Productivity and Welfare in an Overlapping Generations Model with Housing

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Abstract
Productivity and welfare can have an inverse relationship. This paper describes an overlapping generations model where housing is the savings vehicle, and utility is provided by housing and a consumption good. If intratemporal substitution between the consumption good and housing is sufficiently inelastic, aggregate utility is decreasing in productivity. At higher levels of productivity, the relative scarcity of housing aggravates the problem of dynamic efficiency.

Keywords: Housing, income distribution, OLG, savings

1 Introduction

The benefits of productivity growth is one of the few ideas upon which macroeconomists agree, Robert Lucas’ celebrated quote about human capital spillovers being a notable example. However, many outside the economics profession are unconvinced by suggestions that the advance of technology helps all. Popular opinion should not be dismissed.

If there is a fixed supply housing, which is both an essential good and a savings vehicle, productivity and welfare could be inversely related. We study an overlapping generations model where good and housing both provide utility. There is dynamic inefficiency due to over-saving for reasonable parameter choices. An increase in productivity creates more consumption goods but also leads to higher real housing prices exacerbating the over-saving problem. If the preference for smoothing across generations is sufficiently strong, the dynamic inefficiency problem can dominate, and increased productivity leads to reduced welfare.

There are a number of studies of macroeconomic models that include housing, surveyed in Piazzesi and Schneider (2016), but none that examines the effects of productivity growth across an income distribution.

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Iacoviello and Pavan (2013) include a number of interesting features including the option to own or rent, but their primary goal is to establish a calibration matching higher moments in the data. Flavin (2011) does a careful examination and interpretation of the macroeconomics dynamics for varying utility functions that include housing. Flavin and Nakagawa (2008) use a related model to examine the portfolio choice problem. The increase in the real price of housing with an increase in real output matches the stylized facts documented by Knoll et. al. (2017) for post-WWII data in developed countries.

2 OLG model with Housing

A household born in time $t$ maximizes its lifetime utility over the consumption $c_{1,t}$ and housing $q_{1,t}$ in the first period of life (youth) and the same inputs in the second period (old age) $c_{2,t+1}$ and $q_{2,t+1}$.

$$U = u(c_{1,t}, q_{1,t}) + u(c_{2,t+1}, q_{2,t+1})$$

(1)

Single period utility is given by the function

$$u(c, q) = \frac{c^{1-\gamma}}{1-\gamma} + \psi \frac{q^{1-\gamma}}{1-\gamma}$$

$$= U_0 - \theta^{-1} (c^{-\theta} + q^{-\theta})$$

(2)

where the parameter $\gamma > 0$ determines the degree of risk aversion. The chosen utility function is a special case of the function in studies on housing such as Piazzesi et. al. (2016) and Flavin (2011) that allow for both inter- and intra-temporal substitution. In a two period OLG framework, inter-temporal substitution is relatively frictionless so we focus on the function above. In most calibrations the parameter $\gamma$ is greater than one.

Labor is the only input so production of the consumption good take the simple form $Y = aL$ where the parameter $a$ is the measure of productivity and the wage is $w$. There is a unit mass of workers and the labor market is competitive so $L = 1$ and $Y = a = w$. The young and old purchase consumption goods $c_{1,t}$, $c_{2,t}$ at price $p_{c,t}$, and aggregate consumption is then $c_{1,t} + c_{2,t} = Y_t$.

Households rent housing services $q_{1,t}$, $q_{2,t}$ in each period at price $p_{q,t}$. The young also buy a quantity of housing $h_t$ at price $p_{h,t}$ as a savings vehicle, so the aggregate stock of housing is $q_{1,t} + q_{2,t} = h_t$. The old receive rental payments from the young and sell the housing stock to use their savings. Further assume that, in equilibrium, the stock of housing is fixed at $H$ so $h_t = H$.

A fixed stock of housing is a strong assumption though including it as produced good would complicate
the model considerable. Land is a fixed input in housing, and there is no inherent growth in the model, so
the assumption is reasonable for the present model. Relaxing the assumption about the housing stock is
left for future work.

