Capital Allocation Principles

Maochao Xu

Department of Mathematics
Illinois State University
mxu2@ilstu.edu

Capital

Dhaene, et al., 2011, Journal of Risk and Insurance

The level of the capital held by a bank or an insurance company is a key issue for its stakeholders. The regulator, primarily sharing the interests of depositors and policyholders, establishes rules to determine the required capital to be held by the company. The level of this capital is determined such that the company will be able to meet its financial obligations with a high probability as they fall due, even in adverse situations. Rating agencies rely on the level of available capital to assess the financial strength of a company. Shareholders and investors alike are concerned with the risk of their capital investment and the return that it will generate.

Risk measures

Artzner et al., 1999, Mathematical Finance

Risk measure ρ defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$. A coherent risk measure:

1. Monotonicity
$$X_1 \leq X_2 \Rightarrow \rho(X_1) \leq \rho(X_2);$$
2. Positive homogeneity
$$\rho(cX) = c\rho(X), \quad c \geq 0;$$
3. Translation invariance
$$\rho(X + c) = \rho(X) + c, \quad c \geq 0;$$
4. Subadditivity
$$\rho(X + Y) \leq \rho(X) + \rho(Y).$$

Capital allocation Assume we have n risks $X_1, \ldots, X_n$. Then, the aggregate loss is

$$S = \sum_{i=1}^{n} X_i,$$

where this aggregate loss S can be interpreted as:

1. the total loss of a corporate, e.g. an insurance company, with the individual losses corresponding to the losses of the respective business units;
2. the loss from an insurance portfolio, with the individual losses being those arising from the different policies; or
3. the loss suffered by a financial conglomerate, while the different individual losses correspond to the losses suffered by its subsidiaries.

Capital allocation

S is the aggregate loss faced by an insurance company and X_i the loss of business unit i. Assume that the company has already determined the aggregated level of capital and denote this total risk capital by I:

$$I = I_1 + I_2 + \ldots + I_n.$$

What is the optimal allocation strategy?

Allocation formulas

- Haircut allocation
  It is a common industry practice, driven by banking and insurance regulations, to measure stand alone losses by a VaR for a given probability level p. Assume that

$$l_i = \frac{l}{\sum_{j=1}^{n} F_j^{-1}(p)} F_i^{-1}(p);$$

- Quantile allocation-Dhaene et al., 2002, IME
  The comonotonic sum is

$$S^c = \sum_{i=1}^{n} F_i^{-1}(U),$$

where U is a uniform random variable on $(0, 1)$. Then,

$$l_i = F_i^{-1}(F_{S^c}(l));$$

Allocation formulas

- Covariance allocation-Overbeck, 2002

$$l_i = \frac{l}{\mathrm{Var}(S)} \mathrm{Cov}(X_i, S);$$

- CTE allocation

$$l_i = \frac{l}{\mathrm{CTE}_p(S)} \mathrm{E}\left(X_i | S > F_S^{-1}(p)\right),$$

where

$$\mathrm{CET}_p(S) = \mathrm{E}\left(S | S > F_S^{-1}(p)\right).$$

Optimal capital allocation

Decision criterion: Capital should be allocated such that for each business unit the allocated capital and the loss are sufficiently close to each other.

Dhaene, et al. (2011) proposed the following optimization problem to model the capital allocation problem:

$$\min_{\mathbf{l}\in A} \sum_{i=1}^{n} v_i \mathrm{E}\left[\zeta_i \mathbf{D}\left(\frac{X_i - l_i}{v_i}\right)\right], \ s.t. \sum_{i=1}^{n} l_i = l$$

where v_i are nonnegative real numbers such that $\sum_{i=1}^{n} v_i = 1$, and the ζ_i are non-negative random variables such that $\mathrm{E}[\zeta_j] = 1$.

- The non-negative real number v_j is a measure of exposure or business volume of the j-th unit, such as revenue, insurance premium, etc;
- The terms **D** quantify the deviations of the outcomes of the losses X_j from their allocated capital K_j;
- The expectations involve non-negative random variables ζ_j with $\mathrm{E}[\zeta_j] = 1$ that are used as weight factors to the different possible outcomes $\mathbf{D}\,(X_i - l_i)$.

Quadratic optimization

$$\mathbf{D}(x) = x^2.$$

Consider the following optimization:

$$\min_{\mathbf{l} \in A} \sum_{i=1}^{n} \mathrm{E}\left[\zeta_i \frac{(X_i - l_i)^2}{v_i}\right], \ s.t. \sum_{i=1}^{n} l_i = l.$$

Then, the optimal solution is (Dhaene, et al., 2002)

$$l_i = \mathrm{E}\left(\zeta_i X_i\right) + v_i \left(l - \sum_{j=1}^{n} \mathrm{E}(\zeta_j X_j)\right), \quad i = 1, \ldots, n.$$

Convex loss function

We consider how the different capital allocation strategies affect the loss function under the general setup. Specifically, the loss function is defined as

$$\mathbf{L}(\mathbf{l}) = \sum_{i=1}^{n} \phi_i(X_i - l_i), \ \mathbf{l} \in A$$

for some suitable convex functions ϕ_i, where

$$A = \left\{ (l_1, \ldots, l_n) : \sum_{i=1}^{n} l_i = l \right\}.$$

Convex loss function

We also discuss the the following optimisation problem:

$$\min_{\mathbf{l}\in A} \sum_{i=1}^{n} \mathrm{P}\left(\mathbf{L}(\mathbf{l}) \geq t\right), \quad \forall t \geq 0;$$

or equivalently,

$$\min_{\mathbf{l}\in A} \mathrm{E}[\Phi\left(\mathbf{L}(\mathbf{l})\right)],$$

for some increasing function Φ, which could be interpreted as a utility function.

