w_{1/2} = 0.2206 \pm 0.6480$$

which restricts w to ($a_{asc} < 0$ and Result II.3)

$$w \in [0, 0.8687].$$

Since $\mu_P = w\mu_E + (1-w)\mu_B$, μ_P is restricted to

$$\mu_P \in [0.08, 0.1234].$$

Now, we calculate the values for the stock/bond-curve:

$$a_{sbc} = 10.1847, \ b_{sbc} = -1.6577, \ c_{sbc} = 0.0723.$$

Then, the stock/bond-curve is given by

$$\mu_P = \pm\sqrt{0.0982\sigma_P^2 - 0.0005} + 0.0814.$$

Figure 4, p. 26, shows both the asset return shortfall constraint $\mu_P \in [0.08, 0.1234]$ and the stock/bond-curve.

In order to analyze what happens in the graph if we change the minimum acceptable return m and/or the shortfall constraint probability α, we need the following considerations:

Figure 4
The Asset Return Shortfall Constraint for w and the Stock/Bond-Curve
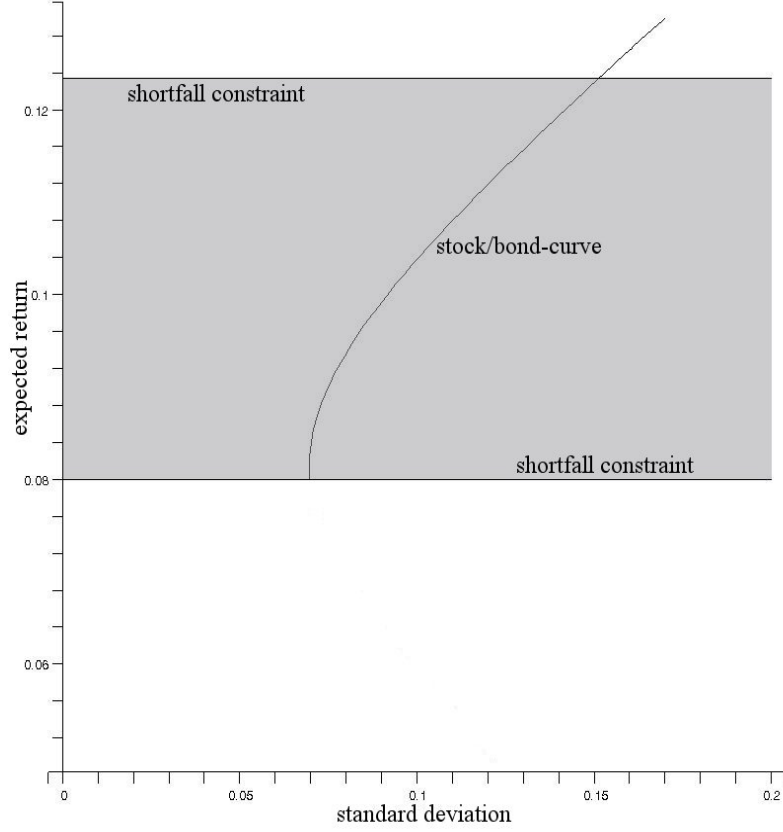
We assume that we have found a portfolio with expected value μ_P^* and standard deviation σ_P^* (we denote this portfolio by (μ_P^*, σ_P^*)) that satisfies the asset return shortfall constraint for a given m^*:

$$\mu_P^* \geq m^* - z_\alpha \cdot \sigma_P^*.$$

Now we decrease the value of m^*. The new value is denoted by m^{new}. Then the

following inequality holds:

$$\mu_P^* \geq m^* - z_\alpha \sigma_P^* > m^{new} - z_\alpha \sigma_P^*.$$

This means that the portfolio (μ_P^*, σ_P^*) satisfies the asset return shortfall constraint

with m^{new}, too.

Now we assume that we have found a portfolio $(\tilde{\mu}_P, \tilde{\sigma}_P)$ that satisfies the asset

return shortfall constraint for a given $\tilde{\alpha}$:

$$\tilde{\mu}_P \geq m - z_{\tilde{\alpha}} \tilde{\sigma}_P.$$

Increasing $\tilde{\alpha}$ yields a new value α^{new}. Since $z_{\tilde{\alpha}} < z_{\alpha^{new}}$, the following inequality holds:

$$\tilde{\mu}_P \geq m - z_{\tilde{\alpha}} \tilde{\sigma}_P > m - z_{\alpha^{new}} \tilde{\sigma}_P.$$

This means that the portfolio $(\tilde{\mu}_P, \tilde{\sigma}_P)$ satisfies the asset return shortfall constraint

with α^{new}, too.

These considerations yield the following result:

Result II.4 (Sensitivity Analysis for the Asset Return)

The set of potential portfolios after a decrease of the minimum acceptable return m

and/or an increase of the asset shortfall constraint probability α is a superset of the set

of potential portfolios before m and/or α is changed.

This result can be generalized for any random return R (which does not

necessarily have to be the asset return). This will be quite useful for the sections

concerning shortfall constraints of the surplus return and the relative return.

Result II.5 (Sensitivity Analysis)

Let R be a random return with expected value μ and standard deviation σ. We assume that

$$\mu \geq m - z_\alpha \sigma$$

is the corresponding shortfall constraint. Then the set of potential portfolios after a decrease of the minimum acceptable return m and/or an increase of the shortfall constraint probability α is a superset of the set of potential portfolios before m and/or α is changed.

Proof. Let (μ^*, σ^*) be a portfolio that satisfies the shortfall constraint for a given m^* and let $m^* > m^{new}$. Then

$$\mu^* \geq m^* - z_\alpha \sigma^* > m^{new} - z_\alpha \sigma^*.$$

This means that the portfolio (μ^*, σ^*) satisfies the shortfall constraint with m^{new}, too.

Let $(\tilde{\mu}, \tilde{\sigma})$ be a portfolio that satisfies the shortfall constraint for a given $\tilde{\alpha}$ and let $\tilde{\alpha} < \alpha^{new}$. Then

$$\tilde{\mu} \geq m - z_{\tilde{\alpha}} \tilde{\sigma} > m - z_{\alpha^{new}} \tilde{\sigma}_P.$$

This means that the portfolio $(\tilde{\mu}, \tilde{\sigma})$ satisfies the shortfall constraint with α^{new}, too.

$\square$

Figure 5, p. 29, shows the impact of a change in m on the graph for the values of

the underlying variables given in Example 2. If m is decreased, the two lines of the

Figure 5
The Impact of m on the Asset Return Shortfall Constraint for w
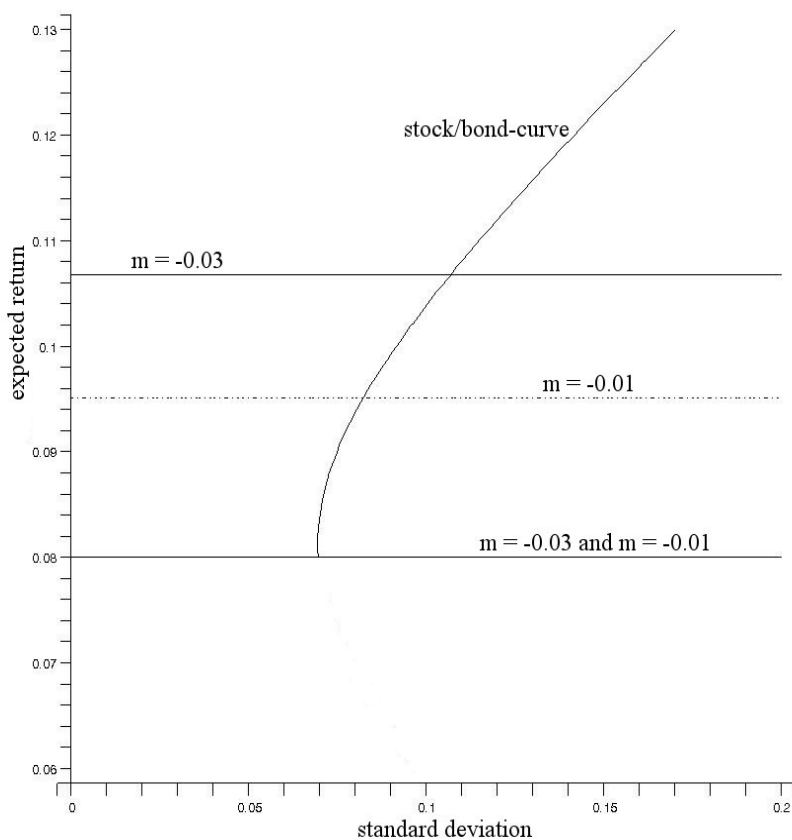
asset return shortfall constraint get closer to each other. This is exactly what Result

II.5 states about changes in m. We would get a very similar figure if we illustrated the

impact of changes in α on the graph.

Before we continue with the next section, it should be noted that we have found

two different representations for the set of potential portfolios that satisfy the asset return shortfall constraint in the μ_P-σ_P-coordinate system: the area above the straight line in figure 1, page 16, and the area between the two horizontal lines in figure 4, page 26. At first glance, one might expect a contradiction in this observation, but this is not the case. Both shortfall constraints produce the same w since the asset return shortfall constraint for w (Result II.3) was derived from the asset return shortfall constraint in Result II.1 by equivalences.

The next figure, figure 6, page 31, shows that if we use the same values for the underlying variables and for α and m (for this example, we chose $\alpha = 0.10$ and $m = -0.07$ and the values in Table 1), both versions of the asset return shortfall constraint restrict μ_P in the same way. Here, version 1 and version 2 refer to Result II.3 and Result II.1, respectively. Since $w \in [0, 1]$, the expected value μ_P must be greater than or equal to 0.08 for both shortfall constraints (because $\mu_P = w\mu_E + (1 - w)\mu_B$, $\mu_E = 0.13$, and $\mu_B = 0.08$). The upper bound is determined by the intersection of the asset return shortfall constraint lines and the stock/bond-curve, but both versions intersect with the stock/bond-curve in the same point.

Figure 6
The Two Versions of the Asset Return Shortfall Constraint and the Stock/Bond-Curve
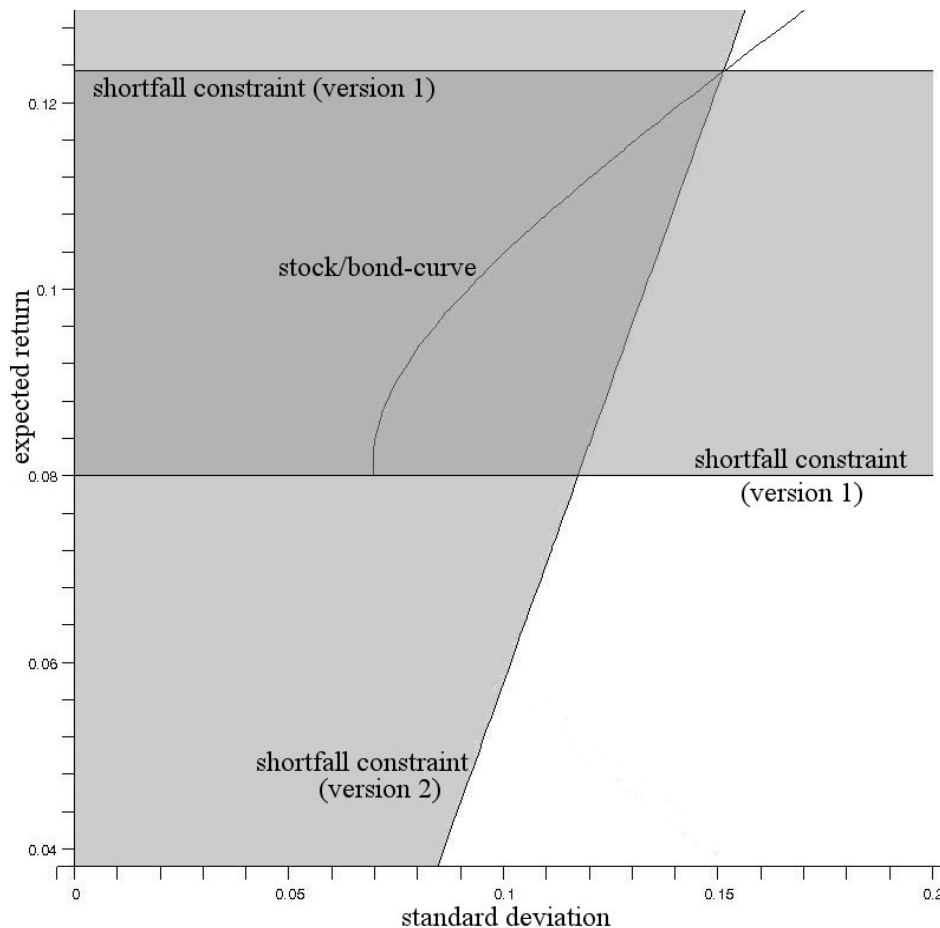
The Liability Model

Up to now, we have only considered managing a pension fund in an asset-only framework. However, a pension fund consists of both assets and liabilities. That is why it is necessary to pay attention to the liability position of the fund as well.

A quite common measure for the liabilities of a pension fund is the accumulated benefit obligation (ABO). The Financial Accounting Standards Board (FASB, 1985)

defines the ABO as of a date as the actuarial present value of benefits attributed by a pension benefit formula to employee service rendered prior to that date and based on current and past compensation levels. With the ABO it is possible to calculate the surplus or deficit of a pension plan:

$$surplus = current\ value\ of\ plan\ assets - ABO$$

if this difference is non-negative or

$$deficit = current\ value\ of\ plan\ assets - ABO$$

if the difference is negative.

Although it can be considered as an approximation of the termination liability of a pension plan, the ABO has one major disadvantage. It does not reflect future salary increases. To solve this problem, the projected benefit obligation (PBO) is introduced. The PBO is defined as the actuarial present value of benefits attributed by a pension benefit formula to employee service rendered prior to that date and based on current, past and future compensation levels (cf. FASB, 1985). Basically the PBO is an ABO, but it includes the additional uncertainty of future pay increases. That is why many pension investors prefer it to the ABO since the approximation of the termination liability tends to be better. When we establish a model for the liabilities in the next subsection, we will see that it consists of two parts where the first one can be regarded as the ABO part. The other part represents the additional PBO features.

It should be noted that there are other liability measures that comprise even more

uncertain events in the future. The "total benefit obligation", for example, includes the future service of current participants of the pension plan. These measures try to make the liability model more realistic, but it would be very difficult to manage the assets against the liabilities since the uncertainty inherent in the liabilities would be too large (cf. Leibowitz et al., 1996).

The Liability Model

In order to include liabilities for asset-liability management, we need a model for the liability return in addition to the model for the asset return that we have already introduced. For the liability model suggested by Leibowitz et al. (1996) we need some notations:

- R_L: return of the liabilities,

- R_B: return of the bond portion in the portfolio,

- R_E: return of the stock portion in the portfolio, and

- ε: error term.

As for the asset return shortfall constraint, we assume that all the underlying random variables are normally distributed. Because R_L will be a linear combination of R_B and R_E, we only need distribution assumptions for R_B, R_E and ε (as we have noted in the section "The Normal Distribution Assumption", the linear combination of normally distributed random variables is normally distributed again):

- $R_B \sim N(\mu_B, \sigma_B^2)$, where μ_B and σ_B^2 are the expected return and the variance of R_B, respectively.

- $R_E \sim N(\mu_E, \sigma_E^2)$, where μ_E and σ_E^2 are the expected return and the variance of R_E, respectively.

- $\varepsilon \sim N(0, \sigma_\varepsilon^2)$ and ε is uncorrelated with either stock or bond returns, i.e. $\rho_{B\varepsilon} = 0$ and $\rho_{E\varepsilon} = 0$.

Then the liability model can be formulated as follows:

$$R_L - \mu_L = a \cdot (R_B - \mu_B) + b \cdot (R_E - \mu_E) + \varepsilon, \tag{2.14}$$

where $a \in \mathbb{R}$, $b \in \mathbb{R}$ and μ_L is the expected liability return.

Since R_L is a linear combination of normally distributed random variables, it is also normally distributed with expected value μ_L and variance σ_L^2:

$$R_L \sim N(\mu_L, \sigma_L^2). \tag{2.15}$$

It is obvious that μ_L must be the expected return of the liabilities since

$$R_L - \mu_L = a \cdot (R_B - \mu_B) + b \cdot (R_E - \mu_E) + \varepsilon$$

$$\Rightarrow \quad E(R_L - \mu_L) = E(a \cdot (R_B - \mu_B) + b \cdot (R_E - \mu_E) + \varepsilon)$$

$$\Rightarrow \quad E(R_L) - \mu_L = a \cdot \underbrace{(E(R_B) - \mu_B)}_{=0} + b \cdot \underbrace{(E(R_E) - \mu_B)}_{=0} + \underbrace{E(\varepsilon)}_{=0}$$

$$\Rightarrow \quad E(R_L) - \mu_L = 0$$

$$\Rightarrow \quad E(R_L) = \mu_L.$$

Here, $E(\cdot)$ is used to denote the expected return of a random variable. The main problem is to find a formula for the variance σ_L^2 of the liability return, i.e. to find an expression of σ_L^2 in terms of known quantities. Leibowitz et al. (1996, pp. 82) give a detailed derivation which is summarized in the following:

At first, the equation (2.14) is split in two pieces:

$$R_L - \mu_L = \underbrace{a \cdot (R_B - \mu_B)}_{=:R_I} + \underbrace{b \cdot (R_E - \mu_E) + \varepsilon}_{=:R_N},$$

where

- R_I is "the portion of the liability return due solely to the change in interest (discount) rates, with the benefit schedule held constant" (Leibowitz et al., 1996, p. 82). R_I has an expected return of

$$\mu_I = a \cdot E(R_B - \mu_B) = 0 \tag{2.16}$$

  and a variance of

$$\sigma_I^2 = a^2 \cdot E\left((R_B - \mu_B)^2\right) = a^2 \cdot \sigma_B^2. \tag{2.17}$$

- R_N is "the portion of the liability return due to 'noise' - that is, any change in the benefit payment schedule itself" (Leibowitz et al., 1996, p. 83). R_N has an expected return of

$$\mu_N = b \cdot E(R_E - \mu_E) + E(\varepsilon) = 0 \tag{2.18}$$

and the covariance between R_N and R_E is

$$
\begin{aligned}
\sigma_{EN} &= E(R_N(R_E - \mu_E)) - E(R_N)E(R_E - \mu_E) \\
&= E(R_N(R_E - \mu_E)) \\
&= E\left(b(R_E - \mu_B)^2\right) + E(\varepsilon(R_E - \mu_E)) \\
&= b\sigma_E^2
\end{aligned}
$$

since ε is uncorrelated with R_E. Thus,

$$
\sigma_{EN} = b\sigma_E^2. \tag{2.19}
$$

From (2.17) and (2.19), we get

$$
a = \frac{\sigma_I}{\sigma_B}, \tag{2.20}
$$

$$
b = \frac{\sigma_{EN}}{\sigma_E^2} = \frac{\sigma_E \sigma_N \rho_{EN}}{\sigma_E^2} = \frac{\sigma_N}{\sigma_E} \cdot \rho_{EN}. \tag{2.21}
$$

Finally, we need a formula for σ_{IN}:

$$
\begin{aligned}
\sigma_{IN} &= E(R_I R_N) - E(R_I)E(R_N) = E(R_I R_N) \\
&= abE((R_B - \mu_B)(R_E - \mu_E)) + a\underbrace{E((R_B - \mu_B)\varepsilon)}_{=0} \\
&= ab\sigma_{EB},
\end{aligned}
$$

or, after plugging in a (2.20) and b (2.21),

$$
\sigma_{IN} = \sigma_I \sigma_N \rho_{EN} \rho_{EB}. \tag{2.22}
$$

Now

$$R_L - \mu_L = \underbrace{a \cdot (R_B - \mu_B)}_{=R_I} + \underbrace{b \cdot (R_E - \mu_E) + \varepsilon}_{=R_N}$$

$$\Rightarrow \quad R_L - \mu_L = R_I + R_N$$

$$\Rightarrow \quad E((R_L - \mu_L)^2) = E((R_I + R_N)^2)$$

$$\Rightarrow \quad \sigma_L^2 = E(R_I^2) + E(R_N^2) + 2E(R_I R_N) \overset{(2.18)}{=} \sigma_I^2 + \sigma_N^2 + 2\sigma_{IN}$$

Using (2.22), we get

$$\sigma_L^2 = \sigma_I^2 + \sigma_N^2 + 2\sigma_I \sigma_N \rho_{EN} \rho_{EB}. \tag{2.23}$$

The following result gives a summary of the important parts of the preceding derivation:

Result II.6 (The Liability Model)

If $R_B \sim N(\mu_B, \sigma_B^2)$, $R_E \sim N(\mu_E, \sigma_E^2)$, $\varepsilon \sim N(0, \sigma_\varepsilon^2)$, and ε is uncorrelated with R_B and R_E, then for the liability model

$$R_L - \mu_L = \underbrace{a \cdot (R_B - \mu_B)}_{=:R_I} + \underbrace{b \cdot (R_E - \mu_E) + \varepsilon}_{=:R_N} \tag{2.24}$$

the following holds:

1. *$R_L \sim N(\mu_L, \sigma_L^2)$ (2.15),*

2. *$\sigma_L^2 = \sigma_I^2 + \sigma_N^2 + 2\sigma_I \sigma_N \rho_{EN} \rho_{EB}$ (2.23),*

3. *$a = \frac{\sigma_I}{\sigma_B}$ (2.20), and*

4. *$b = \frac{\sigma_N}{\sigma_E} \cdot \rho_{EN}$. (2.21)*

Before we started to derive formulas for the liability distribution and the liability model, we noted that the PBO is basically an ABO, but with some additional features. Is this represented by our liability model? The answer is yes. Actually the liability model describes the liability return of a PBO:

According to Leibowitz et al. (1996, p. 41), "the future events reflected in the ABO are primarily demographic (mortality, age at retirement), rather than economic (salary increases)". That is why the bond portion part R_I in (2.24) represents the uncertainty inherent in the ABO.

The stock portion part together with the error term (the "noise" R_N in the liability return) can be considered as the uncertainty in the liability schedule.

The following summarizes this interpretation of the liability model:

$$R_L - \mu_L = \underbrace{a \cdot (R_B - \mu_B)}_{=R_I} + \underbrace{b \cdot (R_E - \mu_E) + \varepsilon}_{=R_N}$$

$$\Leftrightarrow$$

$$R_L - \mu_L = \underbrace{ R_I }_{\text{uncertainty in the ABO}} + \underbrace{ R_N }_{\text{uncertainty in the liability schedule}}$$

The noise factor N of a liability is defined as the portion of the liability volatility unexplained by interest rates. For a PBO, this is the percentage of unexplained liability volatility inherent in future salary increases. With results from Ezra (1991), Leibowitz et al. (1996, p. 68) calculate a noise factor of 7% for the PBO. Of course an ABO has a noise factor of 0% in the liability model since there is no uncertainty in the liability schedule.

The Surplus Return Shortfall Constraint

In the preceding section "The Liability Model", we have defined the surplus as the difference between the current value of the plan assets and the ABO, where the ABO can be substituted by any measure for the liability of a pension plan. Since we have introduced a liability model for the PBO, the PBO will replace the ABO in the definition:

$$surplus = current\ value\ of\ plan\ assets - PBO.$$

Again, a negative surplus means a deficit in the pension plan.

At the beginning of this chapter, we have derived a formula for the asset return shortfall constraint. Although asset-only objectives are sometimes justified, it is often necessary to consider both the risk of the assets and the risk of the liabilities. Only if a fund manager accounts for both risks, can he/she control the overall risk of the pension fund.

This suggests using the surplus return to control the risk. In order to define it, we need some notations:

- A_i: value of the assets at time $i \in \{0, 1\}$,

- L_i: value of the liabilities at time $i \in \{0, 1\}$,

- S_i: surplus at time $i \in \{0, 1\}$,

- $F_0 = \frac{A_0}{L_0}$: funding ratio at time 0,

- R_A: return of the assets with expected value μ_A, and variance σ_A^2, and

- R_L: return of the liabilities with expected value μ_L and variance σ_L^2.

With these notations we can define the surplus return:

Definition 2 (The Surplus Return)

The surplus return R_S is given by the equation

$$R_S = \frac{S_1 - S_0}{L_0}. \tag{2.25}$$

At first glance it might be unreasonable to use L_0 instead of S_0 in the denominator. However, if we used S_0, it would be possible that the denominator be 0 since S_0 could assume a value of 0.

Usually L_0 is preferred to A_0 because one wants to compare the surplus with the liabilities rather than with the assets.

For the next result, we do not need any distribution assumptions for the asset return and liability return.