Budget constraints (young & old) are

\[
p_{c,t}c_{1,t} + p_{q,t}q_{1,t} + p_{h,t}h_t \leq a \quad (3)
\]

\[
p_{c,t+1}c_{2,t+1} + p_{q,t+1}q_{2,t+1} \leq p_{h,t+1}h_t + p_{q,t+1}h_t \quad (4)
\]

2.1 Household optimization

To find the equilibrium prices for the consumption good and housing, one must do the optimization problem
for a representative household then use the aggregation relations for output and housing. Maximizing
lifetime utility (1) with the utility function (2) subject to the budget constraints (3 and 4) and solving out
the Lagrange multiplier determines the following optimality conditions.

\[
k_{k,t} - \left(\psi^{-1} \frac{p_{q,t}}{p_{c,t}}\right) q_{k,t} = \mu
\]

where \(k = 1, 2\).

and the FOC with respect to housing is

\[
q_{2,t+1} \left(\frac{p_{h,t}}{p_{q,t}}\right) = q_{1,t} \left(\frac{p_{h,t+1} + p_{q,t+1}}{p_{q,t+1}}\right)
\]

Combining the optimality conditions with the budget constraints and relations for aggregate variables
yields solutions for equilibrium prices. Using the optimality condition (5), the budget constraints become

\[
p_{c,t} \left(\psi^{-1} \frac{p_{q,t}}{p_{c,t}}\right) q_{1,t} + p_{q,t}q_{1,t} + p_{h,t}H = a \quad (7)
\]

\[
p_{c,t+1} \left(\psi^{-1} \frac{p_{q,t+1}}{p_{c,t+1}}\right) q_{2,t+1} + p_{q,t+1}q_{2,t+1} = p_{h,t+1}H + p_{q,t+1}H
\]

Lagging the budget constraint for the old, adding the constraints and using the aggregate relation \(q_{1,t} + q_{2,t} = H\) and \(a = Y\) yields

\[
p_{c,t} \left(\psi^{-1} \frac{p_{q,t}}{p_{c,t}}\right) = \frac{Y}{H}
\]
Similarly, using the optimality condition (5) to substitute for $q$, in the budget constraints, lagging the budget constraint for the old and aggregating yields the condition

$$
\left( p_{c,t} + B^{-1} p_{q,t} \left( \frac{p_{c,t}}{p_{q,t}} \right)^\delta \right) Y = Y + p_{q,t} H.
$$

Combining the conditions derived from the above two budget constraints and determines the following representation of the prices.

$$
p_{c,t} = 1 \quad \text{(9)}
$$

$$
p_{q,t} = \psi \left( \frac{Y}{H} \right)^\gamma \quad \text{(10)}
$$

Since the price of the consumption good $p_c$ is constant, the price of housing $p_q$ is a real value.

It is immediately evident that the real price of housing services $p_{q,t}$ rises as productivity and output rise and housing $H$ becomes relatively scarce, matching the stylized facts for the post-WWII period in the developed world, documented in Knoll et. al. (2017).

### 2.2 Steady State solution

As is typical in OLG models, the steady state solution is not Pareto optimal. Since there is no discounting and agents have a preference for smoothing, the optimal outcome is to split the consumption good and housing equally between young and old so $c_1^* = c_2^* = \frac{Y}{2}$ and $q_1^* = q_2^* = \frac{H}{2}$.

To solve for the price of housing $p_h$ and other values requires steady state analysis. The first order condition with respect to housing (6) as steady states using the solution $p_c = 1$ becomes

$$
q_2 = q_1 \left( \frac{p_h + p_q}{p_h} \right)^\frac{1}{\gamma}.
$$

It’s apparent that housing is not split equally between young and old and that $q_2 > q_1$. It is straightforward to show that $c_2 > c_1$ as well using the intertemporal household optimality condition (5), so there agents over-save relative to the optimal outcome, a form of dynamic inefficiency.

