Extreme Preference and Distortion Risk Measures on Tail Regions with Nonparametric Inference Method

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Outline

- Learn risk exposure of a financial entity under the occurrence of some other financial variables' extreme scenarios.
  Example:
  - Measure the systemic risk of the expected loss on some financial equity return conditional on the occurrence of an extreme loss in the aggregated return of the financial sector;
  - Measure the relative risk of individual financial entity to some benchmark as a sensitive monitoring index of the market co-movement.
- Traditional risk measures such as Value-at-Risk (VaR) and Expected Shortfall (ES) cannot capture the risk exposure caused by the co-movement of multiple financial variables.
- Use quantitative definition of the exposure on the tail regions to model the dependence of some financial variables

- No formal definition/principle about what is the standard of risk measure when considering the co-movement in financial/insurance industry $\Rightarrow$ hard to extend
- How to consider the choice under the extreme risk which quantitatively evaluates the preference of tail risk ? $\Rightarrow$ Use utility theory and theoretical aspects of distortion functions when facing high risk events
- Asymptotic theory of the risk measures on tail regions

Our Contribution

- Compare risks under some extreme scenarios with different risk levels.
- Consider the risk exposure on the tail regions where a benchmark loss is included in the modeling of extreme scenarios.
- Explore a limiting space of the above risks on tail regions as the extreme scenarios go to extremes.
- Explore distortion risk measure on tail regions and its asymptotic property. Establish asymptotic results using Extreme Value Theory and tail copulas.
- Nonparametric estimator has been proposed and its asymptotic normality is proved.

Preference Relationship Components

- X: the risk random variable
- $\bar{F}_X$: survival distribution
- Γ: a given collection of survival distribution $\bar{F}_X$
- $\mathbb{V}$: the set of all non-negative risk random variables, defined on space Γ. $\mathbb{V}$ always represents the loss.
- (X, Y): a pair of random losses with continuous joint distribution function $F(x, y)$ and marginal distributions $F_1(x) = F(x, \infty)$ and $F_2(y) = F(\infty, y)$.
- $Q_i = F_i^{-1}, i = 1, 2$: the inverse functions of marginal distributions.
- α: the level of the risk, $\alpha \in [0, 1]$

The preference relationship

- The relations, indifference $\sim$ and preference $\succeq$, for any two (survival) distributions $\bar{F}, \bar{G}$ in Γ are

$$\begin{aligned} \bar{F} \sim \bar{G} \quad &\text{if and only if} \quad \bar{F}(t) = \bar{G}(t), \quad \forall\, t \geq 0. \\ \bar{F} \succeq \bar{G} \quad &\text{if and only if} \quad \bar{F}(t) \geq \bar{G}(t), \quad \forall\, t \geq 0. \end{aligned} \tag{1}$$

- The preference $\succeq$ on $\mathbb{V}$ is based on distortion function g is

$$X_1 \succeq X_2 \quad \text{if and only if} \quad \int_{-\infty}^{\infty} g(\bar{F}_{X_1}(t))dt \geq \int_{-\infty}^{\infty} g(\bar{F}_{X_2}(t))dt,$$

The indifference $\sim$ for X_1, X_2 in $\mathbb{V}$ is

$$X_1 \sim X_2 \quad \text{if and only if} \quad \int_{-\infty}^{\infty} g(\bar{F}_{X_1}(t))dt = \int_{-\infty}^{\infty} g(\bar{F}_{X_2}(t))dt.$$

Preference relations on the tail risks

- Suppose Y is a benchmark whose extreme scenarios are of interest.
- Consider the univariate case where extreme scenarios of Y are included as conditional events, and the associated preference relation is called extreme preference when the extreme scenarios go to extreme.
- Technique: Extreme preference depends on the transformation of three spaces in the following sections (from $\Gamma \times \mathbb{C}$, to Γ', then to Γ_0).

Two extensions:

- First, define the scenarios of Y as an event $S(Y, \alpha)$ where α represents the level of the risk.
  Typically, choose $S(Y, \alpha) = \{Y > Q_2(1-\alpha)\}$ as the extreme scenarios.
- Second, truncate the non-tail risk of X below the threshold $Q_1(1-\alpha)$ and consider the ratio of this truncated variable and the threshold.