Result II.7 (Properties of the Surplus Return)

The following properties hold for the surplus return R_S:

$$R_S = F_0 \cdot R_A - R_L, \tag{2.26}$$

$$\mu_S = F_0 \cdot \mu_A - \mu_L, \tag{2.27}$$

$$\sigma_S^2 = F_0^2 \sigma_A^2 + \sigma_L^2 - 2F_0 \sigma_A \sigma_L \rho_{AL}, \tag{2.28}$$

where $R_A = wR_E + (1-w)R_B$, $w \in [0,1]$, μ_S and σ_S are the expected return and the

standard deviation of R_S, respectively. The correlation ρ_{AL} is given by

$$\rho_{AL} = \frac{w\sigma_E\rho_{EL} + (1-w)\sigma_B\rho_{BL}}{\sigma_A} \quad , \; where \qquad (2.29)$$

$$\rho_{EL} = \frac{\sigma_I}{\sigma_L}\rho_{EB} + \frac{\sigma_N}{\sigma_L}\rho_{EN} \;\; and \qquad (2.30)$$

$$\rho_{BL} = \frac{\sigma_I}{\sigma_L} + \frac{\sigma_N}{\sigma_L}\rho_{EN}\rho_{EB}. \qquad (2.31)$$

Proof. The derivations for these formulas are given in Leibowitz et al. (1996, pp. 80) and are summarized in the following:

$$\begin{aligned} R_S \;\; &= \frac{S_1 - S_0}{L_0} = \frac{(A_1 - L_1) - (A_0 - L_0)}{L_0} \\ &= \frac{(1+R_A)A_0 - (1+R_L)L_0 - A_0 + L_0}{L_0} = \frac{A_0 R_A - L_0 R_L}{L_0} \\ &= F_0 R_A - R_L \end{aligned}$$

$$\Rightarrow \;\; \mu_S \;\; = E(F_0 R_A - R_L) = F_0\mu_A - \mu_L$$

$$\sigma_S \;\; = Var(F_0 R_A - R_L) = F_0^2\sigma_A^2 + \sigma_L^2 - 2F_0\sigma_A\sigma_L\rho_{AL}$$

The formulas for ρ_{AL}, ρ_{EL} and ρ_{BL} can be derived as follows:

$$\begin{aligned} \sigma_{AL} \;\; &= Cov(R_A, R_L) = Cov(wR_E + (1-w)R_B, R_L) \\ &= wCov(R_E, R_L) + (1-w)Cov(R_B, R_L) = w\sigma_{EL} + (1-w)\sigma_{BL} \\ \Rightarrow \;\; \rho_{AL} \;\; &= \frac{\sigma_{AL}}{\sigma_A\sigma_L} = \frac{w\sigma_E\sigma_L\rho_{EL} + (1-w)\sigma_B\sigma_L\rho_{BL}}{\sigma_A\sigma_L} \\ &= \frac{w\sigma_E\rho_{EL} + (1-w)\sigma_B\rho_{BL}}{\sigma_A} \end{aligned}$$

$$\sigma_{BL} = E((R_L - \mu_L)(R_B - \mu_B))$$

$$= aE((R_B - \mu_B)^2) + bE((R_E - \mu_E)(R_B - \mu_B)) + E(\varepsilon(R_B - \mu_B))$$

$$\Rightarrow \quad \sigma_{BL} = \frac{\sigma_I}{\sigma_B}\sigma_B^2 + \frac{\sigma_N}{\sigma_E}\rho_{EN}\sigma_{EB} + 0$$

$$\Rightarrow \quad \rho_{BL}\sigma_L\sigma_B = \sigma_I\sigma_B + \frac{\sigma_N}{\sigma_E}\rho_{EN}\rho_{EB}\sigma_B\sigma_E$$

$$\Rightarrow \quad \rho_{BL} = \frac{\sigma_I}{\sigma_L} + \frac{\sigma_N}{\sigma_L}\rho_{EN}\rho_{EB}$$

$$\sigma_{EL} = E((R_L - \mu_L)(R_E - \mu_E))$$

$$= aE((R_B - \mu_B)(R_E - \mu_E)) + bE((R_E - \mu_E)^2) + E(\varepsilon(R_E - \mu_E))$$

$$\Rightarrow \quad \sigma_{EL} = \frac{\sigma_I}{\sigma_B}\sigma_{EB} + \frac{\sigma_N}{\sigma_E}\rho_{EN}\sigma_E^2 + 0$$

$$\Rightarrow \quad \rho_{EL}\sigma_L\sigma_E = \sigma_I\sigma_E\rho_{EB} + \sigma_N\sigma_E\rho_{EN}$$

$$\Rightarrow \quad \rho_{EL} = \frac{\sigma_I}{\sigma_L}\rho_{EB} + \frac{\sigma_N}{\sigma_L}\rho_{EN}$$

$\square$

Since the surplus return combines the asset return and the liability return in one quantity, it qualifies for our goal to manage the assets and liabilities of a pension fund simultaneously.

The surplus return shortfall constraint is quite similar to the asset return shortfall constraint. It allows for controlling the downside risk of the surplus return which again is measured by the corresponding shortfall probability. A fund manager, for example, may have the objective to meet a minimum acceptable surplus return of 3% with a probability of 95%. In general, this constraint can be formulated as follows (cf. Leibowitz et al., 1996, p. 44):

There should be no more than a probability of α that the surplus return will

be less than a minimum acceptable surplus return of m.

Thus, the mathematical expression for the surplus return shortfall constraint is

$$P(R_S < m) \leq \alpha. \tag{2.32}$$

Now we can derive the formula for the shortfall constraint. The next result gives

formulas in terms of μ_S and σ_S and in terms of μ_A and σ_A. The second version allows

for graphing the surplus return shortfall constraint in a μ_A-σ_A-coordinate system. This

is a big advantage since we can graph the shortfall constraint against the

stock/bond-curve in the μ_A-σ_A-coordinate system.

Result II.8 (The Surplus Return Shortfall Constraint)

Let $R_E \sim N(\mu_E, \sigma_E^2)$, $R_B \sim N(\mu_B, \sigma_B^2)$, and $R_L \sim N(\mu_L, \sigma_L^2)$. Then

$$R_S = F_0 R_A - R_L \sim N(\mu_S, \sigma_S^2),$$

and the surplus return shortfall constraint is given by

$$\mu_S \geq m - z_\alpha \cdot \sigma_S. \tag{2.33}$$

An equivalent expression for this constraint is given by the two inequalities

$$\mu_A \geq \frac{\mu_L + m}{F_0}, \tag{2.34}$$

$$a\sigma_A^2 + b\mu_A + c\mu_A^2 \leq d, \tag{2.35}$$

where

$$a = z_\alpha^2 F_0^2, \tag{2.36}$$

$$b = \frac{2F_0 \sigma_L \sigma_B \rho_{BL} z_\alpha^2}{\mu_E - \mu_B} - \frac{2F_0 \sigma_L \sigma_E \rho_{EL} z_\alpha^2}{\mu_E - \mu_B} + 2F_0 \mu_L + 2F_0 m, \tag{2.37}$$

$$c = -F_0^2, \tag{2.38}$$

$$d = \mu_L^2 + m^2 + 2\mu_L m - z_\alpha^2 \sigma_L^2 - \frac{2F_0 z_\alpha^2 \sigma_L \sigma_E \rho_{EL} \mu_B}{\mu_E - \mu_B} + \frac{2F_0 \sigma_L \sigma_B \rho_{BL} z_\alpha^2 \mu_E}{\mu_E - \mu_B}. \tag{2.39}$$

The formulas for ρ_{EL} and ρ_{BL} are given by (2.30) and (2.31).

Proof. Since $R_E \sim N(\mu_E, \sigma_E^2)$ and $R_B \sim N(\mu_B, \sigma_B^2)$, the asset return

$R_A = wR_E + (1 - w)R_B$ is also normally distributed: $R_A \sim N(\mu_A, \sigma_A^2)$. Thus,

$R_S = F_0 R_A - R_L \sim N(\mu_S, \sigma_s^2)$ because it is a linear combination of normally distributed

random variables.

Derivation of the shortfall constraint:

$$P(R_S < m) \leq \alpha$$

$$\Leftrightarrow \quad P\left(\frac{R_S - \mu_P}{\sigma_P} < \frac{m - \mu_S}{\sigma_S}\right) \leq \alpha$$

$$\Leftrightarrow \quad \Phi\left(\frac{m - \mu_S}{\sigma_S}\right) \leq \alpha$$

$$\Leftrightarrow \quad \frac{m - \mu_S}{\sigma_S} \leq z_\alpha$$

$$\Leftrightarrow \quad \mu_S \geq m - z_\alpha \cdot \sigma_S$$

Here, z_α denotes the α-percentile of the standard normal distribution and Φ is the

cumulative distribution function of the standard normal distribution.

Now we have to express this constraint in terms of μ_A and σ_A, i.e. we have to replace μ_S and σ_S in (2.33) by (2.27), and (2.28), respectively:

$$\mu_S \geq m - z_\alpha \sigma_S$$

$$\Leftrightarrow \quad F_0 \mu_A - \mu_L \geq m - z_\alpha \sqrt{F_0^2 \sigma_A^2 + \sigma_L^2 - 2F_0 \sigma_A \sigma_L \rho_{AL}}$$

$$\Leftrightarrow \quad \sqrt{F_0^2 \sigma_A^2 + \sigma_L^2 - 2F_0 \sigma_A \sigma_L \rho_{AL}} \leq \frac{\mu_L + m - F_0 \mu_A}{z_\alpha} \quad (-z_\alpha > 0 \text{ for } \alpha < 0.5)$$

$$\Leftrightarrow \quad \frac{\mu_L + m - F_0 \mu_A}{z_\alpha} \geq 0 \text{ and}$$

$$\left(\sqrt{F_0^2 \sigma_A^2 + \sigma_L^2 - 2F_0 \sigma_A \sigma_L \rho_{AL}} \right)^2 \leq \left(\frac{\mu_L + m - F_0 \mu_A}{z_\alpha} \right)^2$$

$$\stackrel{z_\alpha < 0}{\Longleftrightarrow} \quad \mu_A \geq \frac{\mu_L + m}{F_0} \text{ and}$$

$$\left(\sqrt{F_0^2 \sigma_A^2 + \sigma_L^2 - 2F_0 \sigma_A \sigma_L \rho_{AL}} \right)^2 \leq \left(\frac{\mu_L + m - F_0 \mu_A}{z_\alpha} \right)^2$$

$$\Leftrightarrow \quad \mu_A \geq \frac{\mu_L + m}{F_0} \text{ and}$$

$$\underbrace{z_\alpha^2 (F_0^2 \sigma_A^2 + \sigma_L^2 - 2F_0 \sigma_A \sigma_L \rho_{AL}) \leq (\mu_L + m - F_0 \mu_A)^2}_{(*)}$$

It holds:

$$\mu_A = w\mu_E + (1-w)\mu_B = w(\mu_E - \mu_B) + \mu_B$$

$$\Rightarrow \quad w = \frac{\mu_A - \mu_B}{\mu_E - \mu_B}, \quad (1-w) = \frac{\mu_E - \mu_A}{\mu_E - \mu_B}$$

Now we find an expression for ρ_{AL} with known variables. According to 2.29, we have

$$\rho_{AL} = \frac{w\sigma_E \rho_{El} + (1-w)\sigma_B \rho_{BL}}{\sigma_A}$$

$$\Rightarrow \quad \rho_{AL} = \left(\frac{\mu_A - \mu_B}{\mu_E - \mu_B} \sigma_E \rho_{EL} + \frac{\mu_E - \mu_A}{\mu_E - \mu_B} \sigma_B \rho_{BL} \right) \cdot \frac{1}{\sigma_A}$$

After plugging this result in $(*)$, we get

$$z_\alpha^2 \left[F_0^2 \sigma_A^2 + \sigma_L^2 - 2F_0\sigma_L \left(\frac{\mu_A - \mu_B}{\mu_E - \mu_B} \sigma_E \rho_{EL} + \frac{\mu_E - \mu_A}{\mu_E - \mu_B} \sigma_B \rho_{BL} \right) \right]$$

$$\leq (\mu_L + m - F_0\mu_A)^2$$

$$\Leftrightarrow \quad \overbrace{z_\alpha^2 F_0^2}^{=a} \sigma_A^2 + \mu_A \overbrace{\left(\frac{2F_0\sigma_L\sigma_B\rho_{BL}z_\alpha^2}{\mu_E - \mu_B} - \frac{2F_0\sigma_L\sigma_E\rho_{EL}z_\alpha^2}{\mu_E - \mu_B} + 2F_0(\mu_L + m) \right)}^{=b} \overbrace{-F_0^2}^{=c} \mu_A^2$$

$$\leq \underbrace{\mu_L^2 + m^2 + 2\mu_L m - z_\alpha^2\sigma_L^2 - \frac{2F_0 z_\alpha^2 \sigma_L\sigma_E\rho_{EL}\mu_B}{\mu_E - \mu_B} + \frac{2F_0\sigma_L\sigma_B\rho_{BL}z_\alpha^2\mu_E}{\mu_E - \mu_B}}_{=d}$$

$$\Leftrightarrow \quad a\sigma_A^2 + b\mu_A + c\mu_A^2 \leq d$$

$$\square$$

In order to graph the surplus return shortfall constraint in a μ_A-σ_A-coordinate system, we need to express the expected return μ_A as a function of σ_A:

$$a\sigma_A^2 + b\mu_A + c\mu_A^2 \leq d$$

$$\overset{c \leq 0}{\Leftrightarrow} \quad \left(\mu_A + \frac{b}{2c} \right)^2 - \frac{b^2}{4c^2} \geq \frac{d}{c} - \frac{a}{c}\sigma_A^2$$

$$\Leftrightarrow \quad \mu_A \geq +\sqrt{\frac{d}{c} + \frac{b^2}{4c^2} - \frac{a}{c}\sigma_A^2} - \frac{b}{2c} \quad \text{or}$$

$$\mu_A \leq -\sqrt{\frac{d}{c} + \frac{b^2}{4c^2} - \frac{a}{c}\sigma_A^2} - \frac{b}{2c}$$

Thus, the surplus return shortfall constraint can be written as

$$\mu_A(\sigma_A) \geq \sqrt{\frac{d}{c} + \frac{b^2}{4c^2} - \frac{a}{c}\sigma_A^2} - \frac{b}{2c} \quad \text{or} \tag{2.40}$$

$$\mu_A(\sigma_A) \leq -\sqrt{\frac{d}{c} + \frac{b^2}{4c^2} - \frac{a}{c}\sigma_A^2} - \frac{b}{2c} \tag{2.41}$$

and

$$\mu_A(\sigma_A) \geq \frac{\mu_L + m}{F_0}. \tag{2.34}$$

Example 3

For this example, we need the following values of the underlying variables:

Table 2
Values for Example 3

	Expected Return	Standard Deviation of Returns	Correlation with Bonds	Correlation with Stocks
Assets:				
Stocks	13.0%	17.00%	0.35	1.00
Bonds	8.0%	6.96%	1.00	0.35
Liabilities:				
Basic Schedule	8.0%	15.00%	1.00	
Noise	0.00%	7.00%		0.25

Source: Leibowitz et al. (1996, p.86)

Then, σ_L, ρ_{BL}, and ρ_{EL} can be calculated by using (2.23), (2.31), and (2.30):

$$\sigma_L = 0.171, \ \rho_{BL} = 0.913, \ \rho_{EL} = 0.409. \ (cf. \ Leibowitz \ et \ al., \ 1996, \ p.86)$$

For α, F_0, and m, we choose $\alpha = 0.10$, $F_0 = 1$, and $m = -0.07$. With these values, we get

$$a = 1.6384, \ b = -0.0471, \ c = -1, \ d = -0.0176$$

and

$$\frac{\mu_L + m}{F_0} = 0.06.$$

Thus, the surplus return shortfall constraint for the μ_A-σ_A-coordinate system can be written as

$$\mu_A \geq \sqrt{0.0182 + 1.6384\sigma_A^2} - 0.0236 \ or$$

$$\mu_A \leq -\sqrt{0.0182 + 1.6384\sigma_A^2} - 0.0236$$

$$and$$

$$\mu_A \geq 0.06.$$

This is illustrated in figure 7, p. 48.

<div align="center">

Figure 7
The Surplus Return Shortfall Constraint

</div>
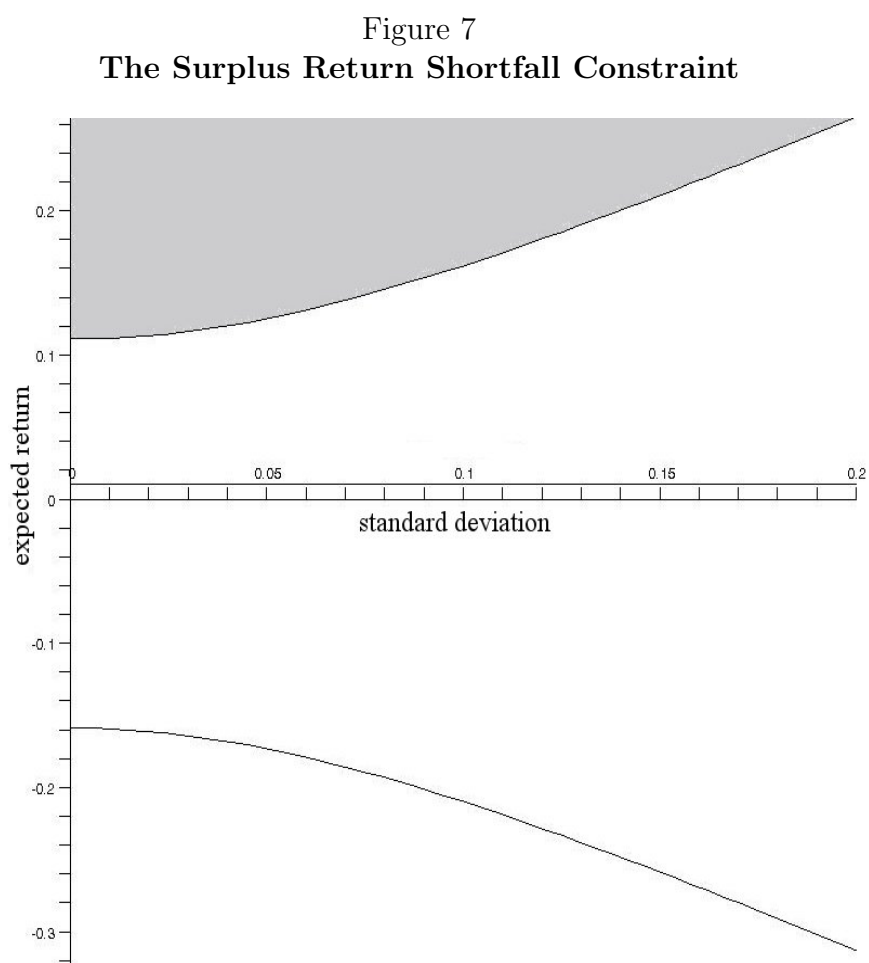
However, $\sigma_A^2 \geq 0$, and that is why we get from the first inequality of this shortfall

constraint

$$\mu_A \geq \sqrt{0.0182 + 1.6384\sigma_A^2} - 0.0236 \geq \sqrt{0.0182 + 1.6384 \cdot 0} - 0.0236 = 0.1113.$$

Therefore, the shortfall constraint reduces to

$$\mu_A \geq \sqrt{0.0182 + 1.6384\sigma_A^2} - 0.0236$$

which is illustrated in figure 8, p. 49.

Figure 8
The Surplus Return Shortfall Constraint without Redundant Equations
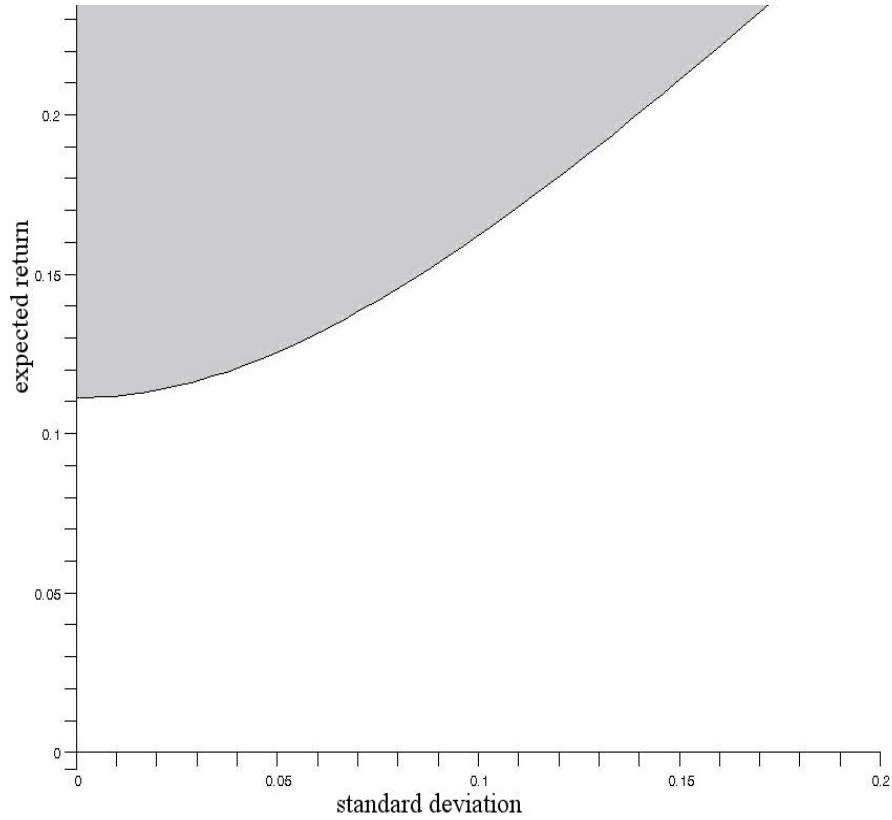
Sensitivity Analysis for the Surplus Return Shortfall Constraint

The sensitivity analysis for the surplus return shortfall constraint is a little bit more complicated than the sensitivity analysis for the asset return shortfall constraint. The reason is that we want to see how the graph in the μ_A-σ_A-coordinate system (and not in the μ_S-σ_S-coordinate system) reacts to changes in the minimum acceptable return m and the shortfall constraint probability α.

The pension fund manager can adjust the surplus return shortfall constraint by increasing or decreasing the minimum acceptable return m or the shortfall constraint probability α. Thus, the amount of risk can be varied and fitted to a desirable risk level. As noted above, the surplus return shortfall constraint qualifies better for asset-liability management since it comprises both the asset return and the liability return. Since the surplus return shortfall constraint

$$\mu_A(\sigma_A) \geq \sqrt{\frac{d}{c} + \frac{b^2}{4c^2} - \frac{a}{c}\sigma_A^2} - \frac{b}{2c} \text{ or}$$
$$\mu_A(\sigma_A) \leq -\sqrt{\frac{d}{c} + \frac{b^2}{4c^2} - \frac{a}{c}\sigma_A^2} - \frac{b}{2c}$$

and

$$\mu_A(\sigma_A) \geq \frac{\mu_L + m}{F_0}$$

has too many variables that can vary (remark: a, b, c, and d are given by (2.36), (2.37), (2.38), and (2.39)), it is impossible to determine directly the impact of a change in m or α on the graph. However, we can apply Result II.5 to the surplus return R_S.

The effects on the graph that Result II.5 implies are illustrated in figure 9, p. 51

and figure 10, page 52. Since we are only interested in portfolios in the first quadrant,

only the curves of the shortfall constraint in this quadrant are displayed.

Figure 9
The Impact of m on the Surplus Return Shortfall Constraint
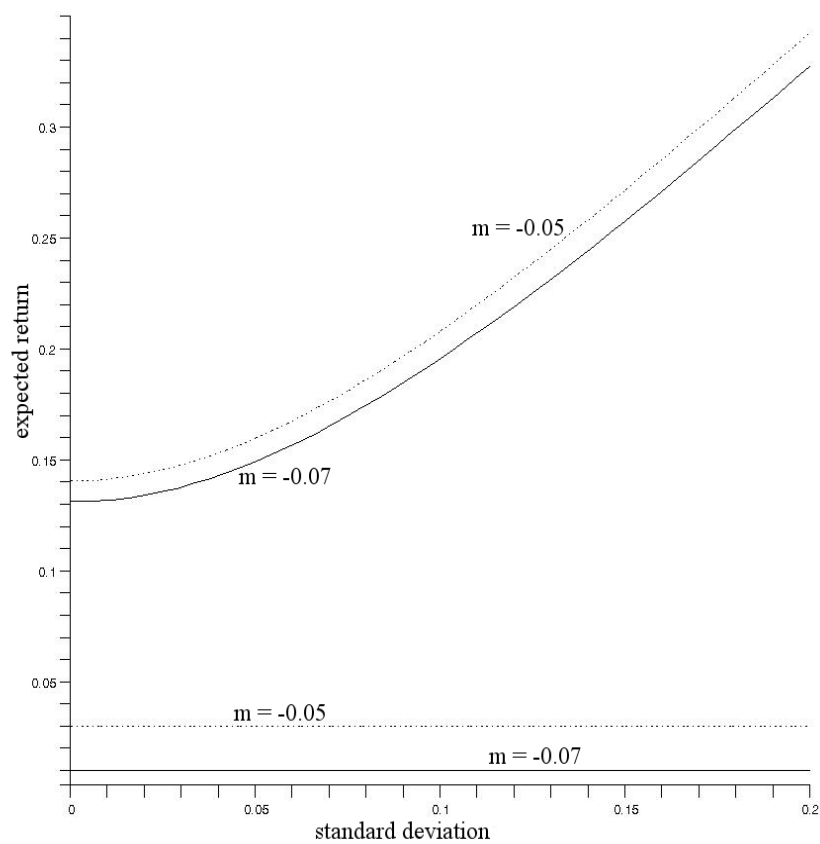
Figure 10
The Impact of α on the Surplus Return Shortfall Constraint
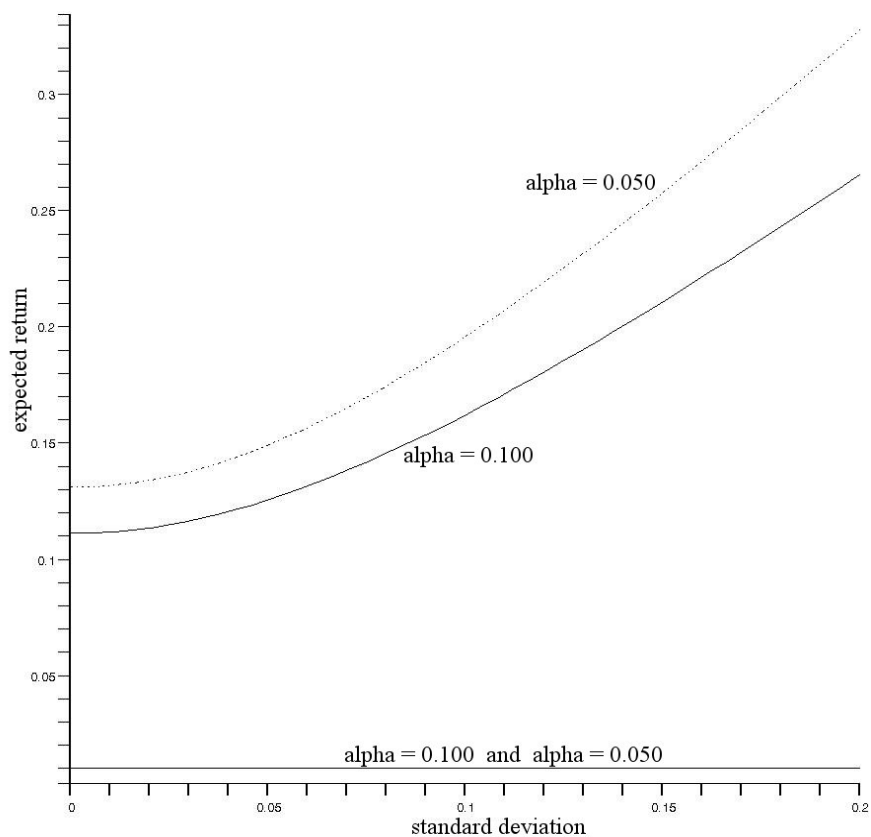
The Surplus Return Shortfall Constraint for w

As for the asset return shortfall constraint, this representation of the surplus

return shortfall constraint in the μ_A-σ_A-coordinate system is not unique. In the next

result, an equivalent condition for the surplus return shortfall constraint in terms of w

is provided:

Result II.9 (The Surplus Return Shortfall Constraint for w)