The intuition behind the primary result of the paper is apparent in the above equation (11). As productivity rise so does output so the ratio $\frac{Y}{H}$ does as well, as the housing stock is fixed. Hence, the equilibrium price of housing $p_q$ [add * to eq values?] also rises with productivity so the ratio in parentheses in the intertemporal optimality condition (11) rises, certeris paribus, exacerbating the dynamic inefficiency problem. Of course, the effect of productivity on $p_h$ could also be relevant, so formal analysis is required.
The above condition along with steady state representations of the two budget constraint conditions (7) and (8) can be solved for $h$.

\[ h = a p_h \left[ \left( \frac{p_h + p_q}{p_h} \right)^{1-\frac{1}{\gamma}} + 1 \right]^{-1} \] (12)

Since, in equilibrium housing equals the total stock $h = H$ and output is determined by productivity $Y = a$, this equation also determines the price of housing $p_h$.

\[ p_h \left[ \left( \frac{p_h + p_q}{p_h} \right)^{1-\frac{1}{\gamma}} + 1 \right] = \frac{Y}{H} \] (13)

Finally, solving out $q_2$ from the budget constrain for the old (8) and the first order condition with respect to housing (11) yields a condition relating for first period housing consumption $q_1$.

\[ q_1 = a \left( \frac{Y}{H} \right)^{-2} \left[ \frac{\frac{Y}{p} - p_h}{1 + \psi \left( \frac{Y}{p} \right)^{\gamma-1}} \right] \] (14)

or using $a = H \left( \frac{Y}{H} \right)$

\[ q_1 = H \left[ \frac{\frac{Y}{p} - p_h}{\frac{Y}{p} + \psi \left( \frac{Y}{p} \right)^{\gamma}} \right] \]

[check for rep agent version] Naturally, households with higher wages consume more housing. However, it is quite possible that an increase in $\frac{Y}{H}$ could mitigate housing consumption due to the higher price $p_q$.

### 2.3 Welfare

Welfare $W$ is aggregate utility $U$ which, with the specified utility function across lifetime utilities, becomes

\[ W = \frac{c_1^{1-\gamma}}{1-\gamma} + \psi \frac{q_1^{1-\gamma}}{1-\gamma} + \frac{c_2^{1-\gamma}}{1-\gamma} + \psi \frac{q_2^{1-\gamma}}{1-\gamma} \] (15)

The optimality condition above allow us to express the integrand solely as a function of $a_i$. With the solutions for $p_c$ (9) and $p_q$ (10), the optimality condition (5) becomes $c = q \left( \frac{Y}{p} \right)$. Using these relations and the first order condition for housing (11) gives the following expression.

\[ W = \frac{1}{1-\gamma} \left[ \left( \frac{Y}{p} \right)^{1-\gamma} + \psi \left[ 1 + \left( \frac{p_h + p_q}{p_h} \right)^{1-\gamma} \right] \right] q_1^{1-\gamma} \]
Using the solution for \( q_1 \) (14) and the condition for \( p_h \) (13) gives

\[
W = \frac{1}{1 - \gamma} \left[ \left( \frac{Y}{\bar{H}} \right)^{1 - \gamma} + \psi \right]^{\gamma} \left( \frac{Y}{\bar{H}} - p_h \right)^{-\gamma} \left( \frac{Y}{\bar{H}} \right)^{\gamma^2} a^{1 - \gamma}
\]

\[
W = \frac{H^{1 - \gamma}}{1 - \gamma} \left[ \left( \frac{Y}{\bar{H}} \right)^{1 - \gamma} + \psi \right]^{\gamma} \left[ \frac{Y}{\bar{H}} - p_h \right] \left( \frac{Y}{\bar{H}} \right)^{(1 - \gamma)^2}
\]

### 3 Productivity and Welfare

We are particularly interested in the response of welfare to productivity, represented by changes in \( a \), which directly affects output \( Y \). For a sufficiently large ratio \( \frac{Y}{\bar{H}} \) and a sufficiently high parameter \( \gamma \) representing the elasticity of substitution between housing and the consumption good, welfare is decreasing in productivity.