Stochastic orders

Assume random variables X and Y have distribution functions F and G, density functions f and g, respectively. X is said to be smaller than Y in the

1. likelihood ratio order, denoted by $X \leq_{\mathrm{lr}} Y$, if $g(x)/f(x)$ is increasing in x for which the ratio is well defined.
2. usual stochastic order, denoted by $X \leq_{\mathrm{st}} Y$, if $F(x) \geq G(x)$ for all x, or equivalently $\mathrm{E}\phi(X) \leq \mathrm{E}\phi(X)$ for all increasing function.
3. increasing and convex order, denoted by $X \leq_{\mathrm{icx}} Y$, if $\mathrm{E}\phi(X) \leq \mathrm{E}\phi(X)$ for all increasing convex function ϕ.

It is known that in the literature:

$$X \leq_{\mathrm{lr}} Y \Longrightarrow X \leq_{\mathrm{st}} Y \Longrightarrow X \leq_{\mathrm{icx}} Y.$$

Let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of components of the vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$.

$\mathbf{x}$ is said to be majorized by $\mathbf{y}$, denoted by $\mathbf{x} \preceq_{\mathrm{m}} \mathbf{y}$, if

$$\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)} \text{ for } j = 1, \ldots, n-1,$$

and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

A real-valued function Φ defined on a set $A \subseteq \Re^n$ is said to be Schur-concave on A if, for any $\mathbf{x}, \mathbf{y} \in A$,

$$\mathbf{x} \succeq_{\mathrm{m}} \mathbf{y} \Longrightarrow \phi(\mathbf{x}) \leq \phi(\mathbf{y}),$$

and ϕ is log-concave on $A = \{\mathbf{x} \in \Re^n : \phi(\mathbf{x}) > \mathbf{0}\}$ if, for any $\mathbf{x}, \mathbf{y} \in A$ and $\alpha \in [0, 1]$,

$$\phi(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \geq [\phi(\mathbf{x})]^{\alpha}[\phi(\mathbf{y})]^{1-\alpha}.$$

Most common univariate parametric densities are log-concave, such as the normal family, all gamma densities with shape parameter ≥ 1, all Weibull densities with exponent ≥ 1, all beta densities with both parameters ≥ 1, the generalized Pareto and the logistic density, see e.g. Bagnoli and Bergstrom (2005).

A real function $f : \Re^n \to \Re$ is said to be arrangement increasing if for all i and j such that $1 \leq i \leq j \leq n$,

$$(x_i - x_j)\left[f(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) - f(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n)\right] \leq 0.$$

Log-concave

Lemma

(Prékopa, 1973; Eaton, 1982) Suppose that $h : \Re^m \times \Re^k \rightarrow \Re_+$ is a log-concave function and that

$$g(\mathbf{x}) = \int_{\Re^k} h(\mathbf{x}, \mathbf{z}) d\mathbf{z}$$

is finite for each $\mathbf{x} \in \Re^m$. Then g is log-concave on $\Re^m$.

Theorem

Let $X_1, X_2, \ldots, X_n$ be independent random variables defined on $\Re_+$ with log-concave density function f. If ϕ_i's are convex functions, then,

$$(l_1, \ldots, l_n) \succeq_{\mathrm{m}} (l_1^*, \ldots, l_n^*) \Rightarrow \sum_{i=1}^{n} \phi_i(X_i - l_i) \geq_{st} \sum_{i=1}^{n} \phi_i(X_i - l_i^*).$$

Ex.

$$(1, 0, \ldots, 0) \succeq_{\mathrm{m}} (1/2, 1/2, \ldots, 0) \succeq_{\mathrm{m}} (1/n, 1/n, \ldots, 1/n)$$

The optimal solution to the following problem

$$\min_{\mathbf{l}\in A} \mathrm{E}\left[\sum_{i=1}^{n} \phi_i(X_i - l_i)\right],$$

is $l^* = (l/n, \ldots, l/n)$.

Examples:

1. Let $\phi_i(x) = k_i x^2$, then

$$\mathbf{L}(\mathbf{D}) = \sum_{i=1}^{n} \mathbf{D}(X_i - l_i) = \sum_{i=1}^{n} k_i (X_i - l_i)^2,$$

   where k_i could be interpreted as the weights attached to different units which reflect the relative importance of the different risks;
2. Let $\phi_i(x) = k_i |x|$, then

$$\mathbf{L}(\mathbf{D}) = \sum_{i=1}^{n} \mathbf{D}(X_i - l_i) = \sum_{i=1}^{n} k_i |X_i - l_i|.$$

What about independent but not necessarily identically distributed random variables?