Let

$$a = F_0^2 \left[\mu_E^2 + \mu_B^2 - 2\mu_E\mu_B - z_\alpha^2(\sigma_E^2 + \sigma_B^2 - 2\sigma_E\sigma_B\rho_{EB}) \right], \tag{2.42}$$

$$b = F_0 \left[2\mu_B\mu_L - 2m\mu_E + 2m\mu_B + 2F_0\mu_E\mu_B - 2\mu_B^2 F_0 - 2\mu_L\mu_E \right.$$

$$\left. + z_\alpha^2(2F_0\sigma_B^2 - 2F_0\sigma_E\sigma_B\rho_{EB} + 2\sigma_L(\sigma_E\rho_{EL} - \sigma_B\rho_{BL})) \right], \tag{2.43}$$

$$c = \mu_L^2 + m^2 + \mu_B^2 F_0^2 + 2\mu_L m - 2\mu_B F_0\mu_L - 2mF_0\mu_B$$

$$- z_\alpha^2(F_0^2\sigma_B^2 + \sigma_L^2 - 2F_0\sigma_L\sigma_B\rho_{BL}). \tag{2.44}$$

Then the surplus return shortfall constraint $P(R_S < m) \le \alpha$ is equivalent to the following conditions if $\mu_S \ge m$:

- *If $a > 0$, then $w \in ((-\infty, w_1] \bigcap [0, 1]) \bigcup ([w_2, +\infty) \bigcap [0, 1])$, where $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and $w_1 \le w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, then $w \in [0, 1]$.*

- *If $a < 0$, then $w \in [w_1, w_2] \bigcap [0, 1]$, where $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and $w_1 \le w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, then there does not exist a w that satisfies the surplus return shortfall constraint.*

- *If $a = 0$ and $b < 0$, then $w \in (-\infty, -\frac{c}{b}] \bigcap [0, 1]$.*

- *If $a = 0$ and $b > 0$, then $w \in [-\frac{c}{b}, +\infty) \bigcap [0, 1]$.*

- *If $a = 0$, $b = 0$ and $c \ge 0$, then $w \in [0, 1]$.*

- *If $a = 0$, $b = 0$ and $c < 0$, then there does not exist a w that satisfies the surplus return shortfall constraint.*

Proof. In the proof for Result II.8 we showed that

$$\mu_S \geq m - z_\alpha \sigma_S$$

$$\Leftrightarrow \quad z_\alpha^2(F_0^2 \sigma_A^2 + \sigma_L^2 - 2F_0 \sigma_A \sigma_L \rho_{AL}) \leq (\mu_L + m - F_0 \mu_A)^2$$

Since we assume $\mu_S = F_0 \mu_A - \mu_L \geq m$ the second condition $\mu_A \geq \frac{\mu_L + m}{F_0}$ is always true. Then

$$z_\alpha^2(F_0^2 \sigma_A^2 + \sigma_L^2 - 2F_0 \sigma_A \sigma_L \rho_{AL}) \leq (\mu_L + m - F_0 \mu_A)^2$$

$$\Leftrightarrow \quad z_\alpha^2 \left[F_0^2 (w^2 \sigma_E^2 + (1-w)^2 \sigma_B^2 + 2w(1-w)\sigma_E \sigma_B \rho_{EB}) + \sigma_L^2 \right.$$

$$\left. -2F_0 \sigma_L (w \sigma_E \rho_{EL} + (1-w)\sigma_B \rho_{EB}) \right]$$

$$\leq \quad \mu_L^2 + m^2 + w^2 \mu_E^2 F_0^2 + (1-w)^2 \mu_B^2 F_0^2 + 2\mu_L m - 2w\mu_L \mu_E F_0$$

$$-2(1-w)\mu_B F_0 \mu_L - 2m F_0 \mu_E w - 2m F_0 (1-w)\mu_B + 2F_0^2 w(1-w)\mu_E \mu_B$$

$$\Leftrightarrow \quad F_0^2 \left[\mu_E^2 + \mu_B^2 - 2\mu_E \mu_B - z_\alpha^2(\sigma_E^2 + \sigma_B^2 - 2\sigma_E \sigma_B \rho_{EB}) \right] \cdot w^2$$

$$+ F_0 \left[2\mu_B \mu_L - 2m\mu_E + 2m\mu_B + 2F_0 \mu_E \mu_B - 2\mu_B^2 F_0 - 2\mu_L \mu_E \right.$$

$$\left. + z_\alpha^2(2F_0 \sigma_B^2 - 2F_0 \sigma_E \sigma_B \rho_{EB} + 2\sigma_L(\sigma_E \rho_{EL} - \sigma_B \rho_{BL})) \right] \cdot w$$

$$+ \mu_L^2 + m^2 + \mu_B^2 F_0^2 + 2\mu_L m - 2\mu_B F_0 \mu_L - 2m F_0 \mu_B$$

$$- z_\alpha^2(F_0^2 \sigma_B^2 + \sigma_L^2 - 2F_0 \sigma_L \sigma_B \rho_{BL}) \geq 0$$

$$\Leftrightarrow \quad aw^2 + bw + c \geq 0$$

If $a > 0$, the function $f(w) = aw^2 + bw + c$ is a parabola which is opened to the top. The nulls of this function are $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $w_1 < w_2$. That is why $f(w) \geq 0 \Leftrightarrow w \leq w_1$ or $w \geq w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, the inequality $aw^2 + bw + c \geq 0$ is true for all w. If $a < 0$, the function $f(w) = aw^2 + bw + c$ is a parabola which is opened to the bottom. The nulls of this function are $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $w_1 < w_2$. That is why $f(w) \geq 0 \Leftrightarrow w_1 \leq w \leq w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, the inequality $aw^2 + bw + c \geq 0$ is false for all w. If $a = 0$ and $b > 0$, this function reduces to a straight line, i.e. $f(w) \geq 0 \Leftrightarrow w \geq -\frac{c}{b}$. If $a = 0$ and $b < 0$, we get $f(w) \geq 0 \Leftrightarrow w \leq -\frac{c}{b}$. If $a = 0$, $b = 0$ and $c \geq 0$, then $f(w) \geq 0$, otherwise $f(w) < 0$. This completes the proof. $\qquad \square$

By using $\mu_A = w\mu_E + (1 - w)\mu_B$, we can find the values of μ_A for which the surplus return shortfall constraint is satisfied. Figure 11, page 56, illustrates this and the fact that the surplus return shortfall constraints in Result II.9 (marked version 2 in the graph) and Result II.8 (marked version 1 in the graph) produce the same restriction for μ_A. This is the case because both shortfall curves have the same intersection point with the stock/bond-curve.

Figure 11
**The Two Versions of the Surplus Return Shortfall Constraint and the
Stock/Bond-Curve**
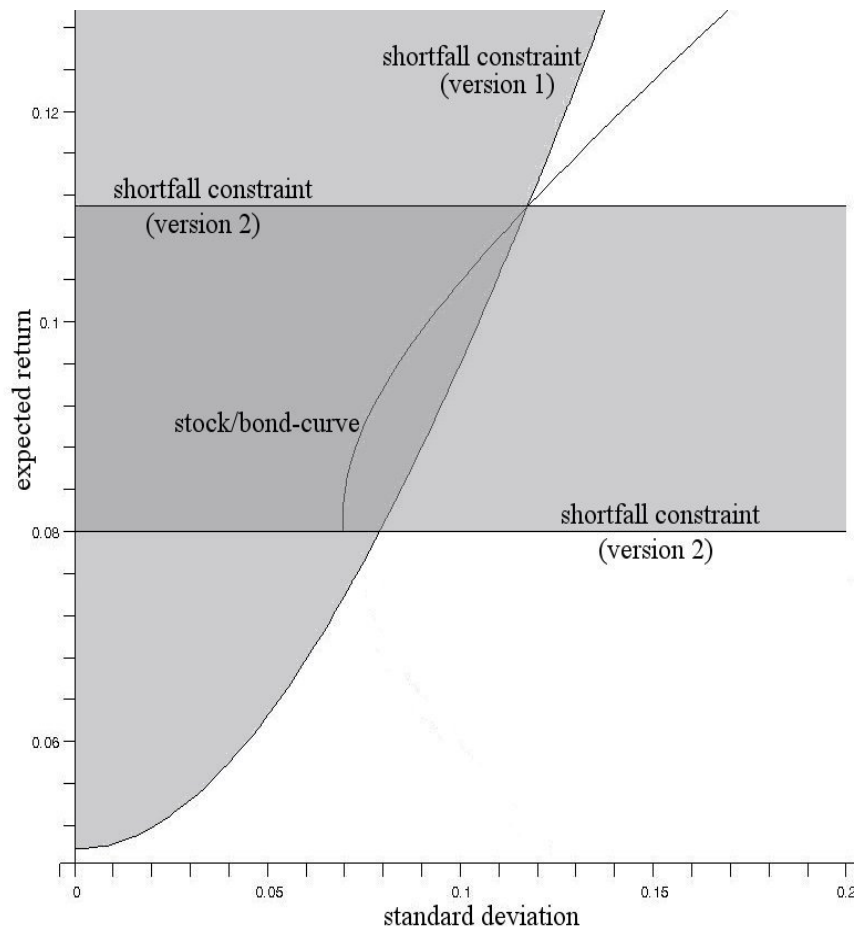
The Relative Return Shortfall Constraint

Up to now, we have considered shortfall constraints in an asset-only framework and in a surplus framework. In this section, we will generalize the idea of these shortfall constraints. This means that another shortfall constraint, the relative return shortfall constraint, will be introduced. We will see that the asset return shortfall constraint and the surplus return shortfall constraints are special cases of the relative return shortfall

constraint.

The relative return consists of two parts. Basically, two different portfolios are considered: the pension fund portfolio and the benchmark portfolio against which the fund manager has to manage the pension fund portfolio. The relative return is then defined as the difference of the two portfolio returns:

$$relative\ return = fund\ portfolio\ return - benchmark\ portfolio\ return.$$

Sometimes it is not possible to get high returns because the market conditions do not allow for them. Then it is not appropriate to say that the pension fund manager did a bad job just because the return of the pension fund is low. That is why a different measure is needed to assess the manager's performance. It is clear that if the relative return is large, the pension fund was managed well relatively to the benchmark portfolio since then the fund portfolio return is large compared to the benchmark portfolio return. If the relative return is small or even negative, the manager didn't manage the fund so well. Thus, the relative return measures the performance of the pension fund manager in comparison to a predetermined benchmark. In a way, the relative return as a measure of performance is more independent of the market since it is difficult to get high values for both the fund portfolio return and the benchmark portfolio return if the market conditions are unprofitable.

As for the asset and surplus return shortfall constraint, the pension fund manager may self-impose a relative return shortfall constraint for managing the pension fund. It is also possible that the constraint is externally imposed on the fund manager, for

example by the company that hired the manager.

In order to control the risk of a low relative return, the fund manager can use a corresponding shortfall constraint. The objective may be to meet a minimum acceptable relative return of 3% with a probability of 95%. In general, this constraint can be formulated as follows (cf. Leibowitz et al., 1996, p. 44):

> *There should be no more than a probability of α that the relative return will*
>
> *be less than a minimum acceptable relative return of m.*

Again, some notations are needed to translate this shortfall constraint into a mathematical expression. For the relative return shortfall constraint, we assume that there is still one stock, but two different bonds to choose from. These bonds have the same expected return, but the standard deviation of the benchmark bond may differ from the standard deviation of the pension fund bond.

- R_A: return of the fund portfolio, where μ_A and σ_A are the expected return and the standard deviation of R_A, respectively.

- R_a: return of the benchmark portfolio, where μ_a and σ_a are the expected return and the standard deviation of R_a, respectively.

- R_B: return of the bond portion in the fund portfolio, where μ_B and σ_B are the expected return and the standard deviation of R_B, respectively.

- R_b: return of the bond portion in the benchmark portfolio, where μ_b and σ_b are the expected return and the standard deviation of R_b, respectively.

- R_E: stock return, where μ_E and σ_E are the expected return and the standard deviation of R_E, respectively.

- R_D: relative return, where μ_D and σ_D are the expected return and the standard deviation of R_D, respectively.

Then the following relationships hold:

$$R_A = w_A R_E + (1 - w_A) R_B, \ w_A \in [0, 1], \tag{2.45}$$

$$R_a = w_a R_E + (1 - w_a) R_b, \ w_a \in [0, 1], \tag{2.46}$$

$$R_D = R_A - R_a = (w_A - w_a) R_E + (1 - w_A) R_B - (1 - w_a) R_b. \tag{2.47}$$

Now we assume that $\mu_B = \mu_b$. Then the expected return μ_D and the variance σ_D^2 of the relative return R_D are given by

$$\mu_D = (w_A - w_a)\mu_E + (1 - w_A)\mu_B - (1 - w_a)\mu_b$$

$$= (w_A - w_a)(\mu_E - \mu_B), \tag{2.48}$$

$$\sigma_D^2 = (w_A - w_a)^2 \sigma_E^2 (1 - \rho_{EB}^2)$$

$$+ [(w_A - w_a)\sigma_E \rho_{EB} + (1 - w_A)\sigma_B - (1 - w_a)\sigma_b]^2. \tag{2.49}$$

For the derivation of the variance formula, more calculations are needed. Leibowitz et al. (1996) give detailed steps to derive this formula. These steps is summarized in the following:

$$\sigma_D^2 = E\left((R_D - \mu_D)^2\right).$$

We can plug in (2.47) and (2.48) and get

$$
\begin{aligned}
\sigma_D^2 \;&=\; E\left[\left((w_A - w_a)(R_E - \mu_E) + (1 - w_A)(R_B - \mu_B) - (1 - w_a)(R_b - \mu_b)\right)^2\right] \\
&=\; (w_A - w_a)^2\sigma_E^2 + (1 - w_A)^2\sigma_B^2 + (1 - w_a)^2\sigma_b^2 + 2(w_A - w_a)(1 - w_A)\sigma_{EB} \\
&\quad -2(w_A - w_a)(1 - w_a)\sigma_{Eb} - 2(1 - w_A)(1 - w_a)\sigma_{Bb}.
\end{aligned}
$$

We assume that $\rho_{Bb} = 1$. Then it may be true that $\rho_{EB} = \rho_{Eb}$ if the stock and bond returns are normally distributed, but there is no proof in Leibowitz et al. (1996) and this reasoning seems to be not obvious. However, it is possible to show that $\rho_{EB} = \rho_{Eb}$ without any distribution assumption:

$$
\rho_{Bb} = 1
$$

$$
\Rightarrow \quad \exists\, \beta > 0 \text{ such that } R_B = \mu_B + \beta(R_b - \mu_b) \quad \text{(cf. Tucker, 1962)}
$$

$$
\Rightarrow \quad Var(R_B) = \sigma_B^2 = Var\left(\mu_B + \beta(R_b - \mu_b)\right) = \beta^2\sigma_b^2
$$

$$
\overset{\beta>0}{\Rightarrow} \quad \beta = \frac{\sigma_B}{\sigma_b}
$$

$$
\Rightarrow \quad \sigma_{EB} = Cov(R_E, R_B) = Cov\left(R_E,\; \mu_B - \frac{\sigma_B}{\sigma_b}\mu_b + \frac{\sigma_B}{\sigma_b}R_b\right)
$$

$$
= \overbrace{Cov\left(R_E,\; \mu_B - \frac{\sigma_B}{\sigma_b}\mu_b\right)}^{=0} + \frac{\sigma_B}{\sigma_b}Cov(R_E, R_b)
$$

$$
= \frac{\sigma_B}{\sigma_b}\sigma_{Eb}
$$

$$
\Rightarrow \quad \sigma_B\sigma_{EB} = \sigma_B\sigma_{Eb}
$$

$$
\Rightarrow \quad \frac{\sigma_{EB}}{\sigma_B\sigma_E} = \frac{\sigma_{Eb}}{\sigma_b\sigma_E}
$$

$$
\Rightarrow \quad \rho_{EB} = \rho_{Eb}
$$

With this result, we can continue to calculate σ_D^2:

$$\sigma_D^2 \; = \; (w_A - w_a)^2\sigma_E^2 + (1 - w_A)^2\sigma_B^2 + (1 - w_a)^2\sigma_b^2$$

$$+2(w_A - w_a)(1 - w_A)\sigma_E\sigma_B\rho_{EB} - 2(w_A - w_a)(1 - w_a)\sigma_E\sigma_b\rho_{Eb}$$

$$-2(1 - w_A)(1 - w_a)\sigma_B\sigma_b$$

Subtracting and adding $(w_A - w_a)^2\sigma_E^2\rho_{EB}^2$ gives

$$\sigma_D^2 \; = \; \left[(w_A - w_a)^2\sigma_E^2 - (w_A - w_a)^2\sigma_E^2\rho_{EB}^2\right] + \left[(w_A - w_a)^2\sigma_E^2\rho_{EB}^2\right.$$

$$+(1 - w_A)^2\sigma_B^2 + (1 - w_a)^2\sigma_b^2 + 2(w_A - w_a)(1 - w_A)\sigma_E\sigma_B\rho_{EB}$$

$$-2(w_A - w_a)(1 - w_a)\sigma_E\sigma_b\rho_{EB} - 2(1 - w_A)(1 - w_a)\sigma_B\sigma_b\bigr]$$

$$= (w_A - w_a)^2\sigma_E^2(1 - \rho_{EB}^2)$$

$$+ \left[(w_A - w_a)\sigma_E\rho_{EB} + (1 - w_A)\sigma_B - (1 - w_a)\sigma_b\right]^2$$

Now we have all the ingredients to find a formula for the shortfall constraint of the relative return. Result II.10 gives the formula in terms of μ_D and σ_D, whereas Result II.11 provides a formula for w:

Result II.10 (The Relative Return Shortfall Constraint)

Let $R_E \sim N(\mu_E, \sigma_E^2)$, $R_B \sim N(\mu_B, \sigma_B^2)$, and $R_b \sim N(\mu_B, \sigma_b^2)$. Then

$$R_D = R_A - R_a \sim N(\mu_D, \sigma_D^2),$$

where μ_D and σ_D^2 are given by (2.48)and (2.49), respectively, and the relative return

shortfall constraint can be written as

$$P(R_D < m) \le \alpha \tag{2.50}$$

$$\Leftrightarrow$$

$$\mu_D \ge m - z_\alpha \cdot \sigma_D. \tag{2.51}$$

Proof. $R_D \sim N(\mu_D, \sigma_D^2)$ since $R_D = R_A - R_a$ is a linear combination of normally distributed random variables. Then $\frac{R_P - \mu_P}{\sigma_P} \sim N(0, 1)$ and the following equivalences hold:

$$P(R_D < m) \le \alpha$$

$$\Leftrightarrow \quad P\left(\frac{R_D - \mu_D}{\sigma_D} < \frac{m - \mu_D}{\sigma_D}\right) \le \alpha$$

$$\Leftrightarrow \quad \Phi\left(\frac{m - \mu_D}{\sigma_D}\right) \le \alpha$$

$$\Leftrightarrow \quad \frac{m - \mu_D}{\sigma_D} \le z_\alpha$$

$$\Leftrightarrow \quad \mu_D \ge m - z_\alpha \cdot \sigma_D$$

Here, z_α denotes the α-percentile of the standard normal distribution and Φ is the cumulative distribution function of the standard normal distribution. $\square$

Result II.11 (The Relative Return Shortfall Constraint for w_A)

Let $R_E \sim N(\mu_E, \sigma_E^2)$, $R_B \sim N(\mu_B, \sigma_B^2)$, and $R_b \sim N(\mu_B, \sigma_b^2)$. Let

$$a \;=\; (\mu_E - \mu_B)^2 - z_\alpha^2 \left[\sigma_E^2 + \sigma_B^2 - 2\sigma_E \sigma_B \rho_{EB}\right], \tag{2.52}$$

$$b = -2m(\mu_E - \mu_B) - 2w_a(\mu_E - \mu_B)^2 - z_\alpha^2 \left[-2w_a\sigma_E^2 - 2\sigma_B^2 \right.$$

$$\left. +2(1 + w_a)\sigma_E\sigma_B\rho_{EB} - 2(1 - w_a)\sigma_E\sigma_b\rho_{EB} + 2(1 - w_a)\sigma_B\sigma_b \right], \quad (2.53)$$

$$c = m^2 + 2mw_a(\mu_E - \mu_B) + w_a^2(\mu_E - \mu_B)^2 - z_\alpha^2 \left[w_a^2\sigma_E^2 \right.$$

$$\left. +\sigma_B^2 + (1 - w_a)^2\sigma_b^2 - 2w_a\sigma_E\sigma_B\rho_{EB} + 2w_a(1 - w_a)\sigma_E\sigma_b\rho_{EB} \right.$$

$$\left. -2(1 - w_a)\sigma_B\sigma_b \right]. \quad (2.54)$$

Then the relative return shortfall constraint is equivalent to the following conditions if

$\mu_D \geq m$:

- *If $a > 0$, then $w \in ((-\infty, w_1] \bigcap [0, 1]) \bigcup ([w_2, +\infty) \bigcap [0, 1])$, where*
  $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ *and $w_1 \leq w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, then*
  $w \in [0, 1]$.

- *If $a < 0$, then $w \in [w_1, w_2] \bigcap [0, 1]$, where $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and $w_1 \leq w_2$. If the*
  root $\sqrt{b^2 - 4ac}$ has no real solution, then there does not exist a w that satisfies the
  relative return shortfall constraint.

- *If $a = 0$ and $b < 0$, then $w \in (-\infty, -\frac{c}{b}] \bigcap [0, 1]$.*

- *If $a = 0$ and $b > 0$, then $w \in [-\frac{c}{b}, +\infty) \bigcap [0, 1]$.*

- *If $a = 0$, $b = 0$ and $c \geq 0$, then $w \in [0, 1]$.*

- *If $a = 0$, $b = 0$ and $c < 0$, then there does not exist a w that satisfies the relative*
  return shortfall constraint.

Proof. We can plug (2.48) for μ_D and (2.49) for σ_D in the inequality (2.51):

$$\mu_D \geq m - z_\alpha \sigma_D$$

$$\overset{\mu_D \geq m, \ z_\alpha < 0}{\Longleftrightarrow} \quad [(w_A - w_a)(\mu_E - \mu_B) + m]^2$$

$$\geq \quad z_\alpha^2 \left[(w_A - w_a)^2 \sigma_E^2 (1 - \rho_{EB}^2) \right.$$

$$+ [(w_A - w_a)\sigma_E \rho_{EB} + (1 - w_A)\sigma_B - (1 - w_a)\sigma_b]^2]$$

$$\Longleftrightarrow \quad \left[(\mu_E - \mu_B)^2 - z_\alpha^2 \left[\sigma_E^2 + \sigma_B^2 - 2\sigma_E \sigma_B \rho_{EB} \right] \right] \cdot w_A^2$$

$$+ \left[-2m(\mu_E - \mu_B) - 2w_a(\mu_E - \mu_B)^2 - z_\alpha^2 \left[-2w_a \sigma_E^2 - 2\sigma_B^2 \right. \right.$$

$$\left. + 2(1 + w_a)\sigma_E \sigma_B \rho_{EB} - 2(1 - w_a)\sigma_E \sigma_b \rho_{EB} + 2(1 - w_a)\sigma_B \sigma_b \right] \cdot w_A$$

$$+ \left[m^2 + 2mw_a(\mu_E - \mu_B) + w_a^2(\mu_E - \mu_B)^2 - z_\alpha^2 \left[w_a^2 \sigma_E^2 \right. \right.$$

$$+ \sigma_B^2 + (1 - w_a)^2 \sigma_b^2 - 2w_a \sigma_E \sigma_B \rho_{EB} + 2w_a(1 - w_a)\sigma_E \sigma_b \rho_{EB}$$

$$\left. -2(1 - w_a)\sigma_B \sigma_b \right] \geq 0$$

$$\Longleftrightarrow \quad aw_A^2 + bw_A + c \geq 0$$

If $a > 0$, the function $f(w) = aw^2 + bw + c$ is a parabola which is opened to the top. The nulls of this function are $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $w_1 < w_2$. That is why $f(w) \geq 0 \Leftrightarrow w \leq w_1$ or $w \geq w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, the inequality $aw^2 + bw + c \geq 0$ is true for all w. If $a < 0$, the function $f(w) = aw^2 + bw + c$ is a parabola which is opened to the bottom. The nulls of this function are $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $w_1 < w_2$. That is why $f(w) \geq 0 \Leftrightarrow w_1 \leq w \leq w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, the inequality $aw^2 + bw + c \geq 0$ is false for all w. If $a = 0$ and $b > 0$, this function reduces to a straight line, i.e. $f(w) \geq 0 \Leftrightarrow w \geq -\frac{c}{b}$. If

$a = 0$ and $b < 0$, we get $f(w) \geq 0 \Leftrightarrow w \leq -\frac{c}{b}$. If $a = 0$, $b = 0$ and $c \geq 0$, then $f(w) \geq 0$,

otherwise $f(w) < 0$. This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Example 4

For this example, we need the following values of the underlying variables:

Table 3
Values for Example 4

	Expected Return	Standard Deviation of Returns	Correlation with Bonds	Correlation with Stocks
Stocks	13.0%	17.00%	0.35	1.00
Bonds	8.0%	6.96%	1.00	0.35

Source: Leibowitz et al. (1996, p.86)

In addition we assume that $w_a = 0$, $\sigma_b = 0.171$, $\alpha = 0.10$ and $m = -0.15$. Since $w_a = 0$, the benchmark portfolio only consists of a bond. The standard deviation σ_b is equal to the standard deviation σ_L of the liability in Example 2. The same holds for the expected value: $\mu_b = \mu_L$. However, we assume now $\rho_{Bb} = 1$. So this differs a little bit from $\rho_{BL} = 0.913$ in Example 2. Nevertheless, we can think of the liability of a pension fund when using this kind of benchmark portfolio. Then we get the following values:

$$a = -0.0392, \ b = 0.0116, \ c = 0.0057$$

and therefore

$$w_{1/2} = 0.1484 \pm 0.4077$$

which restricts w_A to

$$w_A \in [0, 0.5562].$$

There is no need to do a sensitivity analysis. Result II.5 holds for the relative return shortfall constraint, too.

It turns out that the asset return shortfall constraint and the surplus return shortfall constraint are special cases of the relative return shortfall constraint. This will be discussed in the following.

The Asset Return Shortfall Constraint and the Surplus Return Shortfall Constraint as Special Cases of the Relative Return Shortfall Constraint

The result in this subsection will show that we could have started this chapter with a discussion of the relative return shortfall constraint. After that, we could have just mentioned that for certain values of the variables in the relative return shortfall constraint, we would have gotten the asset return shortfall constraint and the surplus return shortfall constraint.

However, there is a reason why the asset return shortfall constraint and the surplus return shortfall constraint were given attention to this large extent. The asset return and the surplus return are important financial ratios. They indicate how profitable a pension fund is. Though this also holds for the relative return, but the relative return

$$R_D = R_A - R_a$$

compares the asset return to the benchmark return, and not to the liability return (except for the case that the benchmark return is assumed to be the liability return). Therefore, as we have mentioned earlier, in particular the surplus return qualifies better for managing the assets against the liabilities (which is one of the pension fund manager's major goals) since the definition of the surplus return includes both the asset return and liability return.