Let \( A = \frac{Y}{\bar{H}} \) which is also equal to \( \frac{a}{\bar{H}} \). Hence, the quantity of interest \( \frac{dW}{da} \) measuring the effect of productivity on welfare is equivalent to \( \frac{dW}{dA} \). For ease of exposition, the focus is on \( \frac{dW}{dA} \) where welfare is written

\[
W = H^{1 - \gamma} \left[ A^{1 - \gamma} + \psi \right]^{\gamma} \left[ \frac{A}{A - p_h} \right] \left( A^{1 - \gamma} \right)^2
\]

Let \( G(A) = \frac{A}{A - p_h} \), then examine \( \frac{dW}{dA} \). Welfare is

\[
W = H^{1 - \gamma} \frac{\exp \left[ \frac{1}{2} \sigma^2 (1 - \gamma)^2 \right]}{1 - \gamma} \left[ A^{1 - \gamma} + \psi \right]^{\gamma} G(A)^\gamma A^{(1 - \gamma)^2}. \tag{16}
\]

The comparative static exercise yields

\[
\frac{dW}{dA} = H^{1 - \gamma} \exp \left[ \frac{1}{2} \sigma^2 (1 - \gamma)^2 \right] \left[ A^{1 - \gamma} + \psi \right]^{\gamma} G(A)^\gamma A^{(1 - \gamma)^2} \times \left\{ \frac{\gamma}{A + \psi A^\gamma} + \frac{\gamma}{1 - \gamma} \frac{G'(A)}{G(A)} + \frac{1 - \gamma}{A} \right\},
\]

so the sign of \( \frac{dW}{dA} \) depends on the sign of \( \{\} \).

For \( A = \frac{Y}{\bar{H}} \), the condition (13) for \( p_h \) can be written

\[
(p_h + \psi A^\gamma)^{\gamma - 1} p_h = (A - p_h)^\gamma \tag{18}
\]
3.1 A special case

An enlightening special is that where the utility function parameter $\gamma$ is set to $\gamma = 2$, which is relatively low compared to choices in the calibrated models with housing of Iacoviello et. al. (2013) and Flavin (2011). It is possible for welfare to be declining productivity and output.

**Proposition 1** For $\gamma = 2$, welfare is decreasing in productivity if $\psi A > 2$, where welfare is given by the expression (16) and productivity is the lognormal distribution parameter $\mu$.

**Proof.** For $\gamma = 2$, the condition (18) determines the solution for $p_h$

$$p_h = \frac{A}{2 + \psi A}$$

and the function $G(A) = \frac{A}{A - p_h}$

$$G(A) = \frac{1}{1 - \frac{1}{2 + \psi A}}.$$

When the parameter $\gamma = 2$, welfare is

$$\frac{dW}{dA} = H^{-1} \exp \left[ \frac{1}{2} \sigma^2 \right] \left[ A^{-1} + \psi \right]^2 G(A)^2 A \times \left\{ \frac{2}{A + \psi A^2} - \frac{G'(A)}{G(A)} \right\} - \frac{1}{A}$$

The term $\frac{G'(A)}{G(A)}$ becomes

$$\frac{G'(A)}{G(A)} = \frac{-\psi}{(1 + \psi A)(2 + \psi A)}$$

so

$$\frac{dW}{dA} = H^{-1} \exp \left[ \frac{1}{2} \sigma^2 \right] \left[ A^{-1} + \psi \right]^2 G(A)^2 A \times \left\{ \frac{2}{A + \psi A^2} + \frac{2d}{(1 + \psi A)(2 + \psi A)} - \frac{1}{A} \right\},$$

but $\{\cdot\}$ can be written

$$\{\cdot\} = \frac{2 - \psi A}{A(2 + \psi A)}.$$

Therefore, for $\gamma = 2$, $\frac{dW}{dA} < 0$ if $\psi A > 2$. ■
3.2 The general case

Given that the ratio of output to the housing stock $A = \frac{Y}{\Pi}$ is sufficiently large, welfare is decreasing in productivity for $\gamma > 1$.

Let the solution for $p_h$ have the form $p_h (A) = \frac{A}{k(A)}$ for the some function $k(A)$. The condition for $p_h$ in equation (18) becomes

$$1 + k(A) \psi A^{\gamma - 1} = (k(A) - 1) \frac{A}{\gamma^2}.$$  

The following result is immediate.

**Lemma 2** As $A \to \infty$, $p_h (A) \to 0$.

**Proof.** The above expression for $k(A)$ can be written

$$\frac{(k(A) - 1) \frac{A}{\gamma^2} - 1}{k(A)} = \psi A^{\gamma - 1}.$$

Since $\frac{A}{\gamma^2} > 1$ for $\gamma > 1$, for sufficiently large $k(A)$, the $(k(A) - 1) \frac{A}{\gamma^2}$ term dominates, so as $A \to \infty$, $k(A) \to \infty$ and $p_h (A) \to 1$. □

The primary general result follows.