Space Construction

- Extreme scenarios event $S(Y, \alpha) = \{Y > Q_2(1 - \alpha)\}$
- Truncate the non-tail risk of X below the threshold $Q_1(1 - \alpha)$ and consider the ratio of this truncated variable and the threshold.
- Construct space $\mathbb{V}$ is the collection, or a subcollection, of X having distribution F with extreme index $\gamma > 0$, which satisfies

$$\lim_{t \to \infty} \frac{\bar{F}(tx)}{\bar{F}(t)} = x^{-1/\gamma}, \quad x > 0. \tag{2}$$

- Γ: the collection of all survival distributions of X

- $\mathbb{C}$: a collection of (survival) copulas of (X, Y), we may say "given benchmark Y" and "given the survival copula of (X, Y)" equivalently in the following as the scenarios of Y is $\{Y > Q_2(1-\alpha)\}$.
- Consider the conditional survival distribution

$$\mathbb{P}\left(\frac{(X-Q_1(1-\alpha))_+}{Q_1(1-\alpha)} > t \,\middle|\, Y > Q_2(1-\alpha)\right) = \begin{cases} \frac{C(\alpha s_\alpha((1+t)^{-1/\gamma}),\alpha)}{\alpha} & t \geq 0 \\ 1 & t < 0 \end{cases}$$

where $a_+ = \max(a, 0)$ for any real value a, b and $s_\alpha(x) = \frac{\bar{F}(Q_1(1-\alpha)x^{-\gamma})}{\alpha}$ such that $s_\alpha(0) = 0$, $s_\alpha(1) = 1$ and $s_\alpha(\infty) = 1/\alpha$.

Define space Γ_α and its elements G_α (Fix α)

- Get G_α by a pair $\bar{F}$ and C such that

$$G_\alpha(x) = \begin{cases} \frac{C(\alpha s_\alpha(x^{-1/\gamma}),\alpha)}{\alpha} & x \geq 1 \\ 1 & 0 \leq x < 1 \end{cases} \tag{3}$$

- $G_\alpha \in \Gamma_\alpha$ is induced by a pair of $\bar{F}$ and C.

- combine $\Gamma' = \cup_\alpha \Gamma_\alpha$ and $\mathbb{V}' = \cup_\alpha \mathbb{V}_\alpha$ and consider the preference relations on them.
- for any $G_{1\alpha_1}, G_{2\alpha_2}$

$$\begin{aligned} G_{1\alpha_1} \sim G_{2\alpha_2} \quad &\text{if and only if} \quad G_{1\alpha_1}(t) = G_{2\alpha_2}(t), \quad \forall\, t \geq 0. \\ G_{1\alpha_1} \succ G_{2\alpha_2} \quad &\text{if and only if} \quad G_{1\alpha_1}(t) \geq G_{2\alpha_2}(t), \quad \forall\, t \geq 0. \end{aligned} \tag{4}$$

Axiom 1 (Neutrality) Let Y be a given loss, X_1 and X_2 belong to $\mathbb{V}$ with respective levels α_1, α_2 and functions $G_{1\alpha_1}$ and $G_{2\alpha_2}$ in Γ' . Then,

$$G_{1\alpha_1} \sim G_{2\alpha_2} \qquad \Longrightarrow \qquad X_1|(Y,\alpha_1) \sim X_2|(Y,\alpha_2) \quad \text{on } \mathbb{V}'$$

where $X|(Y,\alpha)$ means the loss X given the dependence with Y and level α.

Axiom 2 (Complete weak order) $\succeq$ on Γ' is reflexive, transitive, and connected.

Axiom 3 (Continuity) $\succeq$ on Γ' is continuous with respect to L_1-norm.

Axiom 4 (Monotonicity) If for any $t \geq 1$, $G_1(t) \geq G_2(t)$, then $G_1 \succeq G_2$ on Γ'.

Axiom 5 (Dual Independence) If G, G', H belong to Γ' and ω is any real value satisfying $\omega \in [0,1]$, then $G \succeq G'$ implies $\omega G \oplus (1-\omega)H \succeq \omega G' \oplus (1-\omega)H$ where $\oplus$ is the harmonic convex combination operator.