The following result shows the connection between the asset, the surplus, and the relative return shortfall constraint:

Result II.12

The relationship between the relative return shortfall constraint and the asset and the surplus return shortfall constraint can be described by the following two statements:

1. *Assume that the asset portfolio is managed against a benchmark with $R_a = i^*$ (for example $i^* = 0.08$ for a one year treasury bill). Then the relative return shortfall constraint is equivalent to an asset return shortfall constraint with $m^* = m + i^*$ and $\alpha^* = \alpha$.*

2. *Assume that the benchmark is the pension fund liability and assume that the funding ratio is 1: $F_0 = 1$. Then the relative return shortfall constraint is equivalent to a surplus return shortfall constraint with $m^* = m$ and $\alpha^* = \alpha$.*

Proof.   1. $R_D = R_A - R_a = R_A - i^* \Rightarrow P(R_D \leq m) = P(R_A \leq m + i^*)$

2. $R_D = R_A - R_a = R_A - R_L \overset{F_0=1}{=} R_S \Rightarrow P(R_D \le m) = P(R_S \le m)$

□

The Log-normal Distribution Assumption

The last shortfall constraint that will be discussed in this chapter is the funding ratio return shortfall constraint. For this shortfall constraint, we could still try to assume normally distributed returns, but it would be impossible to derive practical formulas.

The natural logarithm applied to a fraction is the difference of the numerator and the denominator, i.e.

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y).$$

This property turns out to be quite useful for the derivation of the shortfall constraint. Since the natural logarithm of log-normally distributed random variables is normally distributed (cf. Klugman et al., 2004, p. 58), we will see that this distribution is a good candidate for the distribution assumption of the returns.

In the section "The Normal Distribution Assumption" we have already noted that the normal distribution has the disadvantage of producing returns that are less than -1. We have seen that this disadvantage can be undone by the log-normal distribution.

The reason why the normal distribution (and not the log-normal distribution) was used for the asset, the surplus, and the relative return shortfall constraint is that it was not practical enough to work with the log-normal distribution. However, it should be

reminded that even if the log-normal distribution is used to model the returns, it does

not reflect reality perfectly:

> But we must also remember that whatever model we select it is only an
>
> approximation of reality. This is reflected in the following modeler's motto
>
> [...]:
>
> All models are wrong, but some models are useful. (Klugman et al., 2004)

The Funding Ratio Return Shortfall Constraint

The funding ratio of a pension plan is defined as the market value of assets

divided by the present value of future liabilities (Leibowitz et al., 1996, p. 191). In

order to discuss what properties the funding ratio return has, it is beneficial to have a

more formal definition of the funding ratio return.

Definition 3 (The Funding Ratio Return)

Let

- A_i: *value of the assets at time* $i \in \{0, 1\}$,

- L_i: *value of the liabilities at time* $i \in \{0, 1\}$,

- $F_0 = \frac{A_0}{L_0}$: *initial funding ratio, and*

- $F_1 = \frac{A_1}{L_1}$: *funding ratio after one period.*

Then the funding ratio return (FRR) is given by the equation

$$FRR = \frac{\frac{A_1}{L_1} - \frac{A_0}{L_0}}{\frac{A_0}{L_0}} = \frac{F_1 - F_0}{F_0}. \tag{2.55}$$

The following result gives an expression of the funding ratio return in terms of the asset return and the liability return:

Result II.13 (The Funding Ratio Return)

Let R_A and R_L be the asset return and the liability return, respectively. Then the funding ratio return can be written as

$$FRR = \frac{1 + R_A}{1 + R_L} - 1. \tag{2.56}$$

Proof.

$$FRR = \frac{\frac{A_1}{L_1} - \frac{A_0}{L_0}}{\frac{A_0}{L_0}} = \frac{A_1}{A_0} \cdot \frac{L_0}{L_1} - 1 = \frac{1 + R_A}{1 + R_L} - 1$$

$\square$

Thus, the funding ratio return does not depend on the initial funding level F_0, but only on the asset return and the liability return. That is why Leibowitz et al. (1996) refer to the funding ratio return as a more "'universal' measure in developing strategic allocations". All the funds that have the same asset return and the same liability return produce the same funding ratio return, regardless of their initial funding ratio.

However, this independence of the initial funding level does not mean that the initial funding level has no influence on the risk tolerance of the fund manager. An

initial funding ratio larger than 1 indicates a surplus position since the assets A_0 must be larger than the liabilities L_0. On the other hand, there is a deficit if the initial funding ratio is less than 1. But with a surplus position it is more likely that the fund manager invests in more risky allocations, whereas with a deficit position, the fund manager probably will not be so risk tolerant. That is why the initial funding ratio will affect the fund manager's decisions when he/she adjusts the funding ratio return shortfall constraint.

The funding ratio return shortfall constraint can be used to control the risk of a pension fund. A fund manager, for example, may have the objective to meet a minimum acceptable funding ratio return of -10% with a probability of 95%. In general, this constraint can be formulated as follows (cf. Leibowitz et al., 1996, p. 44):

There should be no more than a probability of α that the funding ratio return

will be less than a minimum acceptable funding ratio return of m.

In order to derive a formula for the funding ratio return shortfall constraint, we need some properties of the bivariate log-normal distribution. This distribution is defined as follows:

Definition 4 (The Bivariate Log-normal Distribution)

Let $Y = (Y_1, Y_2)^T$ be a multivariate normal random vector with probability density function (cf. Johnson and Wichern, 2002)

$$f_{(Y_1, Y_2)}(y_1, y_2) = \frac{1}{2\pi\sqrt{\sigma_1^2\sigma_2^2(1-\rho_{12}^2)}} \cdot e^{-\frac{1}{2(1-\rho_{12}^2)}\left[\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y_2-\mu_2}{\sigma_2}\right)^2 - 2\rho_{12}\left(\frac{y_1-\mu_1}{\sigma_1}\right)\left(\frac{y_2-\mu_2}{\sigma_2}\right)\right]}$$

where μ_1 and μ_2 are the expected values of Y_1 and Y_2, σ_1 and σ_2 are the standard deviations of Y_1 and Y_2, and ρ_{12} is the correlation between Y_1 and Y_2. Then we say that the random vector $X = \left(e^{Y_1}, e^{Y_2}\right)^T$ has a bivariate log-normal distribution with parameters μ_1, μ_2, σ_1, σ_2, and ρ_{12}.

Thomopoulos (2004) provides the following formulas based on results given in Law and Kelton (2000) and Aitchison and Brown (1957):

Result II.14 (The Bivariate Log-Normal Distribution)

Let $X = (X_1, X_2)$ be a bivariate log-normally distributed random vector and let $Y_1 = \ln(X_1)$ and $Y_2 = \ln(X_2)$. The expected value and the variance of Y_1 and Y_2 are denoted by μ_{Y_1}, σ_{Y_1} and μ_{Y_2}, σ_{Y_2}, respectively. Then the covariance $\sigma_{Y_1 Y_2}$, the expected value μ_{Y_1} and the variance $\sigma_{Y_1}^2$ are given by

$$\sigma_{Y_1 Y_2} = \ln\left(1 + \frac{\sigma_{X_1 X_2}}{|\mu_{X_1} \mu_{X_2}|}\right), \tag{2.57}$$

$$\mu_{Y_1} = \ln\left(\frac{\mu_{X_1}^2}{\sqrt{\mu_{X_1}^2 + \sigma_{X_1}^2}}\right), \tag{2.58}$$

$$\sigma_{Y_1}^2 = \ln\left(1 + \frac{\sigma_{X_1}^2}{\mu_{X_1}^2}\right). \tag{2.59}$$

Now it is possible to find a formula for the funding ratio return shortfall constraint:

Result II.15 (The Funding Ratio Return Shortfall Constraint)

Let $(1 + R_A, 1 + R_L)$ be bivariate log-normally distributed where $1 + \mu_A$, $1 + \mu_L$ and σ_A^2, σ_L^2 are the expected returns and standard deviations of $1 + R_A$ and $1 + R_L$, respectively. Then the funding ratio shortfall constraint can be written as

$$P(FRR < m) \leq \alpha \tag{2.60}$$

$$\Leftrightarrow$$

$$\frac{\ln\left[\dfrac{(m+1)\sqrt{(1+\mu_A)^2 + \sigma_A^2}(1+\mu_L)^2}{\sqrt{(1+\mu_L)^2 + \sigma_L^2}(1+\mu_A)^2}\right]}{\sqrt{\ln\left(\dfrac{\left[(1+\mu_A)^2 + \sigma_A^2\right]\left[(1+\mu_L)^2 + \sigma_L^2\right]}{\left[(1+\mu_A)(1+\mu_L) + \frac{\mu_A - \mu_B}{\mu_E - \mu_B}\sigma_{EL} + \frac{\mu_E - \mu_A}{\mu_E - \mu_B}\sigma_{BL}\right]^2}\right)}} \leq z_\alpha \tag{2.61}$$

where $\sigma_{BL} = \sigma_B \sigma_L \rho_{BL}$ and $\sigma_{EL} = \sigma_E \sigma_L \rho_{EL}$ and ρ_{BL} and ρ_{EL} are given by (2.31) and (2.30). The variance σ_L^2 is given by (2.23).

Proof.

$$P(FRR < m) \overset{ResultII.13}{=} P\left(\frac{1 + R_A}{1 + R_L} - 1 < m\right) = P\left(\frac{1 + R_A}{1 + R_L} \leq m + 1\right)$$

$$= P\left(\underbrace{\ln(1 + R_A)}_{\sim N(\mu_A^*, \sigma_A^{*2})} - \underbrace{\ln(1 + R_L)}_{\sim N(\mu_L^*, \sigma_L^{*2})} < \ln(m + 1)\right)$$

$$= P\left(\underbrace{\frac{\ln(1 + R_A) - \ln(1 + R_L) - (\mu_A^* - \mu_L^*)}{\sqrt{\sigma_A^{*2} + \sigma_L^{*2} - 2\sigma_{AL}^*}}}_{\sim N(0,1)} < \frac{ln(m + 1) - (\mu_A^* - \mu_L^*)}{\sqrt{\sigma_A^{*2} + \sigma_L^{*2} - 2\sigma_{AL}^*}}\right)$$

where μ_A^*, σ_A^{*2} and μ_L^*, σ_L^{*2} are the expected return and the variance of $\ln(1 + R_A)$, $\ln(1 + R_L)$, respectively, and σ_{AL}^* is the covariance between these two random variables.

Then

$$P(FRR < m) \le \alpha$$

$$\Leftrightarrow \frac{ln(m+1) - (\mu_A^* - \mu_L^*)}{\sqrt{\sigma_A^{*\,2} + \sigma_L^{*\,2} - 2\sigma_{AL}^*}} \le z_\alpha$$

$$\Leftrightarrow \quad ln(m+1) - (\mu_A^* - \mu_L^*) \le z_\alpha\sqrt{\sigma_A^{*\,2} + \sigma_L^{*\,2} - 2\sigma_{AL}^*} \qquad (2.62)$$

Now we can use Result II.14 with $X_1 = 1 + R_A$ and $X_2 = 1 + R_L$. Then $\mu_{X_1} = 1 + \mu_A$, $\sigma_{X_1}^2 = \sigma_A^2$, $\mu_{X_2} = 1 + \mu_L$, $\sigma_{X_2}^2 = \sigma_L^2$, and with the formulas in Result II.14 we get

$$\mu_A^* = \ln\left(\frac{(1+\mu_A)^2}{\sqrt{(1+\mu_A)^2 + \sigma_A^2}}\right)$$

$$\mu_L^* = \ln\left(\frac{(1+\mu_L)^2}{\sqrt{(1+\mu_L)^2 + \sigma_L^2}}\right)$$

$$\sigma_A^{*\,2} = \ln\left(1 + \frac{\sigma_A^2}{(1+\mu_A)^2}\right)$$

$$\sigma_L^{*\,2} = \ln\left(1 + \frac{\sigma_L^2}{(1+\mu_L)^2}\right)$$

$$\sigma_{AL}^* = \ln\left(1 + \frac{\sigma_{1+R_A,1+R_L}}{|(1+\mu_A)(1+\mu_L)|}\right)$$

and

$$\sigma_{1+R_A,1+R_L} = Cov(1 + R_A, 1 + R_L) = Cov(R_A, R_L)$$

$$= w\sigma_{EL} + (1-w)\sigma_{BL}, \text{ where } w = \frac{\mu_A - \mu_B}{\mu_E - \mu_B}$$

Since we assume $\mu_A \ge 0$ and $\mu_L \ge 0$, the absolute value in the formula for σ_{AL}^* can be neglected, i.e.

$$\sigma_{AL}^* = \ln\left(1 + \frac{\sigma_{1+R_A,1+R_L}}{(1+\mu_A)(1+\mu_L)}\right).$$

After plugging in all these equations in (2.62), we get (2.61). ◻

Example 5

For this example, we need the following values of the underlying variables:

Table 4
Values for Example 5

	Expected Return	Standard Deviation of Returns	Correlation with Bonds	Correlation with Stocks
Assets:				
Stocks	13.0%	17.00%	0.35	1.00
Bonds	8.0%	6.96%	1.00	0.35
Liabilities:				
Basic Schedule	8.0%	15.00%	1.00	
Noise	0.00%	7.00%		0.25

Source: Leibowitz et al. (1996, p.86)

In Example 3, the values for σ_L, ρ_{BL}, and ρ_{EL} have already been calculated:

$$\sigma_L = 0.171, \ \rho_{BL} = 0.913, \ \rho_{EL} = 0.409.$$

In addition, we choose $\alpha = 0.0275$ and $m = -0.2$. After plugging these values in (2.61),

we get the following inequality for the funding ratio return shortfall constraint:

$$\ln\left(0.8533\frac{\sqrt{(1+\mu_A)^2 + \sigma_A^2}}{(1+\mu_A)^2}\right) \leq 1.96\sqrt{\ln\left(\frac{1.1956\left[(1+\mu_A)^2 + \sigma_A^2\right]}{(1.1005\mu_A + 1.0892)^2}\right)}.$$

This funding ratio return shortfall constraint is illustrated in figure 12, page 76.

In figure 13, page 77, and figure 14, page 78, we can see that if we increase m or

decrease α, the funding ratio return shortfall constraint becomes more strict and the

Figure 12
The Funding Ratio Return Shortfall Constraint
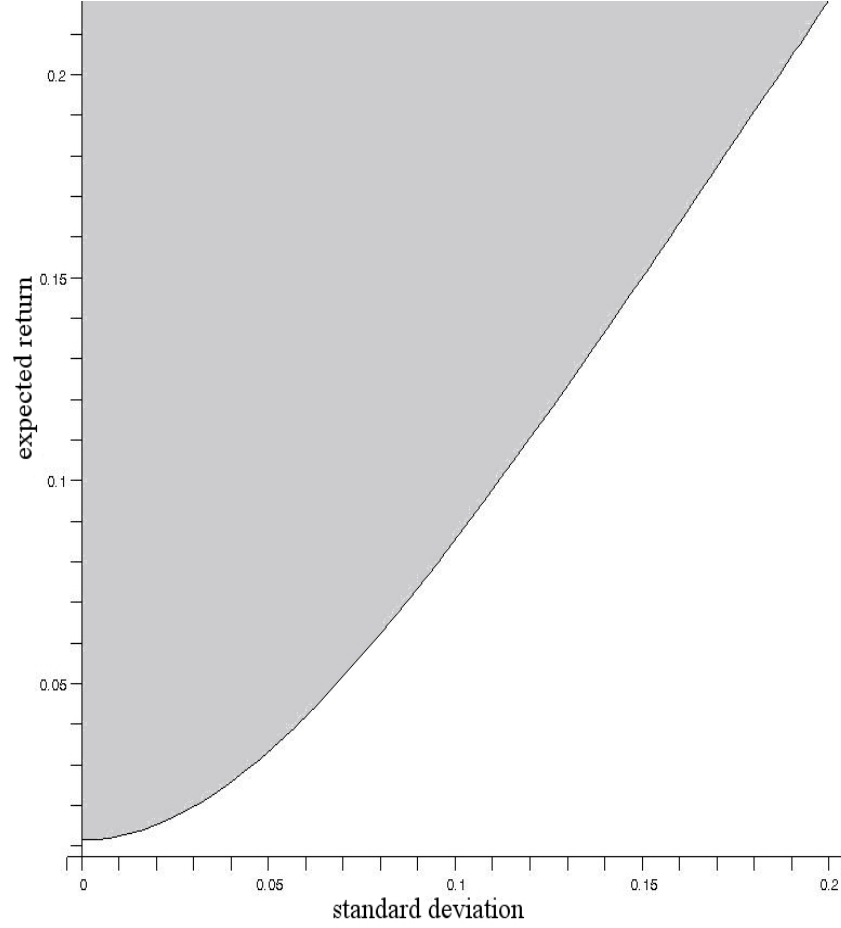
number of portfolios from which the pension fund manager can choose decreases. The

reason is that if we find a portfolio (μ_A^*, σ_A^*) that fulfills the shortfall constraint

condition (2.61)

$$\frac{\ln\left[\frac{(m+1)\sqrt{(1+\mu_A^*)^2 + \sigma_A^{*\,2}(1+\mu_L)^2}}{\sqrt{(1+\mu_L)^2 + \sigma_L^2(1+\mu_A^*)^2}}\right]}{\sqrt{\ln\left(\frac{\left[(1+\mu_A^*)^2 + \sigma_A^{*\,2}\right]\left[(1+\mu_L)^2 + \sigma_L^2\right]}{\left[(1+\mu_A^*)(1+\mu_L) + \frac{\mu_A^* - \mu_B}{\mu_E - \mu_B}\sigma_{EL} + \frac{\mu_E - \mu_A^*}{\mu_E - \mu_B}\sigma_{BL}\right]^2}\right)}} \leq z_\alpha$$

Figure 13
The Impact of α on the Funding Ratio Return Shortfall Constraint
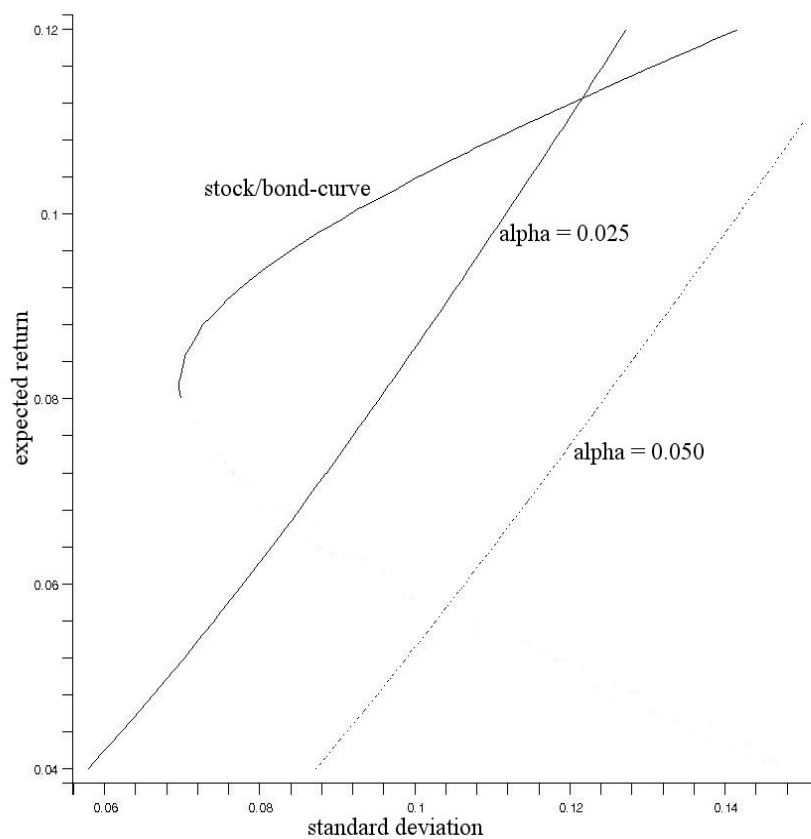
and decrease m or increase α, this inequality is still satisfied because the left side of this inequality decreases or the right side increases.

Figure 14

The Impact of m on the Funding Ratio Return Shortfall Constraint
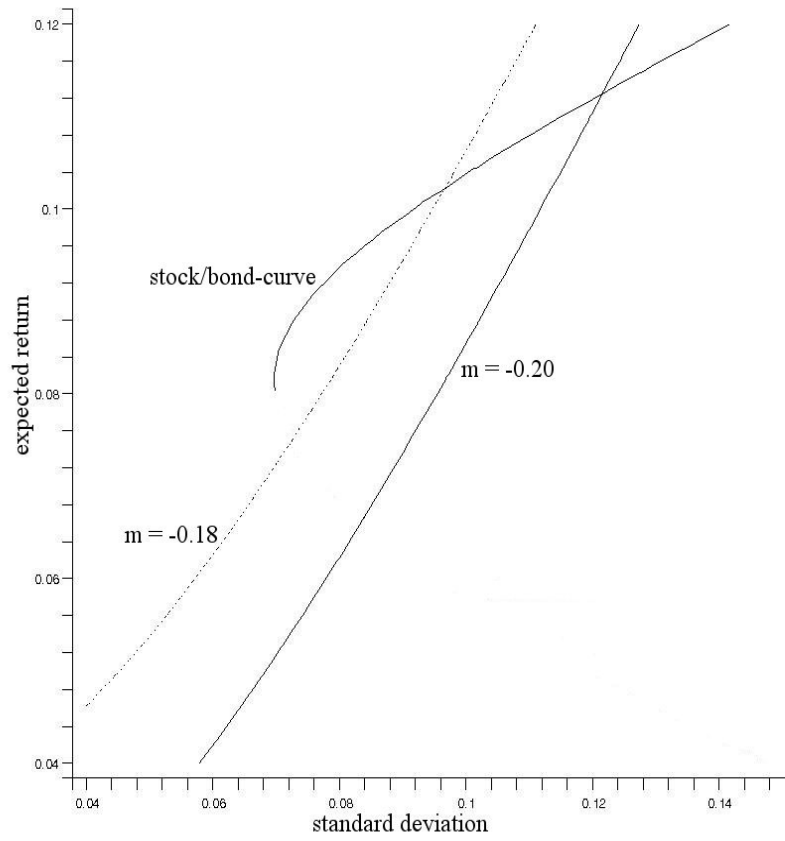
CHAPTER III

TAIL CONDITIONAL EXPECTATION CONSTRAINTS

In the preceding chapter, we used shortfall probability to measure risk. However, there is a major disadvantage with shortfall probability as risk measure. Let us consider an investor who wants to control the investment risk. We have seen that the investor could use a shortfall constraint which restricts the probability that the return of an investment falls below a return that the investor considers as critical. For example, the investor may have the objective to meet a minimum acceptable investment return of 3% with a probability of 95%. With this approach, however, the investor cannot control how much the return falls below 3% in 5% of the cases (with a probability of 5%). Maybe the return mostly assumes values between 0% and 3% in this case, but it is also possible that the return is almost always close to -10% given that it is less than 3%. Of course, the investor would prefer the first scenario to the second scenario, but shortfall probability does not distinguish between these scenarios.

That is why we will use a different risk measure in this chapter that remedies this disadvantage of shortfall probability: tail conditional expectation. This risk measure has already been introduced in chapter II. The tail conditional expectation is defined as

$$TCE_\alpha(R) = E\left(-R| -R \geq VaR_\alpha(R)\right).$$

It has become more and more popular in recent years because it satisfies the (desirable) properties of a coherent risk measure under certain conditions.

Definition 5 (Coherent Risk Measure)

Let $(\Omega, \mathfrak{F}, P)$ be a probability space and let $\mathcal{M}$ be a convex cone (i.e. $X, Y \in \mathcal{M}$, $h > 0 \Rightarrow X + Y \in \mathcal{M}, hX \in \mathcal{M}$) in $L_1(\Omega, \mathfrak{F}, P)$ where

$L_1(\Omega, \mathfrak{F}, P) = \{X | X : \Omega \to \mathbb{R} \text{ is a random variable such that } \int |X| dP < +\infty\}$.

We assume that $X + a \in \mathcal{M}$ for all $a \in \mathbb{R}$. Then a risk measure

$$\rho : \mathcal{M} \to \mathbb{R}$$

is called coherent if it satisfies the following properties:

1. *ρ is monotonous, i.e. $X, Y \in \mathcal{M}$, $X \leq Y$ with probability $1 \Rightarrow \rho(X) \geq \rho(Y)$.*

2. *ρ is sub-additive, i.e. $X, Y \in \mathcal{M} \Rightarrow \rho(X + Y) \leq \rho(X) + \rho(Y)$.*

3. *rho is positively homogeneous, i.e. $X \in \mathcal{M}, h > 0 \Rightarrow \rho(hX) = h\rho(X)$.*

4. *ρ is translation invariant, i.e. $X \in \mathcal{M}, a \in \mathbb{R} \Rightarrow \rho(X + a) = \rho(X) - a$.*

Acerbi and Tasche (2002) show that if the underlying distribution is continuous, the tail conditional expectation TCE fulfills these properties and therefore, TCE is a coherent risk measure.

In order to control the risk of the fund, the pension fund manager can restrict the tail conditional expectation TCE such that TCE won't exceed a value that the manager regards as critical. This restriction of TCE is referred to as tail conditional

expectation constraint. In the following, we will derive formulas for the tail conditional

expectation constraint for the asset return, the surplus return, the relative return, and

the funding ratio return.

The Tail Conditional Expectation Constraint for the Asset Return

At first, we will focus on the tail conditional expectation constraint for the asset

return. For this purpose, we will use the same notations and the same distribution

assumptions as for the asset return shortfall constraint:

- R_P: return of the pension fund portfolio,

- R_B: return of the bond portion in the portfolio, and

- R_E: return of the stock portion in the portfolio.

  We assume that R_B and R_E are normally distributed:

- $R_B \sim N(\mu_B, \sigma_B^2)$, where μ_B and σ_B^2 are the expected return and the variance of

  R_B, respectively.

- $R_E \sim N(\mu_E, \sigma_E^2)$, where μ_E and σ_E^2 are the expected return and the variance of

  R_E, respectively.

The tail conditional expectation is the expected value of the random variable

$-R_P| - R_P \geq VaR_\alpha(R_P)$. That is why it is useful to have a formula for the cumulative

distribution function and the probability density function of this random variable.

These formulas will be derived in the following. At first we concentrate on a random return R which does not necessarily have to be the asset return.