**Proposition 3** For any $\gamma > 1$, $\psi > 0$ there exists a threshold $\tilde{A}$ such that for $A > \tilde{A}$, $\frac{dW}{dA} < 0$.

The proof is in the Appendix (A3).

Hence, qualitatively it is quite possible for welfare to be decreasing in productivity if the ratio of output to the housing stock is sufficiently large. Alternatively, the inverse relationship between welfare and productivity holds if the limit of the housing stock necessitates low wage workers to shift spending to housing to cover the higher costs.

4 Calibration

A calibrated version of the model sheds light on the quantitative importance of the above propositions. Two key stylized facts from Piazzesi and Schneider (2016) guide the analysis.

They show that the fraction of spending on housing against spending on non-durables and services is relatively stable and fluctuates around 0.2. Other papers verify the stability of this relationship over time and across incomes. For the present model, that ratio $\frac{p_{hQ}}{p_{cC}}$ is a function of $A = \frac{Y}{\Pi}$ such that

$$\frac{p_{hQ}}{p_{cC}} = \psi A^{\gamma - 1},$$  

(19)
combining (5) and the solutions for $p_c$ (9) and $p_q$ (10).

Furthermore, Piazzesi and Schneider (2016) show that the stock of housing has fluctuated around 1.5 times GDP for the past few decades, and that is close to its current value, so we take $A = \frac{2}{3}$. Hence, for a given value of the preference parameter $\gamma$, the calibration $\frac{p_a}{p_c} = 0.2$ and $A = \frac{2}{3}$ determines the value of $\psi$. For example, if $\gamma = 2$ then it must be the case that $\psi = 0.3$ to satisfy the condition (19) for the above values.

Figure 1 shows welfare against $A = \frac{Y}{H}$ for different value of $\gamma$ where $\psi$ is determined by (19) for the above values. The welfare values are normalized so that they are equal at the calibrated value $\frac{Y}{H} = \frac{2}{3}$.

The concern that higher productivity could lead to decreasing welfare is apparent for sufficiently large $\gamma$. For $\gamma = 6$, welfare is declining in $\frac{Y}{H}$ for $\frac{Y}{H} > \frac{2}{3}$. If substitution between consumption goods and housing is sufficiently inelastic, higher productivity could lead to lower welfare despite higher real output. This parameter choice $\gamma = 6$ is not unusual. Iacoviello and Pavan (2013) report results for simulations of their model with similar parameter choices as in Figure 1.

Welfare losses from higher productivity are a concern for reasonable calibrations of the model.
5 Conclusion

The benefits of productivity is one of the few generally accepted ideas in macroeconomics, but the issue requires more nuanced thinking. Housing is necessary for virtually everyone, and proper consideration of its role for households demonstrates that higher productivity can raise the real price of housing, aggravating the dynamic inefficiency problem and reducing aggregate welfare. The key assumptions in the model underlying the result are the limited supply of housing and the relative inelasticity of substitution between housing and consumption goods.

The inverse relationship between productivity and welfare holds for reasonable calibrations of the model, though there are many factors to be considered in future work. Some forms of redistribution might re-establish a direct relationship. Considering housing as a produced good and including capital in production are two obvious extensions to consider, so there is much work to be done. However, the overriding message is that consideration of effects across the distribution of incomes can overturn intuitive results from representative agent models.
Appendix

(A1) The following is a detailed development of the model in Sections 3 and 4.