Theorem 1

Theorem

For any fixed level $\alpha_1, \alpha_2 \in (0,1)$*, a preference relation* $\succeq$ *satisfies Axioms A1-A5 if and only if there exists a distortion function* g *such that given the loss* Y *(given any survival copulas respectively), for any* X_1, X_2 *in* $\mathbb{V}$ *with respective* $G_{1\alpha_1}, G_{2\alpha_2}$ *in* Γ'*,*

$$X_1|(Y,\alpha_1) \succeq X_2|(Y,\alpha_2) \quad \textit{on } \mathbb{V}' \quad \textit{if and only if}$$

$$\int_0^\infty g(G_{1\alpha_1}(t))dt \geq \int_0^\infty g(G_{2\alpha_2}(t))dt.$$

Moreover, any variable X *in* $\mathbb{V}$ *is equivalent to its utility on* $\mathbb{V}'$ *given* Y*,* α *and thus* $G_\alpha \in \Gamma'$ *so that*

$$X|(Y,\alpha) \sim [U_\alpha(X;Y);1]\,|(Y,\alpha) \quad \textit{on } \mathbb{V}'. \tag{5}$$

where $U_\alpha(X;Y) := \int_0^\infty g(G_\alpha(t))dt$ *and* $[x;p]$ *is a binary random variable taking* x *with probability* p *and* 0 *with probability* $1-p$*.*

Asymptotic limit of the extreme preference as $\alpha \downarrow 0$

Three questions:

1) Is there a collection of distributions which represent the limits of G_α in Γ' as $\alpha \downarrow 0$?
2) If there exists such a collection Γ_0 for which similar Axioms 1-5 hold as well?
3) If there exists such a collection Γ_0 satisfying some axioms similar to Axioms 1-5, what is the distortion function? Is it consistent with the g in Theorem 1?

Axiom 0' Suppose there exists a measure $R : (0,\infty)^2 \to [0,\infty)$ such that

$$\lim_{\alpha\downarrow 0} \frac{C(\alpha s_\alpha(x),\alpha)}{\alpha} = R(x,1), \quad x \geq 1.$$

Axiom 1' (Neutrality) Let Y be a given loss, X_1 and X_2 belong to $\mathbb{V}$ with respective G_1 and G_2 in Γ_0 . Then,

$$G_1 \sim G_2 \qquad \Longrightarrow \qquad X_1|(Y) \sim X_2|(Y) \quad \text{on } \mathbb{V}_0.$$

Axiom 2' (Complete weak order) $\succeq$ on Γ) is reflexive, transitive, and connected.

Axiom 3' (Continuity) $\succeq$ on Γ_0 is continuous with respect to L_1-norm.

Axiom 4' (Monotonicity) If for any $t \geq 1$, $G_1(t) \geq G_2(t)$, then $G_1 \succeq G_2$ on Γ_0.

Axiom 5' (Dual Independence) If G, G', H belong to Γ_0 and ω is a real value satisfying $\omega \in [0,1]$, then $G \succeq G'$ implies $\omega G \oplus (1-\omega)H \succeq \omega G' \oplus (1-\omega)H$ where $\oplus$ is the harmonic convex combination operator.

Theorem 2

Theorem

A preference relation $\succeq$ satisfies Axioms A0'-A5' if and only if there exists a distortion function g_0 such that given the loss Y (given any survival copulas respectively), for any X_1, X_2 in $\mathbb{V}$ with respective G_1, G_2 in Γ_0,

$$X_1|(Y) \succeq X_2|(Y) \quad \text{on } \mathbb{V}_0 \quad \text{if and only if}$$

$$\int_0^\infty g_0(G_1(t))dt \geq \int_0^\infty g_0(G_2(t))dt.$$

Moreover, any variable X in $\mathbb{V}$ is equivalent to its utility on $\mathbb{V}_0$ given Y and thus $G \in \Gamma_0$ so that

$$X|(Y) \sim [U_0(X;Y);1]\,|(Y) \quad \text{on } \mathbb{V}_0. \tag{6}$$

where $U_0(X;Y) := \int_0^\infty g_0(G(t))dt$ and $[x;p]$ is a binary random variable taking x with probability p and 0 with probability $1-p$.