Result III.1 (The Distribution of the Random Variable $-R| -R \geq VaR_\alpha(R)$)

Let R be a random variable with a continuous distribution and let $v = VaR_\alpha(R)$. Then the cumulative distribution function of $-R| -R \geq v$ is given by

$$F_{-R|-R \geq v}(x) = \begin{cases} 1 - \frac{1}{\alpha}F_R(-x) & \text{if } x \geq v, \\ 0 & \text{if } x < v, \end{cases} \tag{3.1}$$

where $F_R(\cdot)$ denotes the cumulative distribution function of the random variable R. The density function of $-R| -R \geq v$ is given by

$$f_{-R|-R \geq v}(x) = \begin{cases} \frac{1}{\alpha}f_R(-x) & \text{if } x \geq v, \\ 0 & \text{if } x < v, \end{cases} \tag{3.2}$$

where $f_R(\cdot)$ denotes the probability density function of R.

Proof. The Value-at-Risk is defined as

$$VaR_\alpha(R) = -\inf\{x|P(R \leq x) > \alpha\}.$$

Since R has a continuous distribution, the following relationship holds:

$$v = VaR_\alpha(R) \Leftrightarrow \alpha = P(R \leq -v) = F_R(-v). \tag{3.3}$$

Then the formula for the cumulative distribution function can be derived by

straight-forward calculations:

$$
\begin{aligned}
F_{-R|-R \geq v}(x) \; &= P(-R \leq x| - R \geq v) = \frac{P(R \geq -x, R \leq -v)}{P(R \leq -v)} \\
&= \frac{P(-x \leq R \leq -v)}{F_R(-v)} \\
&= \begin{cases} \frac{F_R(-v) - F_R(-x)}{F_R(-v)} & \text{if } x \geq v, \\ 0 & \text{if } x < v \end{cases} \\
&\stackrel{(3.3)}{=} \begin{cases} 1 - \frac{1}{\alpha} F_R(-x) & \text{if } x \geq v, \\ 0 & \text{if } x < v \end{cases}
\end{aligned}
$$

The formula for the density function can be derived as follows:

$$
\begin{aligned}
f_{-R|-R \geq v}(x) \; &= \frac{d}{dx} F_{-R|-R \geq v}(x) \\
&= \begin{cases} \frac{1}{\alpha} f_R(-x) & \text{if } x \geq v, \\ 0 & \text{if } x < v \end{cases}
\end{aligned}
$$

$\square$

With this result, it is possible to find a formula for the TCE constraint for R if R is normally distributed with expected return μ and standard deviation σ.

Result III.2 (The TCE constraint for R)

Let $R \sim N(\mu, \sigma^2)$ and $v = VaR_\alpha(R)$. Then the TCE constraint for R is given by

$$
E(-R| - R \geq v) \leq m
$$

$$
\Leftrightarrow
$$

$$
\mu \geq \frac{e^{-\frac{1}{2} z_{1-\alpha}^2}}{\alpha \sqrt{2\pi}} \cdot \sigma - m. \tag{3.4}
$$

Proof. In the proof for Result III.1, we have already shown that

$$\alpha = F_R(-v).$$

Thus

$$\alpha \quad = F_R(-v) = P\left(\frac{R-\mu}{\sigma} \le \frac{-v-\mu}{\sigma}\right) = \Phi\left(\frac{-v-\mu}{\sigma}\right)$$

$$= 1 - \Phi\left(\frac{v+\mu}{\sigma}\right)$$

$$\Rightarrow \quad \frac{v+\mu}{\sigma} \quad = z_{1-\alpha} \tag{3.5}$$

Now we can calculate the expected value of $-R| - R \ge v$:

$$E(-R| - R \ge v) = \int_{-\infty}^{+\infty} x f_{-R|-R\ge v}(x)dx = \int_{v}^{+\infty} \frac{x}{\alpha} f_R(-x)dx$$

$$= \int_{v}^{+\infty} \frac{x}{\alpha} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{-x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\alpha} \int_{-\infty}^{+\infty} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-(-\mu)}{\sigma}\right)^2} dx - \frac{1}{\alpha} \int_{-\infty}^{v} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{-x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{-\mu}{\alpha} - \frac{1}{\alpha} \int_{-\infty}^{v} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x+\mu}{\sigma}\right)^2} dx \quad \left[u = \frac{x+\mu}{\sigma}, \ du = \frac{1}{\sigma}dx\right]$$

$$= -\frac{\mu}{\alpha} - \frac{1}{\alpha} \int_{-\infty}^{\frac{v+\mu}{\sigma}} \frac{\sigma u - \mu}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$$= -\frac{\mu}{\alpha} - \frac{1}{\alpha} \left[-\frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}\Big|_{-\infty}^{\frac{v+\mu}{\sigma}} - \mu \cdot \Phi\left(\frac{v+\mu}{\sigma}\right)\right]$$

$$\overset{(3.5)}{=} -\frac{\mu}{\alpha} + \frac{1}{\alpha} \left[\frac{\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}z_{1-\alpha}^2} + \mu \overbrace{\Phi(z_{1-\alpha})}^{1-\alpha}\right]$$

$$= -\mu + \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \sigma$$

Therefore

$$E(-R| - R \geq v) \leq m$$

$$\Leftrightarrow \quad -\mu + \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \sigma \leq m$$

$$\Leftrightarrow \quad \mu \geq \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \sigma - m$$

$\square$

Now we can return to the TCE constraint for the asset return. The distribution of the random variable $-R_P| - R_P \geq VaR_\alpha(R_P)$ is provided by the following result.

Result III.3 (The Distribution of $-R_P| - R_P \geq VaR_\alpha(R_P)$)

Let $R_B \sim N(\mu_B, \sigma_B^2)$, $R_E \sim N(\mu_E, \sigma_E^2)$, and let $v = VaR_\alpha(R_P)$. Then the cumulative distribution function of $-R_P| - R_P \geq v$ is given by

$$F_{-R_P|-R_P \geq v}(x) = \begin{cases} 1 - \frac{1}{\alpha}F_{R_P}(-x) & \text{if } x \geq v, \\ 0 & \text{if } x < v, \end{cases} \tag{3.6}$$

where $F_{R_P}(\cdot)$ denotes the cumulative distribution function of the random variable R_P. The density function of $-R_P| - R_P \geq v$ is given by

$$f_{-R_P|-R_P \geq v}(x) = \begin{cases} \frac{1}{\alpha}f_{R_P}(-x) & \text{if } x \geq v, \\ 0 & \text{if } x < v, \end{cases} \tag{3.7}$$

where $f_{R_P}(\cdot)$ denotes the probability density function of R_P. R_P is normally distributed with expected value μ_P and standard deviation σ_P.

Proof. We have already shown that $R_P \sim N(\mu_P, \sigma_P^2)$ (cf. (2.4)). Formulas for μ_P and

σ_P are given by (2.2) and (2.3), respectively. The result follows from Result III.1 since

the normal distribution is a continuous distribution. □

With this result, we can find a formula of the TCE constraint for the asset return.

Result III.4 (The TCE constraint for the asset return)

Let $R_B \sim N(\mu_B, \sigma_B^2)$ and $R_E \sim N(\mu_E, \sigma_E^2)$. Let $v = VaR_\alpha(R_P)$. Then the TCE

constraint for the asset return is given by

$$E(-R_P| - R_P \geq v) \leq m$$

$$\Leftrightarrow$$

$$\mu_P \geq \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \sigma_P - m. \tag{3.8}$$

Proof. The result follows directly from Result III.2 since R_P is normally distributed. □

Example 6

For this example, we choose $\alpha = 0.10$ and $m = 0.10$. Then the TCE constraint for the

asset return is given by

$$\mu_P \geq 1.7585\sigma_P - 0.10.$$

In figure 15, p. 87, this TCE constraint is graphed in a μ_P-σ_P-coordinate system.

Figure 15, p. 87, shows that the TCE constraint for the asset return is

represented by a straight line in the μ_P-σ_P-coordinate system. The slope of this line is

Figure 15
The TCE Constraint for the Asset Return
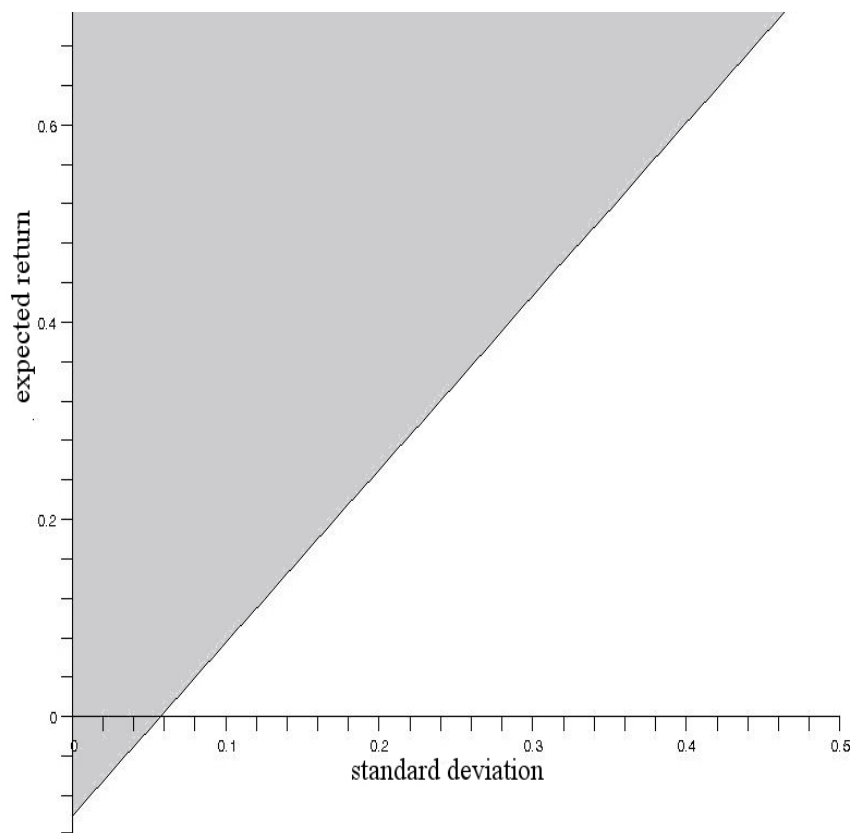
$\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}$ and the y-intercept is $-m$ (cf. (3.8)).

Sensitivity Analysis

As for the asset return shortfall constraint, the pension fund manager can change

two "free" parameters in order to adjust the TCE constraint for the asset return to the

level of risk he/she is willing to accept: these free parameters are α and m. Changes in

those parameters will result in changes in the number of potential portfolios which the

pension fund manager can choose from.

Figure 16
The Impact of α on the TCE Constraint for the Asset Return
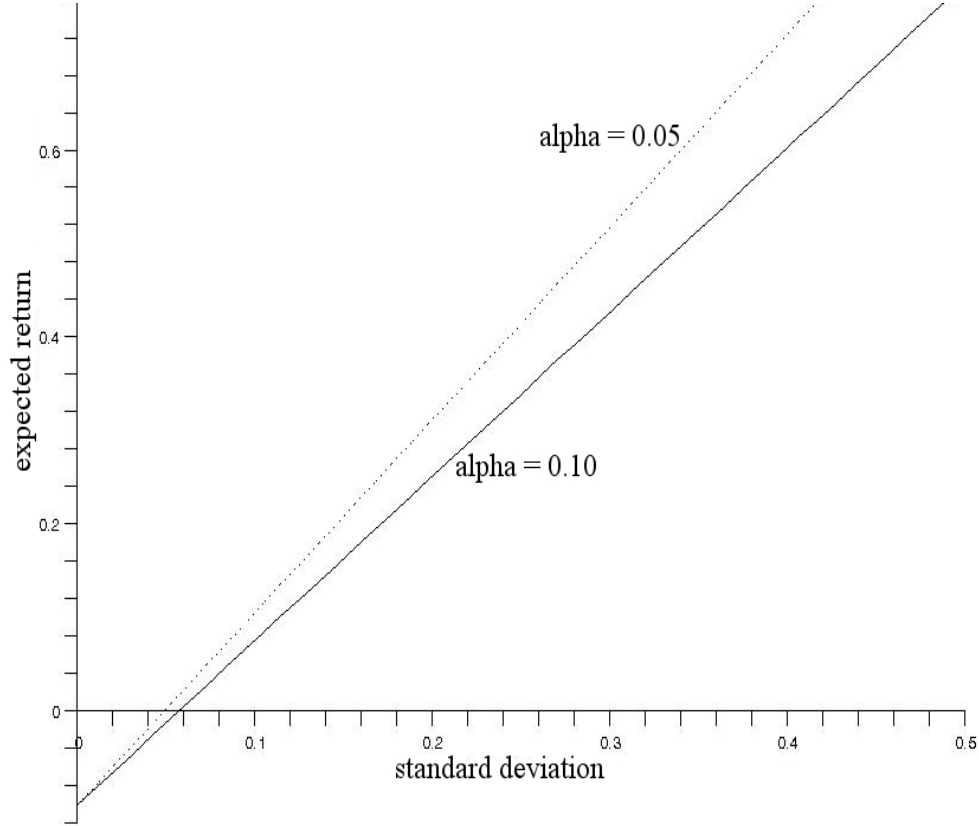
At first, we keep m fixed and analyze what happens when we increase or decrease α. For this purpose, we consider the slope of the line $\mu_P(\sigma) = \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \sigma_P - m$: $\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}$. We assume that $\alpha < 0.5$ since $\alpha \geq 0.5$ is not desirable for the Value-at-Risk: We only want to have small probabilities for which the returns fall below the corresponding Value-at-Risk. If we increase α, the numerator of $\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}$ decreases and the denominator increases. Thus, the slope decreases. With the same reasoning, we can conclude that the slope decreases if we decrease α. This is illustrated in Figure 16, p. 88.

Figure 17
The Impact of m on the TCE Constraint for the Asset Return
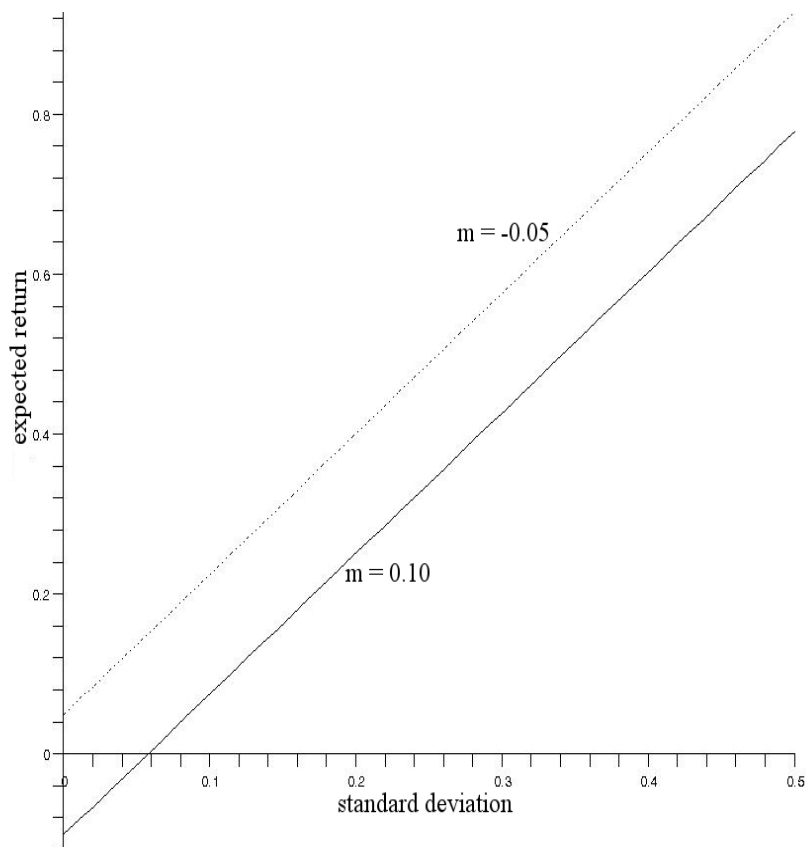
Now we keep α fixed. Since $-m$ is the y-intercept of the line

$\mu_P(\sigma) = \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \sigma_P - m$, the y-intercept decreases if m is increased. The reverse holds

if we decrease m. This is illustrated in Figure 17, p. 89.

The next result corresponds to the result for the sensitivity analysis for the

shortfall constraint. It is a general statement for a random return R. R can be replaced

by R_A so that this result can be applied to the TCE constraint for the asset return.

Result III.5 (Sensitivity Analysis)

Let R be a random return with expected value μ and standard deviation σ. We assume that

$$\mu \geq \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \sigma - m$$

is the corresponding TCE constraint. Then the set of potential portfolios after an increase of m and/or an increase of α is a superset of the set of potential portfolios before m and/or α is changed.

Proof. Let (μ^*, σ^*) be a portfolio that satisfies the TCE constraint for a certain α^* (again, we assume that α is always less than 0.5). Let $\alpha^{new} > \alpha^*$. Then

$$\mu^* \geq \frac{e^{-\frac{1}{2}z_{1-\alpha^*}^2}}{\alpha^*\sqrt{2\pi}} \cdot \sigma^* - m \geq \frac{e^{-\frac{1}{2}z_{1-\alpha^{new}}^2}}{\alpha^{new}\sqrt{2\pi}} \cdot \sigma^* - m$$

That is why the portfolio (μ^*, σ^*) satisfies the new TCE constraint, too.

Let $(\tilde{\mu}, \tilde{\sigma})$ be a portfolio that satisfies the TCE constraint for a certain $\tilde{m}$. Let $m^{new} > \tilde{m}$. Then

$$\tilde{\mu} \geq \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \tilde{\sigma} - \tilde{m} \geq \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \tilde{\sigma} - m^{new}.$$

Therefore, the portfolio $(\tilde{\mu}_P, \tilde{\sigma}_P)$ fulfills the new TCE constraint, too.

These two considerations imply the statement in Result III.5. $\qquad\square$

The TCE Constraint for the Asset Return for w

The next result gives a formula for the TCE constraint for the asset return in terms of w:

Result III.6 (The TCE constraint for the Asset Return for w)

Let

$$a = \mu_E^2 + \mu_B^2 - 2\mu_E\mu_B - \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 \left(\sigma_E^2 + \sigma_B^2 - 2\sigma_E\sigma_B\rho_{EB}\right), \qquad (3.9)$$

$$b = 2\left[\mu_E\mu_B - \mu_B m - \mu_B^2 + \mu_E m + \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 \sigma_B^2\right.$$

$$\left. - \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 \sigma_E\sigma_B\rho_{EB}\right], \qquad (3.10)$$

$$c = \mu_B^2 + m^2 + 2\mu_B m - \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 \sigma_B^2. \qquad (3.11)$$

Then the TCE constraint for the asset return is equivalent to the following conditions if

$\mu_P \geq -m$:

- *If $a > 0$, then $w \in ((-\infty, w_1] \bigcap [0,1]) \bigcup ([w_2, +\infty) \bigcap [0,1])$, where*

  $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ *and $w_1 \leq w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, then*

  $w \in [0,1]$.

- *If $a < 0$, then $w \in [w_1, w_2] \bigcap [0,1]$, where $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and $w_1 \leq w_2$. If the*

  root $\sqrt{b^2 - 4ac}$ has no real solution, then there does not exist a w that satisfies the

  TCE constraint for the asset return.

- *If $a = 0$ and $b < 0$, then $w \in (-\infty, -\frac{c}{b}] \bigcap [0,1]$.*

- *If $a = 0$ and $b > 0$, then $w \in [-\frac{c}{b}, +\infty) \bigcap [0,1]$.*

- *If $a = 0$, $b = 0$ and $c \geq 0$, then $w \in [0,1]$.*

- *If $a = 0$, $b = 0$ and $c < 0$, then there does not exist a w that satisfies the TCE constraint for the asset return.*

Proof. The proof is identical to the one for the asset return shortfall constraint (cf. Result II.3). We just have to replace z_α by $-\dfrac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}$ and m by $-m$.

$$P(R_P < m) \leq \alpha$$

$$\overset{Result III.4}{\Longleftrightarrow} \quad \mu_P \geq \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \sigma_P - m$$

$$\Longleftrightarrow \quad w\mu_E + (1-w)\mu_B$$

$$\geq -m + -\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\sqrt{w^2\sigma_E^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\sigma_E\sigma_B\rho_{EB}}$$

It makes sense to assume $\mu_P \geq -m$ (usually, one chooses $m > 0$ and μ_P should be greater than 0). Then the last inequality is equivalent to

$$(w\mu_E + (1-w)\mu_B + m)^2$$

$$\geq \left(-\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\sqrt{w^2\sigma_E^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\sigma_E\sigma_B\rho_{EB}}\right)^2$$

$$\Leftrightarrow \quad \overbrace{\mu_E^2 + \mu_B^2 - 2\mu_E\mu_B - \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2\left(\sigma_E^2 + \sigma_B^2 - 2\sigma_E\sigma_B\rho_{EB}\right)}^{=a}w^2$$

$$+ 2\overbrace{\left(\mu_E\mu_B + \mu_B m - \mu_B^2 - \mu_E m + \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2\sigma_B^2 - z_\alpha^2\sigma_E\sigma_B\rho_{EB}\right)}^{=b}w$$

$$+ \underbrace{\mu_B^2 + m^2 + 2\mu_B m - \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2\sigma_B^2}_{=c} \geq 0$$

$$\Leftrightarrow \quad aw^2 + bw + c \geq 0$$

If $a > 0$, the function $f(w) = aw^2 + bw + c$ is a parabola which is opened to the top. The nulls of this function are $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $w_1 < w_2$. That is why $f(w) \geq 0 \Leftrightarrow w \leq w_1$ or $w \geq w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, the inequality $aw^2 + bw + c \geq 0$ is true for all w. If $a < 0$, the function $f(w) = aw^2 + bw + c$ is a parabola which is opened to the bottom. The nulls of this function are $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $w_1 < w_2$. That is why $f(w) \geq 0 \Leftrightarrow w_1 \leq w \leq w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, the inequality $aw^2 + bw + c \geq 0$ is false for all w. If $a = 0$ and $b > 0$, this function reduces to a straight line, i.e. $f(w) \geq 0 \Leftrightarrow w \geq -\frac{c}{b}$. If $a = 0$ and $b < 0$, we get $f(w) \geq 0 \Leftrightarrow w \leq -\frac{c}{b}$. If $a = 0$, $b = 0$ and $c \geq 0$, then $f(w) \geq 0$, otherwise $f(w) < 0$. This completes the proof. $\square$

The Tail Conditional Expectation Constraint for the Surplus Return

In chapter II, we have seen that the surplus return shortfall constraint has the same form as the asset return shortfall constraint. The only difference was that the surplus return variables were used instead of the asset return variables. We will see that this is also the case for the TCE constraint for the surplus return.

In order to derive the TCE constraint for the surplus return, we will use the same liability model and the same assumptions as for the surplus return shortfall constraint.

Result III.7 (The Distribution of $-R_S | -R_S \geq VaR_\alpha(R_S)$)

Let $R_B \sim N(\mu_B, \sigma_B^2)$, $R_E \sim N(\mu_E, \sigma_E^2)$, $R_L \sim N(\mu_L, \sigma_L^2)$, and let $v = VaR_\alpha(R_S)$. Then

the cumulative distribution function of $-R_S| - R_S \geq v$ is given by

$$F_{-R_S|-R_S \geq v}(x) = \begin{cases} 1 - \frac{1}{\alpha} F_{R_S}(-x) & \text{if } x \geq v, \\ 0 & \text{if } x < v, \end{cases} \tag{3.12}$$

where $F_{R_S}(\cdot)$ denotes the cumulative distribution function of the random variable R_S.

The density function of $-R_S| - R_S \geq v$ is given by

$$f_{-R_S|-R_S \geq v}(x) = \begin{cases} \frac{1}{\alpha} f_{R_S}(-x) & \text{if } x \geq v, \\ 0 & \text{if } x < v, \end{cases} \tag{3.13}$$

where $f_{R_S}(\cdot)$ denotes the probability density function of R_S. R_S is normally distributed with expected value μ_S and standard deviation σ_S.

Proof. We have already shown in Result II.8 that $R_S \sim N(\mu_S, \sigma_S^2)$. Then this result follows directly from Result III.1. $\qquad \square$

With this result, it is possible to find a formula for the TCE constraint for the surplus return.

Result III.8 (The TCE constraint for the surplus return)

Let $R_B \sim N(\mu_B, \sigma_B^2)$, $R_E \sim N(\mu_E, \sigma_E^2)$, and $R_L \sim N(\mu_L, \sigma_L^2)$. Then the TCE constraint for the surplus return is given by

$$\mu_S \geq \frac{e^{-\frac{1}{2} z_{1-\alpha}^2}}{\alpha \sqrt{2\pi}} \cdot \sigma_S - m. \tag{3.14}$$

An equivalent expression for this constraint is given by the two inequalities

$$\mu_A \geq \frac{\mu_L - m}{F_0}, \tag{3.15}$$

$$a\sigma_A^2 + b\mu_A + c\mu_A^2 \leq d, \tag{3.16}$$

where

$$a = \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \right)^2 F_0^2, \tag{3.17}$$

$$b = \frac{2F_0\sigma_L\sigma_B\rho_{BL} \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \right)^2}{\mu_E - \mu_B} - \frac{2F_0\sigma_L\sigma_E\rho_{EL} \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \right)^2}{\mu_E - \mu_B}$$

$$+ 2F_0\mu_L - 2F_0 m, \tag{3.18}$$

$$c = -F_0^2, \tag{3.19}$$

$$d = \mu_L^2 + m^2 - 2\mu_L m - \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \right)^2 \sigma_L^2 - \frac{2F_0 \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \right)^2 \sigma_L\sigma_E\rho_{EL}\mu_B}{\mu_E - \mu_B}$$

$$+ \frac{2F_0\sigma_L\sigma_B\rho_{BL} \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \right)^2 \mu_E}{\mu_E - \mu_B}. \tag{3.20}$$

The formulas for ρ_{EL} and ρ_{BL} are given by (2.30) and (2.31).

Proof. The first part follows directly from Result III.2 since R_S is normally distributed.

The proof for the second part is identical to the one for Result II.8; we just have

to replace z_α by $-\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}$ and m by $-m$.