The prices for the consumption good and housing are determined by the first order conditions, the budget constraints and the aggregation identities. Maximizing lifetime utility (1) subject to the budget constraints (3 and 4) determines the following Lagrangian with Lagrange multipliers $\lambda$ and $\delta$.

$$
\mathcal{L} = u(c_{1,t}, q_{1,t}) + u(c_{2,t+1}, q_{2,t+1}) \\
+ \lambda (a_i - p_{c,t}c_{1,t} - p_{q,t}q_{1,t} - p_{h,t}h_t) \\
+ \delta (p_{h,t+1}h_{t+1} + p_{q,t+1}q_{t+1} - p_{c,t+1}c_{2,t+1} - p_{q,t}q_{2,t+1})
$$

The F.O.C.s with respect to $c_{1,t}$, $c_{2,t+1}$ and $h_t$ are

$$
u_{c_{1,t}} = \lambda p_{c_1} + \lambda p_q,$$
$$
u_{c_{2,t+1}} = \delta p_{c_2} + \delta p_{q},$$
$$
\gamma (p_{h,t+1} + p_{q,t+1}) = \lambda p_h,$$

With the given utility function (2)

$$
u_c = c^{-\gamma}; u_q = \psi \chi (\chi q)^{-\gamma}$$
$$
u_q/\nu_c = \psi \chi \left( \frac{c}{\chi q} \right)^{\gamma}$$

Combining the optimality conditions with the budget constraints and relations for aggregate variables yields solutions for equilibrium prices. Using the optimality condition (5), the budget constraints become

$$p_{c,t}B \left( \frac{p_{q,t}}{p_{c,t}} \right)^{1/\gamma} q_{1,t} + p_{q,t}q_{1,t} + p_{h,t}h_t = a_i$$
$$p_{c,t+1}B \left( \frac{p_{q,t+1}}{p_{c,t+1}} \right)^{1/\gamma} q_{2,t+1} + p_{q,t+1}q_{2,t+1} = p_{h,t+1}h_{t+1} + p_{q,t+1}h_t$$

Lagging the budget constraint for the old and adding constraints

$$\left( Bp_{c,t} \left( \frac{p_{q,t}}{p_{c,t}} \right)^{1/\gamma} + p_{q,t} \right) (q_{1,t} + q_{2,t}) + p_{h,t}h_t = a_i + (p_{h,t} + p_{q,t}) h_{t-1}$$
Integrating both sides over $a_i$ and using the identities (??), (??) and (??) yields

$$p_{c,t}B \left( \frac{p_{q,t}}{p_{c,t}} \right)^{\frac{1}{\gamma}} = \frac{Y}{H}. \tag{20}$$

Similarly, using the optimality condition (5) to substitute for $q$, in the budget constraints, lagging and adding again

$$\left( p_{c,t} + B^{-1} p_{q,t} \left( \frac{p_{c,t}}{p_{q,t}} \right)^{\frac{1}{\gamma}} \right) (c_{1,t} + c_{2,t}) + p_{h,t} h_t = a_i + (p_{h,t} + p_{q,t}) h_{t-1}$$

Aggregating (again)

$$\left( p_{c,t} + B^{-1} p_{q,t} \left( \frac{p_{c,t}}{p_{q,t}} \right)^{\frac{1}{\gamma}} \right) Y = Y + p_{q,t} H \tag{21}$$

Combining the conditions derived from the budget constraints (20) and (21) determines the following representation of the prices as given by (9) and (10) in the text.

$$p_{c,t} = 1$$

$$p_{q,t} = B^{-\gamma} \left( \frac{Y}{H} \right)^{\gamma}$$

Using steady state analysis with the above results also allow for the derivation of a condition for the price of housing shares used for saving $p_h$, and an expression for savings as a function of household income.

The budget constraint for the old with $q_2$ (8) using steady states is

$$\left( p_c B \left( \frac{p_q}{p_c} \right)^{\frac{1}{\gamma}} + p_q \right) q_2 = (p_h + p_q) h_i.$$ 

Combining the last two by substituting out $q_2$ yields

$$\left( \frac{p_h + p_q}{p_h} \right)^{\frac{1}{\gamma}} \left( p_c B \left( \frac{p_q}{p_c} \right)^{\frac{1}{\gamma}} + p_q \right) q_1 = (p_h + p_q) h_i,$$

but the condition from the budget constrain for young (20) also has $\left( p_c B \left( \frac{p_q}{p_c} \right)^{\frac{1}{\gamma}} + p_q \right) q_1$ so combining gives

$$\left( \frac{p_h + p_q}{p_h} \right)^{\frac{1}{\gamma}} (a_i - p_h h_i) = (p_h + p_q) h_i,$$

which can be solved for $h_i$ as in relation (12) in the text.