Definition

Let $\rho_g : \mathcal{L}(R_+) \to R_+$ be a distortion risk measure for a univariate non-negative random loss Z with distribution function $F_Z \in \mathcal{L}(R_+)$ and non-decreasing distortion function g satisfying $g(0) = 0, g(1) = 1$, the corresponding distortion risk measure is:

$$\rho_g(Z) = \int_0^\infty g(1 - F_Z(x))dx.$$

Definition

The distortion risk measure on tail regions of (X, Y) is the distortion risk measure ρ_g applied to the distribution of $\frac{(X-Q_1(1-\alpha))_+}{Q_1(1-\alpha)} | Y > Q_2(1-\alpha)$ taking into account the marginal threshold, which is

$$\rho_g(X; Y, \alpha) = Q_1(1-\alpha) + \int_{Q_1(1-\alpha)}^{\infty} g\left(\mathbb{P}(X > x | Y > Q_2(1-\alpha))\right) dx$$

where g is a (right-continuous) distortion function.

Nonparametric Estimation

- $(X_1, Y_1), \ldots, (X_n, Y_n)$:independent and identically distributed random vectors with joint distribution F.
  $X_{1:n} \geq X_{2:n} \geq \ldots \geq X_{n:n}, Y_{1:n} \geq Y_{2:n} \geq \ldots \geq Y_{n:n}$.
  $X_{t:n} = X_{\lfloor t \rfloor :n}$ where $\lfloor \cdot \rfloor$ is the flooring integer of any positive real values.
- $F_1(x) = F(x, \infty)$, $F_2(y) = F(\infty, y)$ and $Q_i = F_i^{-1}$, $i = 1, 2$, the generalized inverse function of F_i.
- $\bar{F}_{n1}(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i > x), \quad \bar{F}_{n2}(y) = \frac{1}{n} \sum_{i=1}^{n} I(Y_i > y), \quad x, y \in \mathbb{R}$.: the empirical survival functions

The smoothed and non-smoothed empirical (survival) copulas are given by

$$\begin{cases} \widehat{C}(u,v) = \frac{1}{n}\sum_{i=1}^{n} I(\bar{F}_{n1}(X_i) < u, \bar{F}_{n2}(Y_i) < v), \\ \widetilde{C}(u,v) = \frac{1}{n}\sum_{i=1}^{n} K\left(\frac{1-\bar{F}_{n1}(X_i)/u}{h}\right) K\left(\frac{1-\bar{F}_{n2}(Y_i)/v}{h}\right), \quad u,v \in [0,1]. \end{cases}$$

where $h := h_n > 0$ is the bandwidth.
Two nonparametric estimators of $\rho_g(X;Y,\alpha)$ defined below

$$\begin{cases} \hat{\rho}_g(X;Y,\alpha) = & \frac{1}{n\alpha}\sum_{i=1}^{n}(X_i - X_{n\alpha:n}) I(\bar{F}_{n1(X_i)} < \alpha)\left(g\left(\frac{\widehat{C}(\bar{F}_{n1}(X_i),\alpha)}{\alpha}\right)\right. \\ & +X_{n\alpha:n} \\ \tilde{\rho}_g(X;Y,\alpha) = & \frac{1}{n\alpha}\sum_{i=1}^{n}(X_i - X_{n\alpha:n}) I(\bar{F}_{n1(X_i)} < \alpha)\left(g\left(\frac{\widetilde{C}(\bar{F}_{n1}(X_i),\alpha)}{\alpha}\right)\right. \\ & +X_{n\alpha:n} \end{cases}$$

Assumptions

1.a There exist some $\gamma_1 \in (0,1), \beta_1 \le 0$ and function A which is slowly regular varying with index β_1 and a constant sign near infinity, such that

$$\lim_{t\to\infty} \frac{1}{A(1/\bar{F}_1(t))}\left(\frac{\bar{F}_1(tx)}{\bar{F}_1(t)} - x^{-1/\gamma_1}\right) = x^{-1/\gamma_1}\frac{x^{\beta_1/\gamma_1}-1}{\gamma_1\beta_1}, \quad x > 0.$$

1.b There exist functions $R:(0,\infty)^2 \to [0,\infty)$ and $0<\delta_0<1, L>0, \tau<0, \beta>\gamma_1, T>1$ such that, as $t\to\infty$

$$\sup_{x\in(0,\infty),y\in(0,T]}\left(\frac{tC(t^{-1}x,t^{-1}y)-R(x,y)}{x^{\beta}}\right) = O(t^{\tau}).$$

and

$$\sup_{x,\in(0,\delta_0],y\in(0,1]}\frac{R(x,y)}{(x\wedge y)^{\beta}} \le L.$$

1.c There exists $0<\delta_0'<1, L'>0$ such that

$$|g(u)-g(u')| \le L'|u-u'|, \quad \forall u,u' \in (0,\delta_0').$$

1.d As $n\to\infty$, $\alpha = O(n^{-1+\kappa})$ for some $0<\kappa<\frac{-2\tau}{-2\tau+1}$ and $\sqrt{n\alpha}A(\alpha^{-1}) = o(1)$.