We have to express the TCE constraint in terms of μ_A and σ_A, i.e. we have to

replace μ_S and σ_S in 3.14 by 2.27 and 2.28, respectively:

$$\mu_S \geq -m + \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\sigma_S$$

$$\Leftrightarrow \quad F_0\mu_A - \mu_L \geq -m + \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \sqrt{F_0^2\sigma_A^2 + \sigma_L^2 - 2F_0\sigma_A\sigma_L\rho_{AL}}$$

$$\Leftrightarrow \quad \sqrt{F_0^2\sigma_A^2 + \sigma_L^2 - 2F_0\sigma_A\sigma_L\rho_{AL}} \leq \frac{\mu_L - m - F_0\mu_A}{-\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}} \quad \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} > 0\right)$$

$$\Leftrightarrow \quad \frac{\mu_L - m - F_0\mu_A}{-\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}} \geq 0 \text{ and}$$

$$\left(\sqrt{F_0^2\sigma_A^2 + \sigma_L^2 - 2F_0\sigma_A\sigma_L\rho_{AL}}\right)^2 \leq \left(\frac{\mu_L - m - F_0\mu_A}{-\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}}\right)^2$$

$$\overset{-\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}<0}{\Longleftrightarrow} \quad \mu_A \geq \frac{\mu_L - m}{F_0} \text{ and}$$

$$\left(\sqrt{F_0^2\sigma_A^2 + \sigma_L^2 - 2F_0\sigma_A\sigma_L\rho_{AL}}\right)^2 \leq \left(\frac{\mu_L - m - F_0\mu_A}{-\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}}\right)^2$$

$$\Leftrightarrow \quad \mu_A \geq \frac{\mu_L - m}{F_0} \text{ and}$$

$$\underbrace{\left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 (F_0^2\sigma_A^2 + \sigma_L^2 - 2F_0\sigma_A\sigma_L\rho_{AL}) \leq (\mu_L - m - F_0\mu_A)^2}_{(*)}$$

It holds:

$$\mu_A = w\mu_E + (1-w)\mu_B = w(\mu_E - \mu_B) + \mu_B$$

$$\Rightarrow \quad w = \frac{\mu_A - \mu_B}{\mu_E - \mu_B}, \quad (1-w) = \frac{\mu_E - \mu_A}{\mu_E - \mu_B}$$

Now we find an expression for ρ_{AL} with known variables. According to 2.29, we have

$$\rho_{AL} = \frac{w\sigma_E\rho_{El} + (1-w)\sigma_B\rho_{BL}}{\sigma_A}$$

$$\Rightarrow \quad \rho_{AL} = \left(\frac{\mu_A - \mu_B}{\mu_E - \mu_B}\sigma_E\rho_{EL} + \frac{\mu_E - \mu_A}{\mu_E - \mu_B}\sigma_B\rho_{BL}\right) \cdot \frac{1}{\sigma_A}$$

After plugging this result in $(*)$, we get

$$\left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 \left[F_0^2\sigma_A^2 + \sigma_L^2 - 2F_0\sigma_L\left(\frac{\mu_A-\mu_B}{\mu_E-\mu_B}\sigma_E\rho_{EL} + \frac{\mu_E-\mu_A}{\mu_E-\mu_B}\sigma_B\rho_{BL}\right)\right]$$

$$\leq (\mu_L - m - F_0\mu_A)^2$$

$$\Leftrightarrow \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 F_0^2\sigma_A^2 + \mu_A\left(\frac{2F_0\sigma_L\sigma_B\rho_{BL}\left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2}{\mu_E-\mu_B}\right.$$

$$-\frac{2F_0\sigma_L\sigma_E\rho_{EL}\left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2}{\mu_E-\mu_B} + 2F_0(\mu_L-m)\left.\right) - F_0^2\mu_A^2$$

$$\leq \mu_L^2 + m^2 - 2\mu_L m - \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2\sigma_L^2 - \frac{2F_0\left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2\sigma_L\sigma_E\rho_{EL}\mu_B}{\mu_E-\mu_B}$$

$$+\frac{2F_0\sigma_L\sigma_B\rho_{BL}\left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2\mu_E}{\mu_E-\mu_B}$$

$$\Leftrightarrow a\sigma_A^2 + b\mu_A + c\mu_A^2 \leq d$$

$$\square$$

Example 7

For this example, we need the values in table 5. Then σ_L, ρ_{BL}, and ρ_{EL} can be

calculated by using (2.23), (2.31), and (2.30):

$$\sigma_L = 0.171, \ \rho_{BL} = 0.913, \ \rho_{EL} = 0.409 \ \textit{(cf. Leibowitz et al., 1996, p.86)}$$

For α, F_0, and m, we choose $\alpha = 0.10$, $F_0 = 1$, and $m = 0.20$. With these values, we get

$$a = 3.0922, \ b = -0.3666, \ c = -1, \ d = -0.0189$$

Table 5
Values for Example 7

	Expected Return	Standard Deviation of Returns	Correlation with Bonds	Correlation with Stocks
Assets:				
Stocks	13.0%	17.00%	0.35	1.00
Bonds	8.0%	6.96%	1.00	0.35
Liabilities:				
Basic Schedule	8.0%	15.00%	1.00	
Noise	0.00%	7.00%		0.25

Source: Leibowitz et al. (1996, p.86)

and

$$\frac{\mu_L - m}{F_0} = -0.12.$$

Now we use (2.40), (2.41), and (2.34) to write down the TCE constraint for the

μ_A-σ_A-*coordinate system:*

$$\mu_A \geq \sqrt{0.0525 + 3.0922\sigma_A^2} - 0.1833 \; or$$

$$\mu_A \leq -\sqrt{0.0525 + 3.0922\sigma_A^2} - 0.1833$$

and

$$\mu_A \geq -0.12$$

This TCE constraint is illustrated in Figure 18, p. 99.

We can use Result III.5 to analyze what happens if we change α and m since R_S is

a return with the demanded properties. In figure 18, p. 99, the relevant (upper) curve

Figure 18
The TCE Constraint for the Surplus Return
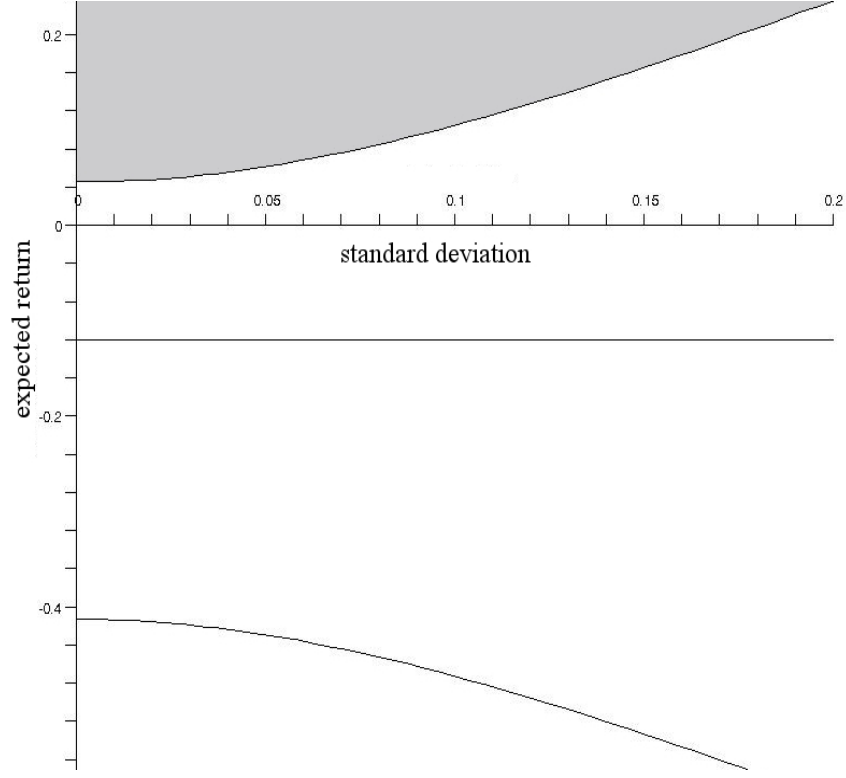
will move upwards if the TCE constraint is made stricter by decreasing α and/or m.

The next result gives a condition for the TCE constraint for the surplus return in terms of w.

Result III.9 (The TCE Constraint for the Surplus Return for w)

Let

$$a = F_0^2 \left[\mu_E^2 + \mu_B^2 - 2\mu_E\mu_B - \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \right)^2 (\sigma_E^2 + \sigma_B^2 - 2\sigma_E\sigma_B\rho_{EB}) \right], \quad (3.21)$$

$$b = F_0 \left[2\mu_B\mu_L + 2m\mu_E - 2m\mu_B + 2F_0\mu_E\mu_B - 2\mu_B^2 F_0 - 2\mu_L\mu_E \right.$$

$$\left. + \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \right)^2 (2F_0\sigma_B^2 - 2F_0\sigma_E\sigma_B\rho_{EB} + 2\sigma_L(\sigma_E\rho_{EL} - \sigma_B\rho_{BL})) \right], \quad (3.22)$$

$$c = \mu_L^2 + m^2 + \mu_B^2 F_0^2 - 2\mu_L m - 2\mu_B F_0\mu_L + 2mF_0\mu_B$$

$$- \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \right)^2 (F_0^2\sigma_B^2 + \sigma_L^2 - 2F_0\sigma_L\sigma_B\rho_{BL}). \quad (3.23)$$

Then the TCE constraint for the surplus return is equivalent to the following conditions

if $\mu_S \geq -m$:

- *If $a > 0$, then $w \in ((-\infty, w_1] \bigcap [0,1]) \bigcup ([w_2, +\infty) \bigcap [0,1])$, where*

  $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and $w_1 \leq w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, then

  $w \in [0,1]$.

- *If $a < 0$, then $w \in [w_1, w_2] \bigcap [0,1]$, where $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and $w_1 \leq w_2$. If the*

  root $\sqrt{b^2 - 4ac}$ has no real solution, then there does not exist a w that satisfies the

  TCE constraint for the surplus return.

- *If $a = 0$ and $b < 0$, then $w \in (-\infty, -\frac{c}{b}] \bigcap [0,1]$.*

- *If $a = 0$ and $b > 0$, then $w \in [-\frac{c}{b}, +\infty) \bigcap [0,1]$.*

- *If $a = 0$, $b = 0$ and $c \geq 0$, then $w \in [0,1]$.*

- *If $a = 0$, $b = 0$ and $c < 0$, then there does not exist a w that satisfies the TCE*

  constraint for the surplus return.

Proof. Again, the proof is the same as the one for the surplus return shortfall

constraint for Result II.9 if we replace z_α by $-\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}$ and m by $-m$. In the proof for

Result III.8 we showed that

$$\mu_S \geq -m + \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\sigma_S$$

$$\Leftrightarrow \quad \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 (F_0^2\sigma_A^2 + \sigma_L^2 - 2F_0\sigma_A\sigma_L\rho_{AL}) \leq (\mu_L - m - F_0\mu_A)^2$$

Since we assume that $\mu_S = F_0\mu_A - \mu_L \geq -m$, the second condition $\mu_A \geq \frac{\mu_L - m}{F_0}$ is

always true. Then

$$\left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 (F_0^2\sigma_A^2 + \sigma_L^2 - 2F_0\sigma_A\sigma_L\rho_{AL}) \leq (\mu_L - m - F_0\mu_A)^2$$

$$\Leftrightarrow \quad \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 \left[F_0^2(w^2\sigma_E^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\sigma_E\sigma_B\rho_{EB}) + \sigma_L^2\right.$$

$$\left. -2F_0\sigma_L(w\sigma_E\rho_{EL} + (1-w)\sigma_B\rho_{EB})\right]$$

$$\leq \quad \mu_L^2 + m^2 + w^2\mu_E^2F_0^2 + (1-w)^2\mu_B^2F_0^2 - 2\mu_Lm - 2w\mu_L\mu_EF_0$$

$$-2(1-w)\mu_BF_0\mu_L + 2mF_0\mu_Ew + 2mF_0(1-w)\mu_B + 2F_0^2w(1-w)\mu_E\mu_B$$

$$\Leftrightarrow \quad F_0^2\left[\mu_E^2 + \mu_B^2 - 2\mu_E\mu_B - \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 (\sigma_E^2 + \sigma_B^2 - 2\sigma_E\sigma_B\rho_{EB})\right] \cdot w^2$$

$$+F_0\left[2\mu_B\mu_L + 2m\mu_E - 2m\mu_B + 2F_0\mu_E\mu_B - 2\mu_B^2F_0 - 2\mu_L\mu_E\right.$$

$$\left. + \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 (2F_0\sigma_B^2 - 2F_0\sigma_E\sigma_B\rho_{EB} + 2\sigma_L(\sigma_E\rho_{EL} - \sigma_B\rho_{BL}))\right] \cdot w$$

$$+\mu_L^2 + m^2 + \mu_B^2F_0^2 - 2\mu_Lm - 2\mu_BF_0\mu_L + 2mF_0\mu_B$$

$$-\left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 (F_0^2\sigma_B^2 + \sigma_L^2 - 2F_0\sigma_L\sigma_B\rho_{BL}) \geq 0$$

$$\Leftrightarrow \quad aw^2 + bw + c \geq 0$$

If $a > 0$, the function $f(w) = aw^2 + bw + c$ is a parabola which is opened to the top.

The nulls of this function are $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $w_1 < w_2$. That is why

$f(w) \geq 0 \Leftrightarrow w \leq w_1$ or $w \geq w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, the

inequality $aw^2 + bw + c \geq 0$ is true for all w. If $a < 0$, the function $f(w) = aw^2 + bw + c$

is a parabola which is opened to the bottom. The nulls of this function are

$w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $w_1 < w_2$. That is why $f(w) \geq 0 \Leftrightarrow w_1 \leq w \leq w_2$. If the root

$\sqrt{b^2 - 4ac}$ has no real solution, the inequality $aw^2 + bw + c \geq 0$ is false for all w. If

$a = 0$ and $b > 0$, this function reduces to a straight line, i.e. $f(w) \geq 0 \Leftrightarrow w \geq -\frac{c}{b}$. If

$a = 0$ and $b < 0$, we get $f(w) \geq 0 \Leftrightarrow w \leq -\frac{c}{b}$. If $a = 0$, $b = 0$ and $c \geq 0$, then $f(w) \geq 0$,

otherwise $f(w) < 0$. This completes the proof. $\qquad \square$

The Tail Conditional Expectation Constraint for the Relative Return

In this section, we will give two results for the TCE constraint of the relative

return that correspond to the ones for the relative return shortfall constraint. It is quite

intuitive that we can transfer these results for the TCE since this worked for the asset

return and the surplus return. However, it is not obvious that the TCE constraint for

the asset return and the surplus return are special cases of the TCE constraint for the

relative return: The TCE is a conditional expectation where the condition depends on

the Value-at-Risk of the random variables. This will be discussed in the subsection

"The TCE Constraint for the Asset Return and the Surplus Return as Special Cases of

the TCE Constraint for the Relative Return".

Again, we use the same notations and assumptions for the underlying variables as for the relative return shortfall constraint. At first we start with the result for the distribution of $-R_D| - R_D \geq VaR_\alpha(R_D)$.

Result III.10 (The Distribution of $-R_D| - R_D \geq VaR_\alpha(R_D)$)

Let $R_B \sim N(\mu_B, \sigma_B^2)$, $R_E \sim N(\mu_E, \sigma_E^2)$, $R_b \sim N(\mu_B, \sigma_b^2)$ (we assume $\mu_B = \mu_b$), and let $v = VaR_\alpha(R_D)$ where $R_D = R_A - R_a$. Then the cumulative distribution function of $-R_D| - R_D \geq v$ is given by

$$F_{-R_D|-R_D \geq v}(x) = \begin{cases} 1 - \frac{1}{\alpha}F_{R_D}(-x) & \text{if } x \geq v, \\ 0 & \text{if } x < v, \end{cases} \qquad (3.24)$$

where $F_{R_D}(\cdot)$ denotes the cumulative distribution function of the random variable R_D. The density function of $-R_D| - R_D \geq v$ is given by

$$f_{-R_D|-R_D \geq v}(x) = \begin{cases} \frac{1}{\alpha}f_{R_D}(-x) & \text{if } x \geq v, \\ 0 & \text{if } x < v, \end{cases} \qquad (3.25)$$

where $f_{R_D}(\cdot)$ denotes the probability density function of R_D. R_D is normally distributed with expected value μ_D and standard deviation σ_D.

Proof. We have already shown in Result II.10 that $R_D \sim N(\mu_D, \sigma_D^2)$. Then the result follows directly from Result III.1. $\qquad \square$

The following result gives a formula for the TCE constraint of the relative return.

Result III.11 (The TCE constraint for the relative return)

Let $R_B \sim N(\mu_B, \sigma_B^2)$, $R_E \sim N(\mu_E, \sigma_E^2)$, and $R_b \sim N(\mu_B, \sigma_b^2)$. Then the TCE constraint

for the relative return is given by

$$\mu_D \geq \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \sigma_D - m. \tag{3.26}$$

Proof. The result follows directly from Result III.2 since R_D is normally distributed. $\square$

Again, we can apply Result III.5 by replacing R by R_D to see what happens if m or α is changed.

It is useful to have the following version of the TCE constraint for the relative return for w_A:

Result III.12 (The TCE Constraint for the Relative Return for w_A)

Let $R_E \sim N(\mu_E, \sigma_E^2)$, $R_B \sim N(\mu_B, \sigma_B^2)$ and $R_b \sim N(\mu_B, \sigma_b^2)$. Let

$$a = (\mu_E - \mu_B)^2 - \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 \left[\sigma_E^2 + \sigma_B^2 - 2\sigma_E\sigma_B\rho_{EB}\right], \tag{3.27}$$

$$b = 2m(\mu_E - \mu_B) - 2w_a(\mu_E - \mu_B)^2 - \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 \left[-2w_a\sigma_E^2 - 2\sigma_B^2\right.$$

$$\left. +2(1+w_a)\sigma_E\sigma_B\rho_{EB} - 2(1-w_a)\sigma_E\sigma_b\rho_{EB} + 2(1-w_a)\sigma_B\sigma_b\right], \tag{3.28}$$

$$c = m^2 - 2mw_a(\mu_E - \mu_B) + w_a^2(\mu_E - \mu_B)^2 - \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 \left[w_a^2\sigma_E^2\right.$$

$$\left. +\sigma_B^2 + (1-w_a)^2\sigma_b^2 - 2w_a\sigma_E\sigma_B\rho_{EB} + 2w_a(1-w_a)\sigma_E\sigma_b\rho_{EB}\right.$$

$$\left. -2(1-w_a)\sigma_B\sigma_b\right]. \tag{3.29}$$

Then the relative return shortfall constraint is equivalent to the following conditions if

$$\mu_D \geq -m:$$

- If $a > 0$, then $w \in ((-\infty, w_1] \bigcap [0, 1]) \bigcup ([w_2, +\infty) \bigcap [0, 1])$, where

  $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and $w_1 \leq w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, then

  $w \in [0, 1]$.

- If $a < 0$, then $w \in [w_1, w_2] \bigcap [0, 1]$, where $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ and $w_1 \leq w_2$. If the

  root $\sqrt{b^2 - 4ac}$ has no real solution, then there does not exist a w that satisfies the

  TCE constraint for the relative return.

- If $a = 0$ and $b < 0$, then $w \in (-\infty, -\frac{c}{b}] \bigcap [0, 1]$.

- If $a = 0$ and $b > 0$, then $w \in [-\frac{c}{b}, +\infty) \bigcap [0, 1]$.

- If $a = 0$, $b = 0$ and $c \geq 0$, then $w \in [0, 1]$.

- If $a = 0$, $b = 0$ and $c < 0$, then there does not exist a w that satisfies the TCE

  constraint for the relative return.

Proof. Again, the proof is the same as the one for the relative return shortfall

constraint for Result II.11 if we replace z_α by $-\frac{e^{-\frac{1}{2} z_{1-\alpha}^2}}{\alpha \sqrt{2\pi}}$ and m by $-m$.

We can plug (2.48) for μ_D and (2.49) for σ_D in the inequality (3.26). Since

$\mu_D \geq -m$ and $-\frac{e^{-\frac{1}{2} z_{1-\alpha}^2}}{\alpha \sqrt{2\pi}} < 0$, we get:

$$\mu_D \geq -m + \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\sigma_D$$

$$\Leftrightarrow \quad [(w_A - w_a)(\mu_E - \mu_B) - m]^2$$

$$\geq \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 \left[(w_A - w_a)^2\sigma_E^2(1 - \rho_{EB}^2)\right.$$

$$\left.+[(w_A - w_a)\sigma_E\rho_{EB} + (1 - w_A)\sigma_B - (1 - w_a)\sigma_b]^2\right]$$

$$\Leftrightarrow \quad \left[(\mu_E - \mu_B)^2 - \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 \left[\sigma_E^2 + \sigma_B^2 - 2\sigma_E\sigma_B\rho_{EB}\right]\right] \cdot w_A^2$$

$$+ \left[2m(\mu_E - \mu_B) - 2w_a(\mu_E - \mu_B)^2 - \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 \left[-2w_a\sigma_E^2 - 2\sigma_B^2\right.\right.$$

$$\left.\left.+2(1 + w_a)\sigma_E\sigma_B\rho_{EB} - 2(1 - w_a)\sigma_E\sigma_b\rho_{EB} + 2(1 - w_a)\sigma_B\sigma_b]\right] \cdot w_A$$

$$+ \left[m^2 - 2mw_a(\mu_E - \mu_B) + w_a^2(\mu_E - \mu_B)^2 - \left(\frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}}\right)^2 \left[w_a^2\sigma_E^2\right.\right.$$

$$+\sigma_B^2 + (1 - w_a)^2\sigma_b^2 - 2w_a\sigma_E\sigma_B\rho_{EB} + 2w_a(1 - w_a)\sigma_E\sigma_b\rho_{EB}$$

$$\left.\left.-2(1 - w_a)\sigma_B\sigma_b]\right] \geq 0$$

$$\Leftrightarrow \quad aw_A^2 + bw_A + c \geq 0$$

If $a > 0$, the function $f(w) = aw^2 + bw + c$ is a parabola which is opened to the top. The nulls of this function are $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $w_1 < w_2$. That is why $f(w) \geq 0 \Leftrightarrow w \leq w_1$ or $w \geq w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, the inequality $aw^2 + bw + c \geq 0$ is true for all w. If $a < 0$, the function $f(w) = aw^2 + bw + c$ is a parabola which is opened to the bottom. The nulls of this function are $w_{1/2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where $w_1 < w_2$. That is why $f(w) \geq 0 \Leftrightarrow w_1 \leq w \leq w_2$. If the root $\sqrt{b^2 - 4ac}$ has no real solution, the inequality $aw^2 + bw + c \geq 0$ is false for all w. If

$a = 0$ and $b > 0$, this function reduces to a straight line, i.e. $f(w) \geq 0 \Leftrightarrow w \geq -\frac{c}{b}$. If

$a = 0$ and $b < 0$, we get $f(w) \geq 0 \Leftrightarrow w \leq -\frac{c}{b}$. If $a = 0$, $b = 0$ and $c \geq 0$, then $f(w) \geq 0$,

otherwise $f(w) < 0$. This completes the proof. $\qquad\qquad\qquad\qquad\qquad$ $\square$

The TCE Constraint for the Asset Return and the Surplus Return as Special Cases of the TCE Constraint for the Relative Return

In chapter II we showed that under certain conditions, the asset return shortfall constraint and the surplus return shortfall constraint are special cases of the relative return shortfall constraint. The same is true for the TCE constraint. The next result provides this relationship between the TCE constraint of the asset return and surplus return and the TCE constraint of the relative return.

Result III.13

Let $R_A \sim N(\mu_A, \sigma_A^2)$, $R_B \sim N(\mu_B, \sigma_B^2)$, $R_b \sim N(\mu_B, \sigma_b^2)$, $R_L \sim N(\mu_L, \sigma_L^2)$. Then the relationship between the TCE of the relative return and the TCE of the asset and the surplus return can be described by the following two statements:

1. *Assume that the asset portfolio is managed against a benchmark with $R_a = i^*$ (for example $i^* = 0.08$ for a one year treasury bill). Then the TCE constraint for the relative return is equivalent to a TCE constraint for the asset return with*

   $m^* = m - i^*$ and $\alpha^* = \alpha$.

2. *Assume that the benchmark is the pension fund liability and assume that the*

funding ratio is 1: $F_0 = 1$. Then the TCE constraint for the relative return is

equivalent to a TCE constraint for the surplus return with $m^* = m$ and $\alpha^* = \alpha$.

Proof. 1. In the proof for Result III.4, we showed that

$$E(-R_P | - R_P \geq v) = -\mu_P + \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \sigma_P.$$

After replacing the variables for R_P by the corresponding ones for R_A and R_D, we

get

$$E(-R_A | - R_A \geq v) = -\mu_A + \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \sigma_A, \tag{3.30}$$

$$E(-R_D | - R_D \geq v) = -\mu_D + \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \sigma_D. \tag{3.31}$$

Since $R_D = R_A - R_a = R_A - i^*$, we get

$$\mu_D = \mu_A - i^*, \quad \sigma_D = \sigma_A.$$

Thus

$$E(-R_D | - R_D \geq v)$$

$$\overset{(3.30)}{=} \quad -\mu_D + \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \sigma_D = -(\mu_A - i^*) + \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \sigma_A$$

$$= \quad -\mu_A + \frac{e^{-\frac{1}{2}z_{1-\alpha}^2}}{\alpha\sqrt{2\pi}} \cdot \sigma_A + i^*$$

$$= \quad E(-R_A | - R_A \geq v) + i^*$$

Therefore

$$E(-R_D | - R_D \geq v) \leq m$$

$$\Leftrightarrow \quad E(-R_A | - R_A \geq v) \leq m - i^*$$

2. $\quad R_D = R_A - R_a = R_A - R_L \stackrel{F_0=1}{=} R_S$

$\quad \Rightarrow E(-R_D| - R_D \geq v) = E(-R_S| - R_S \geq v)$

$\square$

The Tail Conditional Expectation Constraint for the Funding Ratio Return

In this section, we will derive a formula for the TCE constraint for the funding ratio return (FRR). As for the corresponding shortfall constraint, it is only possible to express this constraint as an implicit function in terms of μ_A and σ_A.