Lemma

Let $X_1, X_2, \ldots, X_n$ be independent random variables defined on $\Re_+$ with arrangement increasing density function f. If ϕ is a convex function, then,

$$(l_1, \ldots, l_n) \succeq_{\mathrm{m}} (l_1^*, \ldots, l_n^*)$$

implies

$$\sum_{i=1}^{n} \phi(X_i - l_{(i)}) \leq_{st} \sum_{i=1}^{n} \phi(X_i - l_{(n-i+1)}).$$

Lemma

If $g(x_1, x_2)$ *is log-concave on* $\Re^2_+$ *and*

$$g(x_{(2)}, x_{(1)}) \geq g(x_{(1)}, x_{(2)}) \text{ for all } (x_1, x_2) \in \Re^2_+,$$

then

$$(x_1, x_2) \preceq_{\mathrm{m}} (y_1, y_2) \Longrightarrow g(x_{(1)}, x_{(2)}) \geq g(y_{(1)}, y_{(2)}).$$

Theorem

Let $X_1, X_2, \ldots, X_n$ be independent random variables defined on $\Re_+$ with log-concave density functions $f_1, f_2, \ldots, f_n$, respectively. If $X_1 \leq_{lr} X_2 \leq_{lr} \ldots \leq_{lr} X_n$, then, for any convex function ϕ,

$$(l_1, \ldots, l_n) \succeq_{\mathrm{m}} (l_1^*, \ldots, l_n^*)$$

implies

$$\sum_{i=1}^{n} \phi(X_i - l_{(n-i+1)}) \geq_{st} \sum_{i=1}^{n} \phi(X_i - l_i^*).$$

Theorem

Let $X_1, X_2, \ldots, X_n$ be independent random variables defined on $\Re_+$. If $X_1 \leq_{lr} X_2 \leq_{lr} \ldots \leq_{lr} X_n$, then, for any convex function ϕ,

$$(l_1, \ldots, l_n) \succeq_{\mathrm{m}} (l_1^*, \ldots, l_n^*)$$

implies

$$\sum_{i=1}^{n} \phi(X_i - l_{(n-i+1)}) \geq_{icx} \sum_{i=1}^{n} \phi(X_i - l_i^*).$$

Assume that $\mathbf{l}^* = (l_1^*, \dots, l_n^*)$ is a solution to the following problem:

$$\min_{\mathbf{l} \in A} \mathrm{E}\left[\Phi\left(\sum_{i=1}^{n} \phi(X_i - l_i)\right)\right], \tag{1}$$

where ϕ is a convex function, and Φ is an increasing function. Now, we are interested in the structure of $\mathbf{l}^*$.

Theorem

Let $X_1, X_2, \ldots, X_n$ be independent random variables defined on $\Re_+$. If $\mathbf{l}^*$ is a solution to Problem 1, then, for each pair (i, j),

$$X_i \leq_{lr} X_j \Longrightarrow l_i^* \leq l_j^*.$$

Example:

Now, consider a portfolio containing m risk classes, and class i contains n_i independent and identically distributed risks $X_{i,1}, \ldots, X_{i,n_i}$ distributed as X_i with gamma density function having parameters (k_i, θ), where k_i is the shape parameter and θ is the scale parameter. Then the aggregate loss is

$$S = \sum_{i=1}^{m} S_i = \sum_{i=1}^{m} \sum_{j=1}^{n_i} X_{i,j}.$$

Hence, the loss function is

$$\mathbf{L}(\mathbf{l}) = \sum_{i=1}^{m} \phi\left(S_i - n_i l_i\right),$$

where l_i is the capital allocated to each risk $X_{i,j}$ in class i. It is well-known that S_i is a gamma random variable with parameters $(n_i k_i, \theta)$. It is easy to check that if $k_i \leq k_j$, then $X_i \leq_{\mathrm{lr}} X_j$. Hence, under the optimal capital allocation scheme, one should have

$$n_i k_i \leq n_j k_j \Longrightarrow n_i l_i \leq n_j l_j.$$

Optimal allocation of policy limits

Assume that a policyholder has a total policy limit $l = l_1 + \ldots + l_n$, which can be allocated arbitrarily among the n risks $X_1, \ldots, X_n$. Then, the total retained loss of the policyholder is

$$\sum_{i=1}^{n} (X_i - l_i)_+,$$

where $x_+ = \max\{x, 0\}$.

For example, the compensation package of many big companies includes a commonly called “Flexible Spending Account Programme", which allows employees to allocate pre-tax dollars toward specific expenses such as healthcare, medical costs or dependent care. This is essentially a form of allocating policy limits.

If we choose $\phi_i(x) = \max\{x, 0\} = x_+$, which is a convex function, then the loss function becomes

$$\mathbf{L}(\mathbf{l}) = \sum_{i=1}^{n} \mathbf{D}(X_i - l_i) = \sum_{i=1}^{n} (X_i - l_i)_+.$$

Hence, the optimal allocation of policy limits becomes minimizing the loss function.

Thank you!