Theorem

Suppose the set $B = \{x \in (0,1] \mid g \text{ is discontinuous at } R(x,1)\}$ has zero Lebesque measure. Under the Assumption (1.a)-(1.c), the distortion risk measure satisfies

$$\lim_{\alpha \downarrow 0} \frac{\rho_g(X;Y,\alpha)}{Q_1(1-\alpha)} = 1 + \int_1^\infty g(R(x^{-1/\gamma_1},1))dx. \quad (7)$$

Theorem

Suppose the set $B = \{x \in (0,1]) \mid g \text{ is discontinuous at } R(x,1)\}$ has zero Lebesque measure. Under the Assumption (1.a)-(1.d), the distortion risk measure satisfies

$$\lim_{n \to \infty} \sqrt{n\alpha} \left| \frac{\rho_g(X;Y,\alpha)}{Q_1(1-\alpha)} - \rho_0 \right| = 0. \quad (8)$$

Assumptions:

2.a The function R in Assumption (1.b) has a continuous first-order partial derivatives $R_1(x, 1) = \partial R(x, 1)/\partial x$.

2.b $R(1, 1) > 0$ and $R_1(1, 1) > 0$.

2.c The kernel distribution function K has symmetric density k with support $[-1, 1]$ and the kernel density k has bounded first derivative.

2.d The bandwidth $h := h_n > 0$ satisfies $n\alpha h^2 \to \infty$ and $n\alpha h^4 \to 0$.

Theorem

Let $\alpha = \alpha_n$ be an intermediate sequence such that $\alpha_n \to 0$ and $n\alpha_n \to \infty$ as $n \to \infty$. Suppose the set $B = \{x \in (0,1] \mid g$ is discontinuous at $R(x,1)\}$ has zero Lebesque measure. Under the Assumptions **??** *and (2.a), (2.b),*

$$\sqrt{n\alpha}\left(\frac{\hat{\rho}_g(X;Y,\alpha)}{\rho_g(X;Y,\alpha)} - 1\right) \xrightarrow{d} N(0, \sigma^2_{R,g,\gamma_1}).$$

with $\sigma^2_{R,g,\gamma_1} = Var(\Phi_{R,g,\gamma_1})/\rho_0^2$ and

$$\Phi_{R,g,\gamma_1} = \gamma_1 \int_0^1 \frac{\mathbb{W}_R(R^{\leftarrow}(x),1)}{(R^{\leftarrow}(x))^{\gamma_1+1}R_1(R^{\leftarrow}(x),1)} dg(x - \gamma_1 \frac{\mathbb{W}_R(R^{\leftarrow}(1),1)}{(R^{\leftarrow}(1))^{\gamma_1+1}R_1(R^{\leftarrow}(1),1)}.$$

Theorem continued

Theorem

is non-degenerate. $R^{\leftarrow}$ *is the (right-continuous) generalized inverse function of* $R(\cdot, 1)$ *and* $\mathbb{W}_R$ *is a R-Brownian motion, i.e. a zero-mean Gaussian process with covariance function with*

$$\mathbb{E}(\mathbb{W}_R(u_1, v_1)\mathbb{W}_R(u_2, v_2)) = R(u_1 \wedge u_2, v_1 \wedge v_2),$$

$$(u_i, v_i) \in (0, \infty]^2 \backslash \{\infty, \infty\}, \; i = 1, 2.$$

Moreover, if Assumption (2.c) and (2.d) are also satisfied, then

$$\sqrt{n\alpha}\left(\frac{\tilde{\rho}_g(X; Y, \alpha) - \hat{\rho}_g(X; Y, \alpha)}{\rho_g(X; Y, \alpha)}\right) \xrightarrow{P} 0.$$

Q & A ?

Thank you very much for your attention!