At first we need to know the distribution of the random variable

$-FRR| - FRR \geq VaR_\alpha(FRR).$

Result III.14 (The Distribution of $-FRR| - FRR \geq VaR_\alpha(FRR)$)

Let $v = VaR_\alpha(FRR)$. We assume that FRR has a continuous distribution. Then the cumulative distribution function of $-FRR| - FRR \geq VaR_\alpha(FRR)$ can be written as

$$F_{-FRR|-FRR \geq v}(x) = \begin{cases} 1 - \frac{1}{\alpha} F_{\frac{1+R_A}{1+R_L}}(1-x) & \text{if } 1 \geq x \geq v, \\ 0 & \text{if } x < v, \\ 1 & \text{otherwise.} \end{cases} \tag{3.32}$$

The probability density function is given by

$$f_{-FRR|-FRR \geq v}(x) = \begin{cases} \frac{1}{\alpha} f_{\frac{1+R_A}{1+R_L}}(1-x) & \text{if } 1 \geq x \geq v, \\ 0 & \text{otherwise.} \end{cases} \tag{3.33}$$

Proof. The cumulative distribution function can be derived by straight-forward

derivations:

$$F_{-FRR|-FRR \geq v}(x) = P\left(-FRR \leq x| -FRR \geq v\right)$$

$$= P\left(-\left(\frac{1+R_A}{1+R_L} - 1\right) \leq x \middle| -\left(\frac{1+R_A}{1+R_L} - 1\right) \geq v\right)$$

$$= \frac{P\left(v \leq 1 - \frac{1+R_A}{1+R_L} \leq x\right)}{P\left(1 - \frac{1+R_A}{1+R_L} \geq v\right)} = \frac{P\left(1 - x \leq \frac{1+R_A}{1+R_L} \leq 1 - v\right)}{P\left(\frac{1+R_A}{1+R_L} \leq 1 - v\right)}$$

$$= \frac{F_{\frac{1+R_A}{1+R_L}}(1-v) - F_{\frac{1+R_A}{1+R_L}}(1-x)}{F_{\frac{1+R_A}{1+R_L}}(1-v)}$$

$$= \begin{cases} 1 - \frac{F_{\frac{1+R_A}{1+R_L}}(1-x)}{F_{\frac{1+R_A}{1+R_L}}(1-v)} & \text{if } 1 \geq x \geq v, \\ 0 & \text{if } x < v, \\ 1 & \text{otherwise} \end{cases}$$

It holds:

$$F_{\frac{1+R_A}{1+R_L}}(1-v) = P\left(\frac{1+R_A}{1+R_L} - 1 \leq -v\right) = P(FRR \leq -v) = \alpha \qquad (3.34)$$

Thus,

$$F_{-FRR|-FRR \geq v}(x) = \begin{cases} 1 - \frac{1}{\alpha} F_{\frac{1+R_A}{1+R_L}}(1-x) & \text{if } 1 \geq x \geq v, \\ 0 & \text{if } x < v, \\ 1 & \text{otherwise.} \end{cases}$$

The density function can be obtained by taking the derivative of the cumulative distribution function.

$$\begin{aligned} f_{-FRR|-FRR \geq v}(x) &= \frac{d}{dx} F_{-FRR|-FRR \geq v}(x) \\ &= \begin{cases} \frac{1}{\alpha} f_{\frac{1+R_A}{1+R_L}}(1-x) & \text{if } 1 \geq x \geq v, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$\square$

Now we can derive a formula for the TCE constraint for FRR.

Result III.15 (The TCE constraint for FRR)

Let $(1 + R_A, 1 + R_L)$ be bivariate log-normally distributed and let $(1 + \mu_A, \sigma_A^2)$ and

$(1 + \mu_L, \sigma_L^2)$ be the corresponding expected values and variances of $1 + R_A$ and $1 + R_L$,

respectively. Let $v = VaR_\alpha(FRR)$. Then the TCE constraint for the funding ratio

return is given by the following inequality:

$$
\Phi \left(z_\alpha - \sqrt{\ln \left(\frac{((1 + \mu_A)^2 + \sigma_A^2)((1 + \mu_L)^2 + \sigma_L^2)}{[(1 + \mu_A)(1 + \mu_L) + \sigma_A \sigma_L \rho_{AL}]^2} \right)} \right)
$$
$$
\geq \frac{\alpha \left[(1 + \mu_A)(1 + \mu_L) + \sigma_A \sigma_L \rho_{AL} \right] (1 + \mu_L)^2}{(1 + \mu_A)^2 \left[(1 + \mu_L)^2 + \sigma_L^2 \right]} (1 - m), \tag{3.35}
$$

where μ_A and σ_A^2 are the expected return and the variance of

$R_A = w\mu_E + (1 - w)\mu_B,\ w \in [0, 1]$ and ρ_{AL} is given by (2.29):

$$
\rho_{AL} = \frac{w\sigma_E \rho_{EL} + (1 - w)\sigma_B \rho_{BL}}{\sigma_A}.
$$

Proof. Since $1 + R_A$ and $1 + R_E$ are bivariate log-normally distributed, it holds:

$$
\ln \left(\frac{1 + R_A}{1 + R_L} \right) = \ln(1 + R_A) - \ln(1 + R_L) \sim N(\mu, \sigma^2).
$$

(remark: We will derive formulas for μ and σ later.)

Thus, $\frac{1 + R_A}{1 + R_L}$ is log-normally distributed with parameters μ and σ. Now we can

calculate the expected value of $-FRR| - FRR \geq v$:

$$E(-FRR| - FRR \geq v) = \int_{-\infty}^{+\infty} x f_{-FRR|-FRR \geq v}(x) dx$$

$$\overset{(3.35)}{=} \int_{v}^{1} \frac{x}{\alpha} f_{\frac{1+R_A}{1+R_L}}(1-x) dx$$

$$= \frac{1}{\alpha} \int_{v}^{1} \frac{x}{\sqrt{2\pi}\sigma(1-x)} e^{-\frac{1}{2}\left(\frac{\ln(1-x)-\mu}{\sigma}\right)^2} dx \quad [u = 1-x, \; du = -dx]$$

$$= \frac{-1}{\alpha} \int_{1-v}^{0} \frac{1-u}{\sqrt{2\pi}\sigma u} e^{-\frac{1}{2}\left(\frac{\ln(u)-\mu}{\sigma}\right)^2} du$$

$$= \frac{1}{\alpha} \left[\int_{0}^{1-v} \frac{1}{\sqrt{2\pi}\sigma u} e^{-\frac{1}{2}\left(\frac{\ln(u)-\mu}{\sigma}\right)^2} du - \int_{0}^{1-v} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\ln(u)-\mu}{\sigma}\right)^2} du \right]$$

$$= \frac{1}{\alpha} \left[F_{\frac{1+R_A}{1+R_L}}(1-v) - \int_{0}^{1-v} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{\ln(u)-\mu}{\sigma}\right)^2} du \right]$$

$$= \left[\text{substitution} : x = \frac{\ln(u)-\mu}{\sigma}, \; dx = \frac{1}{u\sigma} du \right]$$

$$\overset{(3.34)}{=} 1 - \frac{1}{\alpha} \int_{-\infty}^{\frac{\ln(1-v)-\mu}{\sigma}} \frac{e^{\sigma x + \mu}}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$= 1 - \frac{1}{\alpha} \int_{-\infty}^{\frac{\ln(1-v)-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left((x-\sigma)^2 - \sigma^2 - 2\mu\right)} dx$$

$$= [\text{substitution} : u = x - \sigma, \; du = dx]$$

$$= 1 - \frac{e^{\frac{1}{2}\sigma^2 + \mu}}{\alpha} \int_{-\infty}^{\frac{\ln(1-v)-\mu}{\sigma} - \sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$$= 1 - \frac{e^{\frac{1}{2}\sigma^2 + \mu}}{\alpha} \Phi\left(\frac{\ln(1-v)-\mu}{\sigma} - \sigma\right) \tag{3.36}$$

It holds:

$$\alpha \overset{(3.34)}{=} P\left(\frac{1+R_A}{1+R_L} \leq 1-v\right) = P(\ln(1+R_A) - \ln(1+R_L) \leq \ln(1-v))$$

$$= P\left(\frac{\ln(1+R_A) - \ln(1+R_L) - \mu}{\sigma} \leq \frac{\ln(1-v)-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{\ln(1-v)-\mu}{\sigma}\right)$$

Thus,

$$\frac{\ln(1-v) - \mu}{\sigma} = z_\alpha$$

$$\Leftrightarrow \quad ln(1-v) = z_\alpha \sigma + \mu$$

Now we can continue to calculate $E(-FRR| - FRR \geq v)$:

$$E(-FRR| - FRR \geq v) \overset{(3.36)}{=} 1 - \frac{e^{\frac{1}{2}\sigma^2 + \mu}}{\alpha} \Phi\left(\frac{\ln(1-v) - \mu}{\sigma} - \sigma\right)$$

$$= 1 - \frac{e^{\frac{1}{2}\sigma^2 + \mu}}{\alpha} \Phi\left(\frac{z_\alpha \sigma + \mu - \mu}{\sigma} - \sigma\right)$$

$$= 1 - \frac{e^{\frac{1}{2}\sigma^2 + \mu}}{\alpha} \Phi\left(z_\alpha - \sigma\right) \tag{3.37}$$

We can calculate σ^2 and μ by using Result II.14. Let (μ_A^*, σ_A^*) and (μ_L^*, σ_L^*) be the expected values and standard deviations of $\ln(1 + R_A)$ and $\ln(1 + R_L)$. Then

$$\sigma^2 = \text{Var}\left(\ln(1 + R_A) - \ln(1 + R_L)\right) = \sigma_A^{*2} + \sigma_L^{*2} - 2\sigma_{AL}^*$$

$$= \ln\left(\frac{((1 + \mu_A)^2 + \sigma_A^2)((1 + \mu_L)^2 + \sigma_L^2)}{[(1 + \mu_A)(1 + \mu_L) + \sigma_A \sigma_L \rho_{AL}]^2}\right)$$

$$\mu = \mu_A^* - \mu_L^* = \ln\left(\frac{(1 + \mu_A)^2 \sqrt{(1 + \mu_L)^2 + \sigma_L^2}}{(1 + \mu_L)^2 \sqrt{(1 + \mu_A)^2 + \sigma_A^2}}\right)$$

After plugging this in (3.37), we get for the TCE constraint for FRR:

$$E(-FRR| - FRR \geq v) \leq m$$

$$\Leftrightarrow \quad \Phi\left(z_\alpha - \sqrt{\ln\left(\frac{((1 + \mu_A)^2 + \sigma_A^2)((1 + \mu_L)^2 + \sigma_L^2)}{[(1 + \mu_A)(1 + \mu_L) + \sigma_A \sigma_L \rho_{AL}]^2}\right)}\right)$$

$$\geq \frac{\alpha\left[(1 + \mu_A)(1 + \mu_L) + \sigma_A \sigma_L \rho_{AL}\right](1 + \mu_L)^2}{(1 + \mu_A)^2\left[(1 + \mu_L)^2 + \sigma_L^2\right]}(1 - m)$$

$$\square$$

Example 8

For this example, we need the following values of the underlying variables:

<div style="text-align:center">

Table 6
Values for Example 8

</div>

	Expected Return	Standard Deviation of Returns	Correlation with Bonds	Correlation with Stocks
Assets:				
Stocks	13.0%	17.00%	0.35	1.00
Bonds	8.0%	6.96%	1.00	0.35
Liabilities:				
Basic Schedule	8.0%	15.00%	1.00	
Noise	0.00%	7.00%		0.25

<div style="text-align:center">

Source: Leibowitz et al. (1996, p.86)

</div>

In Example 3, the values for σ_L, ρ_{BL} and ρ_{EL} have already been calculated:
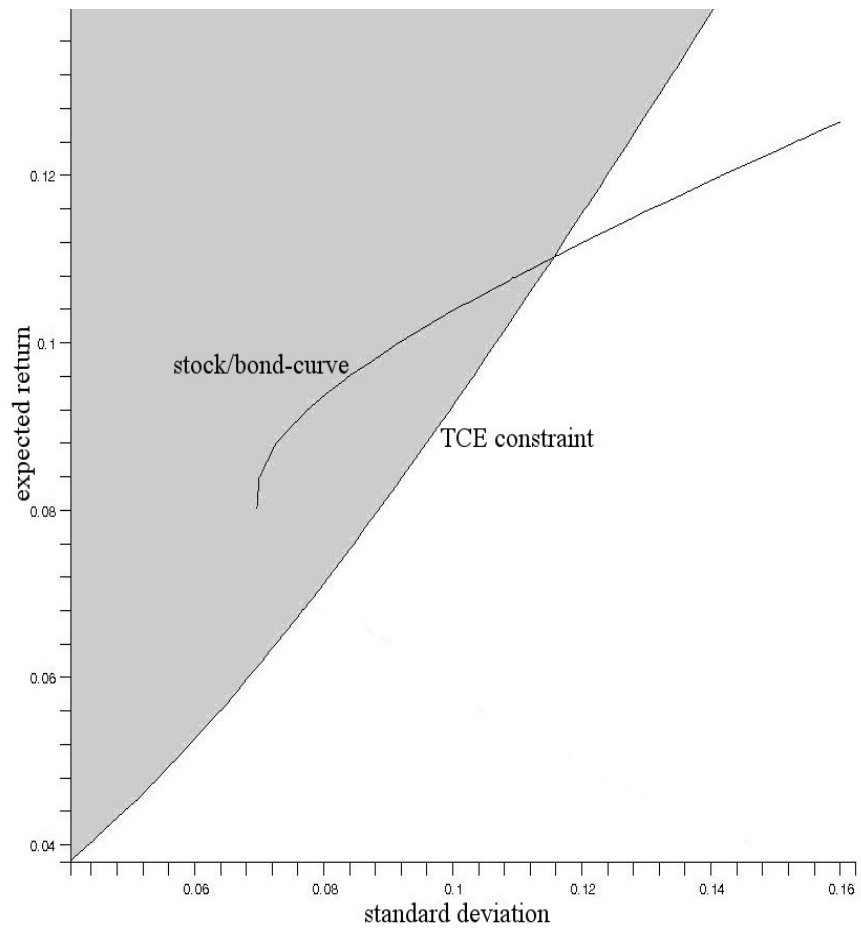
$$\sigma_L = 0.171, \ \rho_{BL} = 0.913, \ \rho_{EL} = 0.409.$$

In addition, we choose $\alpha = 0.10$ and $m = 0.17$. After plugging these values in (3.35), we get the following inequality for the TCE of the funding ratio return:

$$\Phi\left(-1.28 - \sqrt{\ln\left(\frac{1.1956((1+\mu_A)^2 + \sigma_A^2)}{(1.1005\mu_A + 1.0892)^2}\right)}\right) \geq 0.0810 \cdot \frac{1.1005\mu_A + 1.0892}{(1+\mu_A)^2}.$$

This TCE constraint for FRR is illustrated in figure 19, page 115.

Figure 19

The TCE Constraint for FRR

CHAPTER IV

THE OPTIMIZATION PROBLEM

The Optimization Problem

In this chapter, we want to discuss a strategy which a pension fund manager can use in order to decide what percentage to invest in stocks and what percentage to invest in bonds. For this purpose, we will use the results from chapter II and chapter III.

We showed that a pension fund manager can control the risk of a pension fund by using one or more shortfall or tail conditional expectation constraints. But these restrictions just reduced the fund manager's potential choices. We haven't analyzed yet which portfolio the manager should choose after this reduction of choices.

With the constraints in chapter II and chapter III, the pension fund manager can adjust the risk of the pension fund to a level which he/she is willing to accept. After setting the risk level of the pension fund, it is desirable to get an expected return as high as possible (under the restriction of the shortfall constraints and the TCE constraints).

One possibility is now to focus on the asset return and to maximize the expected value of this return. However, the fund manager could also try to maximize the expected value of the surplus return, the relative return, or the funding ratio return.

We denote the optimization problem by (P). Then, (P) can be formulated as follows:

$$(P) \begin{cases} \mu_A = w\mu_E + (1-w)\mu_B \rightarrow max \\[2ex] \text{under the constraints} \\ P(R_A < m) \le \alpha_A \qquad\qquad\qquad \text{asset return shortfall constraint} \\ P(R_S < m) \le \alpha_S \qquad\qquad\qquad \text{surplus return shortfall constraint} \\ P(R_D < m) \le \alpha_D \qquad\qquad\qquad \text{relative return shortfall constraint} \\ P(FRR < m) \le \alpha_{FRR} \qquad\qquad \text{funding ratio return shortfall} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{constraint} \\[2ex] E(-R_A| - R_A \ge VaR_{\alpha_A}(R_A)) \le m_A \quad TCE \text{ constraint for the asset return} \\ E(-R_S| - R_S \ge VaR_{\alpha_S}(R_S)) \le m_S \quad TCE \text{ constraint for the surplus} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{return} \\[1ex] E(-R_D| - R_D \ge VaR_{\alpha_D}(R_D)) \le m_D \quad TCE \text{ constraint for the relative} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{return} \\[1ex] E(-FRR| - FRR \ge VaR_{\alpha_{FRR}}(FRR)) \quad TCE \text{ constraint for the funding} \\ \qquad\qquad\qquad\qquad \le m_{FRR} \qquad\qquad \text{ratio return} \end{cases}$$

This does not mean that all shortfall and TCE constraints have to be used. A subset of these constraints can also be selected to restrict the maximization of μ_A. If the formulas from the preceding chapters are used, we should think about the different assumptions for the underlying distributions: We assumed normally distributed returns for the shortfall and TCE constraints for the asset, the surplus, and the relative return, but we used log-normally distributed returns for the shortfall and TCE constraint for the funding ratio return. Because of these different assumptions, these shortfall and TCE constraints shouldn't be combined.

If we assume that $\mu_E \geq \mu_B$, the following holds:

$$\mu_A = w\mu_E + (1-w)\mu_B = w\underbrace{(\mu_E - \mu_B)}_{\geq 0} + \mu_B \rightarrow max$$

$$\Leftrightarrow$$

$$w \rightarrow max. \tag{4.1}$$

We note that maximizing the expected return of the surplus return and the relative return would lead to the same optimization problem since

$$\mu_S = F_0\mu_A - \mu_L = F_0(w\underbrace{(\mu_E - \mu_B)}_{\geq 0} + \mu_B) - \mu_L \rightarrow max$$

$$\Leftrightarrow$$

$$w \rightarrow max$$

and

$$\mu_D = \mu_A - \mu_a = w\underbrace{(\mu_E - \mu_B)}_{\geq 0} + \mu_B - \mu_a \rightarrow max$$

$$\Leftrightarrow$$

$$w \rightarrow max.$$

Thus, (P) can be written as

$$(P) \begin{cases} w \to max \\ \\ \text{under the constraints} \\ P(R_A < m) \leq \alpha_A \\ P(R_S < m) \leq \alpha_S \\ P(R_D < m) \leq \alpha_D \\ P(FRR < m) \leq \alpha_{FRR} \\ E(-R_A| - R_A \geq VaR_{\alpha_A}(R_A)) \leq m_A \\ E(-R_S| - R_S \geq VaR_{\alpha_S}(R_S)) \leq m_S \\ E(-R_D| - R_D \geq VaR_{\alpha_D}(R_D)) \leq m_D \\ E(-FRR| - FRR \geq VaR_{\alpha_{FRR}}(FRR)) \leq m_{FRR} \end{cases}$$

The Solution for the Optimization Problem

The Solution for the Shortfall and TCE Constraints for the Asset, the Surplus, and the Relative Return

In this section, we will solve the optimization problem (P). For this purpose, we focus on the shortfall and TCE constraints for the asset, the surplus, and the relative return first. As we have noted in the preceding section, the shortfall and TCE constraints for the funding ratio return should be considered separately.

The goal is now to make w as large as possible given that a selection of the six shortfall and TCE constraints is satisfied. It is very useful that we have derived equivalent conditions for the corresponding shortfall and TCE constraints for w in Result II.3, Result II.9, Result II.11, Result III.6, Result III.9 and Result III.12 because they tell us which w fulfill the corresponding constraints. The intersection of all the sets that these results give us represents the set of w that satisfy all constraints. Now

we only have to find the maximum of this set and this maximum is the solution for the optimization problem.

This procedure can be summarized in the following algorithm:

1. Calculate the restrictions for all imposed shortfall and TCE constraints using Result II.3, Result II.9, Result II.11, Result III.6, Result III.9 and Result III.12.

2. Calculate the intersection of the resulting sets. This intersection is denoted by Ω.

3. Calculate $\omega = \max(\Omega)$. Then ω is the solution for the optimization problem.

Example 9

For this example, we need the following values of the underlying variables:

Table 7
Values for Example 9

	Expected Return	Standard Deviation of Returns	Correlation with Bonds	Correlation with Stocks
Assets:				
Stocks	13.0%	17.00%	0.35	1.00
Bonds	8.0%	6.96%	1.00	0.35
Liabilities:				
Basic Schedule	8.0%	15.00%	1.00	
Noise	0.00%	7.00%		0.25

Source: Leibowitz et al. (1996, p.86)

Then, σ_L, ρ_{BL} and ρ_{EL} can be calculated by using (2.23), (2.31) and (2.30):

$$\sigma_L = 0.171, \ \rho_{BL} = 0.913, \ \rho_{EL} = 0.409. \ \textit{(cf. Leibowitz et al., 1996, p.86)}$$

We consider a pension fund manager who wants to control the risk of a pension fund by using a surplus return shortfall constraint and a TCE constraint for the asset return.

Then the three steps of the optimization algorithm are:

1. *The fund manager decides to use $\alpha = 0.10$ and $m = -0.15$ for the surplus return shortfall constraint. In order to use Result II.9, we need to calculate the variables a, b, and c with the formulas given in this result:*

$$a = -0.0392, \quad b = 0.0206, \quad c = 0.0023.$$

   Therefore

$$w_{1/2} = 0.2634 \pm 0.3564 \Rightarrow w \in [0, 0.8039].$$

   For the TCE constraint of the asset return, the pension fund manager chooses $\alpha = 0.10$ and $m = 0.10$. Then the values for a, b, and c for Result III.6 are

$$a = -0.0762, \quad b = 0.0223, \quad c = 0.0174.$$

   Therefore

$$w_{1/2} = 0.1466 \pm 0.5000 \Rightarrow w \in [0, 0.6466].$$

2. $\Omega = [0, 0.8039] \bigcap [0, 0.6466] = [0, 0.6466]$

3. $\omega = \max(\Omega) = \max([0, 0.6466]) = 0.6466$

Thus, the pension fund manager should choose the portfolio that consists of 65% stocks and 35% bonds. The expected value of this portfolio is
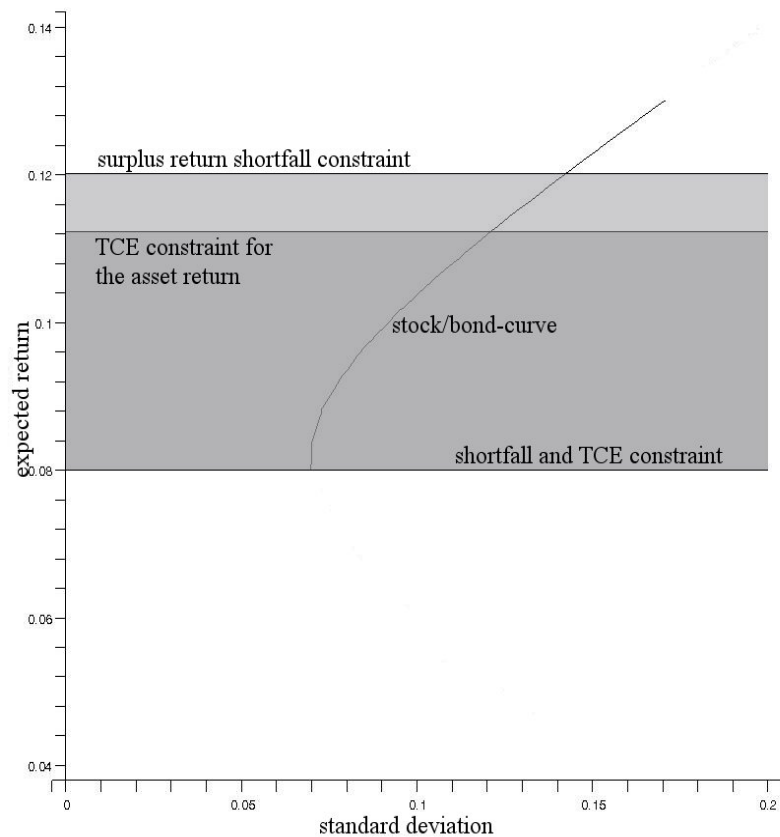
$$\mu_P = w\mu_E + (1 - w)\mu_B = 11.25\%$$

and the standard deviation is (cf. (2.3))

$$\sigma_P = \sqrt{w^2\sigma_E^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\sigma_E\sigma_B\rho_{EB}} = 12.12\%.$$

The surplus return shortfall constraint and the TCE constraint for the asset return together with the stock/bond-curve are illustrated in figure 20, page 122.

Figure 20
The Surplus Return Shortfall Constraint, the TCE Constraint for the Asset Return and the Stock/Bond-Curve
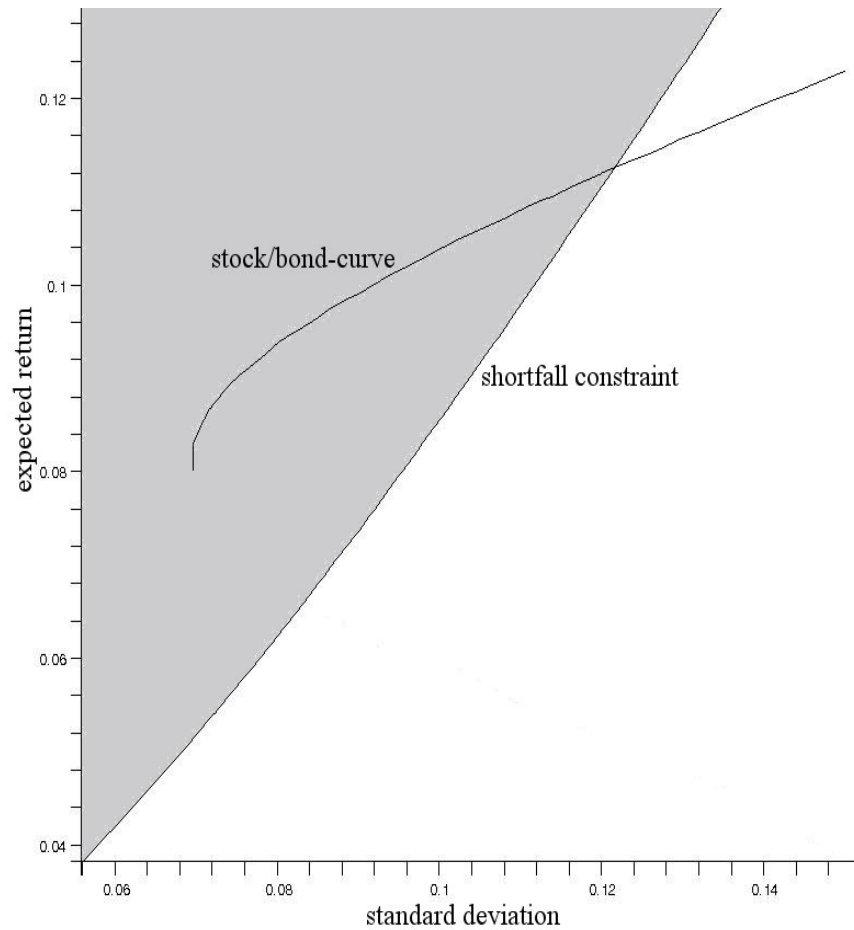
The Solution for the Shortfall and TCE Constraint for the Funding Ratio Return

Now we will focus on the shortfall and TCE constraint for the funding ratio return. For these two constraints, we didn't derive formulas in terms of w in chapter II and chapter III. The reason is that in Result II.13 and Result III.15, the shortfall and TCE constraints are given in an implicit form which does not allow for solving these inequalities for w. That is why we need a different approach to solve the optimization problem.

For this purpose, we consider the following figure 21, page 124.