$$h_i = a_i p_h^{-1} \left[ \left( \frac{p_h + p_q}{p_h} \right)^{1-\frac{1}{\gamma}} + 1 \right]^{-1}$$
(A2) The following is a more detailed development of the welfare considerations for the model presented in Section 4. We can express welfare (15) solely as a integral function of \( a_i \). With the solutions for \( p_q \) and \( p_c \), the optimality condition (5) becomes \( c = q \left( \frac{Y}{\psi} \right) \) so the above expression for welfare can be written (suppressing the \( a_i \)'s)

\[
W = \frac{1}{1 - \gamma} \left[ \left( \frac{Y}{\psi} \right)^{1-\gamma} + \psi \chi^{1-\gamma} \right] \int_0^\infty \left( q_1^{1-\gamma} + q_2^{1-\gamma} \right) dF(a_i)
\]

The first order condition for housing with steady state variables (11) with the above relation gives

\[
W = \frac{1}{1 - \gamma} \left[ \left( \frac{Y}{\psi} \right)^{1-\gamma} + \psi \chi^{1-\gamma} \right] \left[ 1 + \left( \frac{p_h + p_q}{p_h} \right)^{1-\gamma} \right] \int_0^\infty q_1^{1-\gamma} dF(a_i)
\]

Using the solution for \( q_1 \) (14) and the condition for \( p_h \) (13) gives

\[
W = \frac{1}{1 - \gamma} \left[ \left( \frac{Y}{\psi} \right)^{1-\gamma} + \psi \chi^{1-\gamma} \right] \left[ \frac{Y}{\psi - p_h} \right] \left( \frac{Y}{\psi} \right)^{-2(1-\gamma)} \left[ \frac{\frac{Y}{\psi} - p_h}{1 + B^{-\gamma} \left( \frac{Y}{\psi} \right)^{-\gamma-1}} \right] \int_0^\infty a_i^{1-\gamma} dF(a_i)
\]

\[
W = \frac{1}{1 - \gamma} \left[ \left( \frac{Y}{\psi} \right)^{1-\gamma} + B^{-\gamma} \right] \left[ \frac{Y}{\psi - p_h} \right] \left( \frac{Y}{\psi} \right)^{2(1-\gamma)(1+\gamma)} \left[ \frac{\frac{Y}{\psi} - p_h}{\left( \frac{Y}{\psi} \right)^{1-\gamma} + B^{-\gamma}} \right] \int_0^\infty a_i^{1-\gamma} dF(a_i)
\]

\[
W = \frac{1}{1 - \gamma} \left[ \left( \frac{Y}{\psi} \right)^{1-\gamma} + B^{-\gamma} \right] \gamma \left( \frac{Y}{\psi} - p_h \right)^{-\gamma} \left( \frac{Y}{\psi} \right)^{\gamma} \int_0^\infty a_i^{1-\gamma} dF(a_i)
\]

Using the lognormal distribution \( F(a_i) = L(a_i, \mu, \sigma) \), the integral expression is simply \( \int_0^\infty a_i^{1-\gamma} dF(a_i) = Y^{1-\gamma} \exp \left[ \frac{1}{2} \sigma^2 (1 - \gamma)^2 \right] \), so

\[
W = \frac{H^{1-\gamma} \exp \left[ \frac{1}{2} \sigma^2 (1 - \gamma)^2 \right]}{1 - \gamma} \left[ \left( \frac{Y}{\psi} \right)^{1-\gamma} + B^{-\gamma} \right] \gamma \left( \frac{Y}{\psi - p_h} \right)^{-\gamma} \left( \frac{Y}{\psi} \right)^{\gamma} \left( \frac{Y}{\psi} \right)^{1-\gamma}
\]

\[
W = \frac{H^{1-\gamma} \exp \left[ \frac{1}{2} \sigma^2 (1 - \gamma)^2 \right]}{1 - \gamma} \left[ \left( \frac{Y}{\psi} \right)^{1-\gamma} + B^{-\gamma} \right] \gamma \left( \frac{Y}{\psi - p_h} \right)^{-\gamma} \left( \frac{Y}{\psi} \right)^{1-\gamma+\gamma^2}
\]