We can see that the intersection point of the FRR shortfall constraint and the stock/bond-curve determines that part of the stock/bond-curve which we can choose portfolios from: If the stock/bond-curve lies to the left of the shortfall constraint curve, all the portfolios represented by this part of the stock/bond-curve satisfy the shortfall constraint; otherwise (i.e. if the stock/bond-curve lies to the right of the shortfall constraint curve), these portfolios do not fulfill the constraint. Thus, the idea is now to determine all the intersection points of the stock/bond-curve and the TCE or shortfall constraint curve in the first quadrant. From these intersection points, the μ_A-value is taken to form consecutive intervals in terms of μ_A. For each interval, we check if the stock/bond-curve lies to the left or to the right of the constraint curve. After that, we will exclude all the intervals where the TCE or shortfall constraint is not satisfied. The union of the remaining intervals represents the set of all the portfolios we can choose from (in terms of μ_A). By using $\mu_A = w\mu_E + (1-w)\mu_B$ which we can solve for w

Figure 21
The FRR Shortfall Constraint and the Stock/Bond-Curve

$(w = \frac{\mu_A - \mu_B}{\mu_E - \mu_B})$, we can determine the values of w that fulfill the TCE or shortfall constraint.

This procedure can be summarized in the following algorithm:

1. Determine all the intersection points of the stock/bond-curve and the shortfall and TCE constraint curve in the first quadrant. The number of these points is denoted by n.

2. Arrange the μ_A-values of these points in ascending order:

$\mu_A^1 < \mu_A^2 < ... < \mu_A^{n-1} < \mu_A^n$. From these values, form the intervals

$[\mu_B, \mu_A^1], (\mu_A^1, \mu_A^2], ..., (\mu_A^{n-1}, \mu_A^n], (\mu_A^n, \mu_E]$. Again, we assume that $\mu_B \leq \mu_E$.

3. Determine whether the stock/bond-curve lies to the left or to the right of the

    TCE or shortfall constraint curve. Exclude all the intervals for which the

    stock/bond-curve lies to the right of at least one of the constraint curves.

4. Let $\mathcal{M}$ be the union of all the remaining intervals. Then $\mu_A^* = \sup(\mathcal{M})$ is the

    solution to the optimization problem (more precise: $\omega = \frac{\mu_A^* - \mu_B}{\mu_E - \mu_B}$ is the solution to

    the optimization problem).

In order to verify that this algorithm is valid, we need to show that the

constraints for the funding ratio return can be represented by a continuous curve in the

μ_A-σ_A-coordinate system and that all the portfolios to the left of this curve satisfy the

constraints. The next two results provide conditions that assure that the algorithm can

be applied.

Result IV.1 (Properties of the Funding Ratio Return Shortfall Constraint)

The following holds for the funding ratio return shortfall constraint:

1. *Assume that for $\mu_A = 0$, the funding ratio shortfall constraint (2.61) has a*

solution $\sigma_A^{sol} \geq 0$ *(if we replace $\leq$ by $=$ in the inequality), i.e.*

$$\ln \left[\frac{(m+1)\sigma_A(1+\mu_L)^2}{\sqrt{(1+\mu_L)^2 + \sigma_L^2}} \right]$$
$$= z_\alpha \sqrt{\ln \left(\frac{\sigma_A^2 \left[(1+\mu_L)^2 + \sigma_L^2 \right]}{\left[(1+\mu_L) - \frac{\mu_B}{\mu_E - \mu_B}\sigma_{EL} + \frac{\mu_E}{\mu_E - \mu_B}\sigma_{BL} \right]^2} \right)} \qquad (4.2)$$

has a solution $\sigma_A^{sol} \geq 0$ and assume that $\frac{\sigma_{EL} - \sigma_{BL}}{\mu_E - \mu_B} > 0$ and $\mu_L > 0$. Then there

exists a continuous increasing function $\mu_A : [\sigma_A^{sol}, +\infty) \to \mathbb{R}$ that represents the

funding ratio return shortfall constraint in the μ_A-σ_A-coordinate system. The same

is true if we assume that for $\sigma_A = 0$, (2.61) has a solution $\mu_A^{sol} \geq 0$ if we replace

$\leq$ by $=$ in the inequality. Then there exists a continuous increasing function

$\mu_A : [0, +\infty) \to \mathbb{R}$ that represents the funding ratio return shortfall constraint.

2. *All portfolios to the left of the graph of this function and to the right of the*

   μ_A-axis satisfy the funding ratio return shortfall constraint in the

   μ_A-σ_A-coordinate system. All other portfolios in the first quadrant do not fulfill

   the shortfall constraint.

Proof.   1. Let σ_A^{sol} be the solution for (4.2). We consider the equation (cf. the

   inequality (2.61))

$$\ln \left[\frac{(m+1)\sqrt{(1+\mu_A)^2 + \sigma_A^2}(1+\mu_L)^2}{\sqrt{(1+\mu_L)^2 + \sigma_L^2}(1+\mu_A)^2} \right]$$
$$= z_\alpha \sqrt{\ln \left(\frac{\left[(1+\mu_A)^2 + \sigma_A^2 \right] \left[(1+\mu_L)^2 + \sigma_L^2 \right]}{\left[(1+\mu_A)(1+\mu_L) + \frac{\mu_A - \mu_B}{\mu_E - \mu_B}\sigma_{EL} + \frac{\mu_E - \mu_A}{\mu_E - \mu_B}\sigma_{BL} \right]^2} \right)}$$

$$\Leftrightarrow \quad \ln\left[\frac{(m+1)\sqrt{(1+\mu_A)^2+\sigma_A^2}(1+\mu_L)^2}{\sqrt{(1+\mu_L)^2+\sigma_L^2(1+\mu_A)^2}}\right]$$

$$-z_\alpha\sqrt{\ln\left(\frac{[(1+\mu_A)^2+\sigma_A^2]\,[(1+\mu_L)^2+\sigma_L^2]}{\left[(1+\mu_A)(1+\mu_L)+\frac{\mu_A-\mu_B}{\mu_E-\mu_B}\sigma_{EL}+\frac{\mu_E-\mu_A}{\mu_E-\mu_B}\sigma_{BL}\right]^2}\right)}=0$$

$$\Leftrightarrow \quad \ln\left[\frac{(m+1)\sqrt{\frac{1}{(1+\mu_A)^2}+\frac{\sigma_A^2}{(1+\mu_A)^4}}(1+\mu_L)^2}{\sqrt{(1+\mu_L)^2+\sigma_L^2}}\right]$$

$$-z_\alpha\sqrt{\ln\left(\frac{\left[(1+\frac{\sigma_A^2}{(1+\mu_A)^2}\right]\,[(1+\mu_L)^2+\sigma_L^2]}{\left[(1+\mu_L)+\frac{\mu_A}{1+\mu_A}\frac{\sigma_{EL}-\sigma_{BL}}{\mu_E-\mu_B}\right]^2}\right)}=0 \tag{4.3}$$

Let us consider the function

$$h(\mu_A,\sigma_A) \;=\; \ln\left[\frac{(m+1)\sqrt{\frac{1}{(1+\mu_A)^2}+\frac{\sigma_A^2}{(1+\mu_A)^4}}(1+\mu_L)^2}{\sqrt{(1+\mu_L)^2+\sigma_L^2}}\right]$$

$$-z_\alpha\sqrt{\ln\left(\frac{\left[(1+\frac{\sigma_A^2}{(1+\mu_A)^2}\right]\,[(1+\mu_L)^2+\sigma_L^2]}{\left[(1+\mu_L)+\frac{\mu_A}{1+\mu_A}\frac{\sigma_{EL}-\sigma_{BL}}{\mu_E-\mu_B}\right]^2}\right)}$$

If we increase σ_A, both summands increase since $-z_\alpha>0$ (we assume $\alpha<0.5$).

Therefore $h(\mu_A,\sigma_A)$ increases.

If we increase μ_A, both summands decrease because of $-z_\alpha>0$, $\frac{\sigma_{EL}-\sigma_{BL}}{\mu_E-\mu_B}>0$ and the following consideration:

$$\frac{d}{d\mu_A}\left(\frac{\mu_A}{1+\mu_A}\right)=\frac{1}{(1+\mu_A)^2}>0.$$

This means that $\frac{\mu_A}{1+\mu_A}$ increases if μ_A is increased. Therefore $h(\mu_A,\sigma_A)$ decreases.

In addition:

$$h(\mu_A,\sigma_A)\to+\infty \quad \text{for } \sigma_A\to+\infty$$

$$h(\mu_A,\sigma_A)\to-\infty \quad \text{for } \mu_A\to+\infty$$

From this consideration and the fact that $h(\cdot, \cdot)$ is continuous, we can conclude the following:

We assumed that $(\sigma_A^{sol}, 0)$ satisfies (4.3). If we increase μ_A, we have to increase σ_A^{sol} as well so that (4.3) is still fulfilled. The resulting σ_A is unique. Thus, for each $\mu_A > 0$ we can find a $\sigma_A > 0$ such that (4.3) is satisfied. The curve consisting of these points must be increasing and continuous and can be represented by a function $\mu_A : [\sigma_A^{sol}, +\infty) \to \mathbb{R}$ (because $h(\cdot, \cdot)$ is continuously differentiable and because of the theorem of implicit functions (cf. Heuser, 2000, p. 292)). For the second part, we can just replace the point $(\sigma_A^{sol}, 0)$ by $(0, \mu_A^{sol})$ and repeat the proof.

2. The inequality $h(\mu_A, \sigma_A) \leq 0$ is equivalent to (2.61) and $h(\cdot, \cdot)$ decreases if we decrease σ_A and increases if we increase σ_A. That is why all portfolios to the left of the curve and to the right of the μ_A-axis satisfy the funding ratio return shortfall constraint and all other portfolios in the first quadrant do not fulfill the shortfall constraint.

□

Result IV.2 (Properties of the TCE constraint for FRR)

The following holds for the TCE constraint for the funding ratio return:

1. *Assume that for $\mu_A = 0$, the TCE constraint for FRR (3.35) has a solution $\sigma_A^{sol} \geq 0$ (if we replace $\leq$ by $=$ in the inequality), i.e.*

$$\Phi\left(z_\alpha - \sqrt{\ln\left(\frac{\sigma_A^2((1+\mu_L)^2 + \sigma_L^2)}{[(1+\mu_L) + \sigma_A\sigma_L\rho_{AL}]^2}\right)}\right)$$
$$= \frac{\alpha\left[(1+\mu_L) + \sigma_A\sigma_L\rho_{AL}\right](1+\mu_L)^2}{[(1+\mu_L)^2 + \sigma_L^2]}(1-m) \qquad (4.4)$$

has a solution $\sigma_A^{sol} \geq 0$ and assume that $0 < \frac{\sigma_{EL} - \sigma_{BL}}{\mu_E - \mu_B} < 1$ and $\mu_L > 0$. Then there

exists a continuous increasing function $\mu_A : [\sigma_A^{sol}, +\infty) \to \mathbb{R}$ that represents the

TCE constraint for the funding ratio return in the μ_A-σ_A-coordinate system. The

same is true if we assume that for $\sigma_A = 0$, (3.35) has a solution $\mu_A^{sol} \geq 0$ if we

replace $\leq$ by $=$ in the inequality. Then there exists a continuous increasing

function $\mu_A : [0, +\infty) \to \mathbb{R}$ that represents the TCE constraint for the funding

ratio return in the μ_A-σ_A-coordinate system.

2. All portfolios to the left of the graph of this function and to the right of the

   μ_A-axis satisfy the TCE constraint for the funding ratio return. All other

   portfolios in the first quadrant do not fulfill the TCE constraint.

Proof. 1. Let σ_A^{sol} be the solution for (4.4). We consider the equation (cf. the

   inequality (3.35))

$$\Phi\left(z_\alpha - \sqrt{\ln\left(\frac{((1+\mu_A)^2 + \sigma_A^2)((1+\mu_L)^2 + \sigma_L^2)}{[(1+\mu_A)(1+\mu_L) + \sigma_A\sigma_L\rho_{AL}]^2}\right)}\right)$$
$$= \frac{\alpha\left[(1+\mu_A)(1+\mu_L) + \sigma_A\sigma_L\rho_{AL}\right](1+\mu_L)^2}{(1+\mu_A)^2\left[(1+\mu_L)^2 + \sigma_L^2\right]}(1-m)$$

$$\Leftrightarrow \; \Phi\left(z_\alpha - \sqrt{\ln\left(\frac{((1+\mu_A)^2 + \sigma_A^2)((1+\mu_L)^2 + \sigma_L^2)}{\left[(1+\mu_A)(1+\mu_L) + \frac{\mu_A - \mu_B}{\mu_E - \mu_B}\sigma_{EL} + \frac{\mu_E - \mu_A}{\mu_E - \mu_B}\sigma_{BL}\right]^2}\right)}\right)$$

$$-\frac{\alpha\left[(1+\mu_A)(1+\mu_L) + \frac{\mu_A - \mu_B}{\mu_E - \mu_B}\sigma_{EL} + \frac{\mu_E - \mu_A}{\mu_E - \mu_B}\sigma_{BL}\right](1+\mu_L)^2}{(1+\mu_A)^2\left[(1+\mu_L)^2 + \sigma_L^2\right]}(1-m) = 0$$

$$\Leftrightarrow \; \Phi\left(z_\alpha - \sqrt{\ln\left(\frac{\left(1 + \frac{\sigma_A^2}{(1+\mu_A)^2}\right)((1+\mu_L)^2 + \sigma_L^2)}{\left[(1+\mu_L) + \frac{\mu_A}{1+\mu_A}\frac{\sigma_{EL} - \sigma_{BL}}{\mu_E - \mu_B}\right]^2}\right)}\right)$$

$$-\frac{\alpha\left[(1+\mu_L) + \frac{\mu_A}{1+\mu_A}\frac{\sigma_{EL} - \sigma_{BL}}{\mu_E - \mu_B}\right](1+\mu_L)^2}{(1+\mu_A)\left[(1+\mu_L)^2 + \sigma_L^2\right]}(1-m) = 0 \qquad (4.5)$$

Let us consider the function

$$h(\mu_A, \sigma_A) \;=\; \Phi\left(z_\alpha - \sqrt{\ln\left(\frac{\left(1 + \frac{\sigma_A^2}{(1+\mu_A)^2}\right)((1+\mu_L)^2 + \sigma_L^2)}{\left[(1+\mu_L) + \frac{\mu_A}{1+\mu_A}\frac{\sigma_{EL} - \sigma_{BL}}{\mu_E - \mu_B}\right]^2}\right)}\right)$$

$$-\frac{\alpha\left[(1+\mu_L) + \frac{\mu_A}{1+\mu_A}\frac{\sigma_{EL} - \sigma_{BL}}{\mu_E - \mu_B}\right](1+\mu_L)^2}{(1+\mu_A)\left[(1+\mu_L)^2 + \sigma_L^2\right]}(1-m)$$

If we increase σ_A, $h(\mu_A, \sigma_A)$ decreases.

If we increase μ_A, we can see that the first summand increases. The second one increases as well because

$$\frac{d}{d\mu_A}\left(\frac{(1+\mu_L) + \frac{\mu_A}{1+\mu_A}\frac{\sigma_{EL} - \sigma_{BL}}{\mu_E - \mu_B}}{1+\mu_A}\right)$$

$$= \; -\frac{1+\mu_L}{(1+\mu_A)^2} + \frac{[1+\mu_A - 2\mu_A]\frac{\sigma_{EL} - \sigma_{BL}}{\mu_E - \mu_B}}{(1+\mu_A)^3}$$

$$= \; \frac{\overbrace{-1 + \frac{\sigma_{EL} - \sigma_{BL}}{\mu_E - \mu_B}}^{<0} - \mu_L\mu_A - \mu_L - \mu_A\left(1 + \frac{\sigma_{EL} - \sigma_{BL}}{\mu_E - \mu_B}\right)}{(1+\mu_A)^3} < 0$$

This means that $\frac{(1+\mu_L) + \frac{\mu_A}{1+\mu_A}\frac{\sigma_{EL} - \sigma_{BL}}{\mu_E - \mu_B}}{1+\mu_A}$ decreases if μ_A is increased. Therefore

$h(\mu_A, \sigma_A)$ increases.

In addition:

For $\sigma_A \to +\infty$:

$$h(\mu_A, \sigma_A) \to -\frac{\alpha \left[(1 + \mu_L) + \frac{\mu_A}{1+\mu_A}\frac{\sigma_{EL}-\sigma_{BL}}{\mu_E-\mu_B}\right](1+\mu_L)^2}{(1+\mu_A)\left[(1+\mu_L)^2 + \sigma_L^2\right]}(1-m) < 0$$

For $\mu_A \to +\infty$:

$$h(\mu_A, \sigma_A) \to \Phi\left(z_\alpha - \sqrt{\ln\left(\frac{((1+\mu_L)^2 + \sigma_L^2)}{\left[(1+\mu_L) + \frac{\sigma_{EL}-\sigma_{BL}}{\mu_E-\mu_B}\right]^2}\right)}\right) > 0$$

From this consideration and the fact that $h(\cdot, \cdot)$ is continuous, we can conclude

the following:

We assumed that $(\sigma_A^{sol}, 0)$ satisfies (4.5). If we increase μ_A, we have to increase

σ_A^{sol} as well so that (4.5) is still fulfilled. The resulting σ_A is unique. Thus, for

each $\mu_A > 0$ we can find a $\sigma_A > 0$ such that (4.5) is satisfied. The curve consisting

of these points must be increasing and continuous and can be represented by a

function $\mu_A : [\sigma_A^{sol}, +\infty) \to \mathbb{R}$ (because $h(\cdot, \cdot)$ is continuously differentiable and

because of the theorem of implicit functions (cf. Heuser, 2000, p. 292)).

For the second part, we can just replace the point $(\sigma_A^{sol}, 0)$ by $(0, \mu_A^{sol})$ and repeat

the proof.

2. The inequality $h(\mu_A, \sigma_A) \geq 0$ is equivalent to (2.61) and $h(\cdot, \cdot)$ increases if we

decrease σ_A and decreases if we increase σ_A. That is why all portfolios to the left

of the curve and to the right of the μ_A-axis satisfy the TCE constraint for the

funding ratio return and all other portfolios in the first quadrant do not fulfill the

TCE constraint.

□

Example 10

For this example, we need the following values of the underlying variables:

Table 8
Values for Example 10

	Expected Return	Standard Deviation of Returns	Correlation with Bonds	Correlation with Stocks
Assets:				
Stocks	13.0%	17.00%	0.35	1.00
Bonds	8.0%	6.96%	1.00	0.35
Liabilities:				
Basic Schedule	8.0%	15.00%	1.00	
Noise	0.00%	7.00%		0.25

Source: Leibowitz et al. (1996, p.86)

In Example 3, the values for σ_L, ρ_{BL}, and ρ_{EL} have already been calculated:

$$\sigma_L = 0.171, \ \rho_{BL} = 0.913, \ \rho_{EL} = 0.409.$$

We consider a pension fund manager who wants to control the risk of a pension fund by using a shortfall constraint and a TCE constraint for the funding ratio return. We assume that the fund manager chooses $\alpha = 0.0275$ and $m = -0.2$ for the shortfall constraint (cf. Example 5) and $\alpha = 0.10$ and $m = 0.17$ for the TCE constraint (cf. Example 8). Then we get the following inequalities for the funding ratio return shortfall

constraint

$$\ln\left(0.8533\frac{\sqrt{(1+\mu_A)^2+\sigma_A^2}}{(1+\mu_A)^2}\right) \leq 1.96\sqrt{\ln\left(\frac{1.1956\left[(1+\mu_A)^2+\sigma_A^2\right]}{(1.1005\mu_A+1.0892)^2}\right)}$$

and for the TCE constraint

$$\Phi\left(-1.28-\sqrt{\ln\left(\frac{1.1956((1+\mu_A)^2+\sigma_A^2)}{(1.1005\mu_A+1.0892)^2}\right)}\right) \geq 0.0810 \cdot \frac{1.1005\mu_A+1.0892}{(1+\mu_A)^2}.$$

The stock/bond-curve is given by

$$\mu_P = \pm\sqrt{0.0982\sigma_P^2 - 0.0005} + 0.0814. \quad \text{(cf. Example 2)}$$

This is illustrated in figure 22, page 134.

It holds $0 < \frac{\sigma_{EL}-\sigma_{BL}}{\mu_E-\mu_B} = 0.0205$ and $\mu_L = 0.08 > 0$. For this example, we will not give a formal proof that there exists a solution $\mu_A^{sol} \geq 0$ for the equations in Result IV.1 and Result IV.2. Figure 22, page 134, indicates that this is the case because both curves of the constraints intersect with the σ_A-axis. Then the four steps of the optimization algorithm are:

1. *From figure 22, we can find the intersection points $(0.114, 0.110)$ and $(0.120, 0.112)$.*

2. *Since $0.110 < 0.112$, the corresponding intervals are $(0.080, 0.110]$, $(0.110, 0.112]$ and $(0.112, 0.130]$.*

3. *For the intervals $(0.110, 0.112]$ and $(0.112, 0.130]$, the stock/bond-curve lies to the right of at least one of the shortfall constraint curve and the TCE constraint curve and therefore, these intervals are excluded.*

Figure 22
The FRR Shortfall Constraint, the TCE Constraint for FRR and the Stock/Bond-Curve
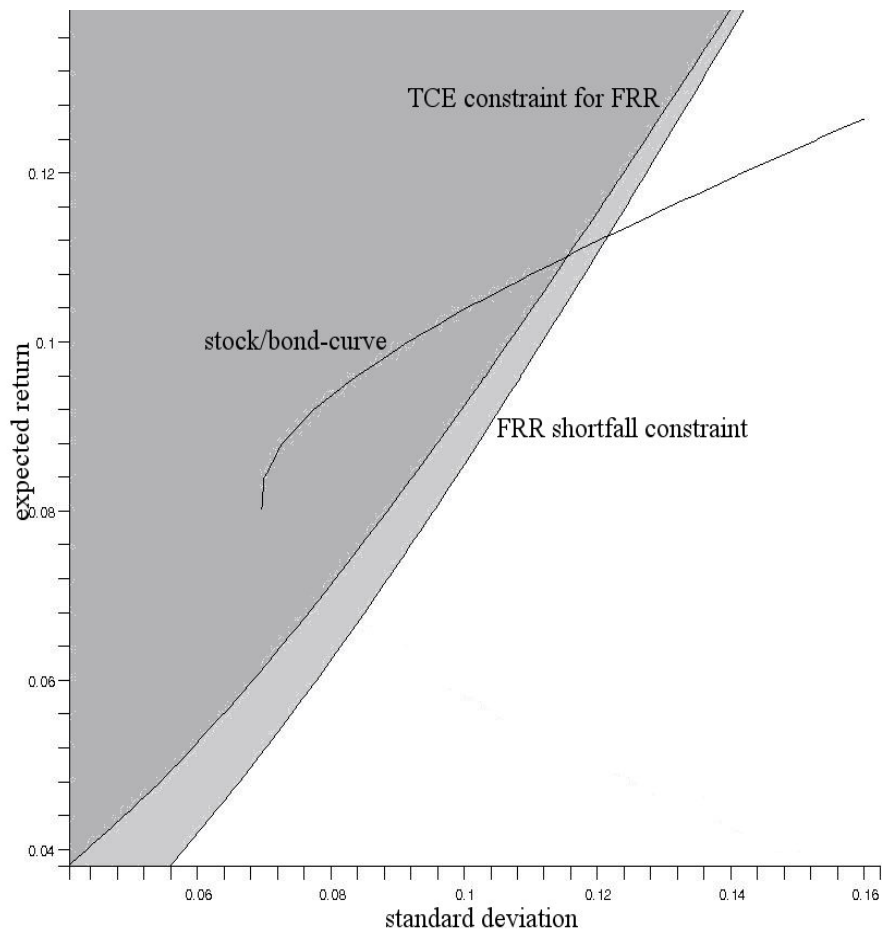
4. $\mathcal{M} = [0.080, 0.110)$. Then $\mu_A^* = \sup[0.08, 0.110) = 0.110$ and $\omega = \frac{0.110-0.08}{0.13-0.08} = 0.60$.

Thus, the pension fund manager should invest 60% in stocks and 40% in bonds. The expected value of this portfolio is

$$\mu_P = w\mu_E + (1-w)\mu_B = 11.00\%$$

and the standard deviation is (cf. (2.3))

$$\sigma_P = \sqrt{w^2\sigma_E^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\sigma_E\sigma_B\rho_{EB}} = 11.47\%.$$

CHAPTER V

SUMMARY AND CONCLUSIONS

In this thesis, we have provided tools which can support a pension fund manager in managing a pension fund. We have seen how a pension fund manager can use these tools to control the risk level of the fund and how he/she can come to an investment decision based on the risk level he/she is willing to accept.

In chapter I, a brief overview of the different types of pension plans was given. We decided to consider a defined benefit plan with a fund since for this type of plans, the employer bears the investment risk. Companies often hire a pension fund manager to manage the pension fund.

In chapter II, an overview of risk measures was given. Based on shortfall probability as risk measure, we developed the concept of shortfall constraints. With a shortfall constraint, the pension fund manager can restrict the shortfall probability of falling below a certain return such that this probability will not exceed a value that the manager regards as critical. Under the assumption of normally distributed returns and log-normally distributed returns, respectively, we derived formulas for the shortfall constraints of the asset return, the surplus return, the relative return, and the funding ratio return. We also discussed what happens if the fund manager changes the free parameters "shortfall constraint probability" and "minimum acceptable return".

In chapter III, we noticed that the shortfall probability has the disadvantage of failing to measure to which extent the return falls below the minimum acceptable return given that the return has already fallen below this minimum return. That is why we decided to work with another risk measure: tail conditional expectation. Again, formulas for the corresponding tail conditional expectation constraints of the asset return, the surplus return, the relative return, and the funding ratio return were derived under the assumption of normally and log-normally distributed returns.

Finally in chapter IV, an investment strategy was provided based on the shortfall and tail conditional expectation constraints given in chapter II and chapter III.

For all the calculations in this thesis, we assumed that there is only one stock and one bond to choose from. This reduced the investment decision as to what percentage to invest in the stock and what percentage to invest in the bond. Of course, the pension fund manager can invest in a large variety of different stocks and bonds and other types of investment in reality. Therefore, an extension of the discussed model to one with several stocks and bonds is encouraged.

Another aspect that should be considered critically is the assumption of normally and log-normally distributed returns. Although we got some valuable insights into the shortfall and tail conditional expectation constraint concept, we just approximated the reality by using these distribution assumptions. Therefore, from a practical point of view, one could try to find optimal portfolios by simulation rather than modeling the returns with the normal and log-normal distribution.

Finally, the concept of shortfall and TCE constraints is not restricted to controlling the risk of a pension fund. Any investor who needs to manage assets against liabilities can use these constraints to adjust the risk level he/she is willing to accept.

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