\[
W = \frac{H^{1-\gamma} \exp \left[ \frac{1}{2} \sigma^2 (1 - \gamma)^2 \right]}{1 - \gamma} \left[ \left( \frac{Y}{\psi} \right)^{1-\gamma} + B^{-\gamma} \right] \gamma \left( \frac{Y}{\psi - p_h} \right)^{\gamma} \left( \frac{Y}{\psi} \right)^{1-2\gamma+\gamma^2}
\]

\[
W = \frac{H^{1-\gamma} \exp \left[ \frac{1}{2} \sigma^2 (1 - \gamma)^2 \right]}{1 - \gamma} \left[ \left( \frac{Y}{\psi} \right)^{1-\gamma} + B^{-\gamma} \right] \gamma \left( \frac{Y}{\psi - p_h} \right)^{\gamma} \left( \frac{Y}{\psi} \right)^{(1-\gamma)^2}
\]

The expression for welfare at the end of section 2.3 is equivalent.
The proof of Prop 3 follows.

**Proof.** From (17), the expression for \( \frac{dW}{dA} \) for \( \gamma > 1 \) can be written

\[
\frac{dW}{dA} = H^{1-\gamma} \exp \left[ \frac{1}{2} \sigma^2 (1 - \gamma)^2 \right] \left[ A^{1-\gamma} + d \right] \gamma G(A) \gamma A^{(1-\gamma)^2-1} \times \frac{\gamma}{1 + dA^{\gamma-1}} + \frac{\gamma}{1 - \gamma} \frac{AG'(A)}{G(A)} + 1 - \gamma
\]

In the expression for \( \frac{dW}{dA} \) above, as \( A \to \infty \) the first term in \( \{ \cdot \} \), \( \frac{\gamma}{1 + dA^{\gamma-1}} \to 0 \) for \( \gamma > 1 \). The more difficult issue is the second term.

In terms of \( k(A) \), the function \( G(A) = \frac{k(A)}{k(A) - 1} \). Hence, the derivative \( G'(A) = \frac{-k'(A)}{(k(A) - 1)^2} \) and

\[
\frac{G'(A)}{G(A)} = \frac{-k'(A)}{k(A)(k(A) - 1)}.
\]

Differentiating the condition for \( k(A) \) yields

\[
k'(A) dA^{\gamma-1} + k(A) (\gamma - 1) dA^{\gamma-2} = k'(A) (k(A) - 1)^{\frac{1}{\gamma - 1}},
\]

so

\[
k'(A) = \frac{(\gamma - 1) dA^{\gamma-2}}{\frac{\gamma}{\gamma - 1} (k(A) - 1)^{\frac{1}{\gamma - 1}} - dA^{\gamma-1}}
\]

Furthermore,

\[
\left( \frac{\gamma}{1 - \gamma} \right) \frac{AG'(A)}{G(A)} = \left( \frac{1}{k(A) - 1} \right) \left( \frac{\gamma d}{\frac{\gamma}{\gamma - 1} (k(A) - 1)^{\frac{1}{\gamma - 1}} A^{1-\gamma} - d} \right) = \frac{\gamma d}{\frac{\gamma}{\gamma - 1} (k(A) - 1)^{\frac{1}{\gamma - 1}} A^{1-\gamma} - d (k(A) - 1)} = \frac{\gamma d}{\frac{1}{\gamma - 1} (A^{1-\gamma} + dk(A)) - d (k(A) - 1)} = \frac{\gamma d}{\frac{1}{\gamma - 1} A^{1-\gamma} + \frac{1}{\gamma - 1} dk(A) - 1}
\]

(check this) So for \( \gamma > 1 \), as \( A \to \infty \), \( k(A) \to \infty \) so \( \left( \frac{\gamma}{1 - \gamma} \right) \frac{AG'(A)}{G(A)} \to 0 \). Therefore, as \( A \to \infty \),

\[
\frac{dW}{dA} \to 1 - \gamma,
\]

and the proposition holds by the properties of limits. \( \blacksquare \)
References

Flavin, Marjorie (2011). Housing, adjustment costs and macroeconomic dynamics, working paper.


