

# Informational Efficiency and Endogenous Rational Bubbles

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## Abstract

In a model where rational bubbles form and collapse endogenously, properly specified tests of return predictability have little power to reject deviations from the efficient markets model. A weighted replicator dynamic describes how agents switch between a forecast based on fundamentals, a rational bubble forecast and a weighted average of the two. A significant portion of the population may adopt the rational bubble forecast, which is inconsistent with the efficient markets model but satisfies informational efficiency. Tests on simulated data show excess variance in the price and unpredictable returns.

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# 1 Introduction

Rejection of the hypothesis that asset returns are forecastable could be due to an informationally inefficient market or an incorrect model, the joint hypothesis problem discussed in Fama (1970). The interpretation of the opposite finding is also problematic. Evidence that returns are not forecastable, informational efficiency (IE), does not imply that asset prices are solely determined by fundamentals, as in the standard efficient markets model (EMM)<sup>1</sup>.

This paper describes an asset pricing model that is informationally efficient, but the asset price could depend on extraneous information. A representative agent rational bubble model, such as Blanchard (1979) has these features, but it is unclear how agents would coordinate on such forecasts. The present model includes heterogeneous forecasting strategies where rational bubbles endogenously form and collapse, representing a deviation from the efficient markets model. However, properly specified tests of IE have very little power to reject such deviations. Therefore, tests of return forecastability give evidence about the forecastability of returns and little else.

The endogenous rational bubbles produce excess variance in the returns as found in financial data<sup>2</sup>. Under the standard EMM, where the price is determined by discounted expected future dividends, Cochrane (2007) shows that the variance of the price-dividend ratio can be decomposed into the unforecastable components of the dividend innovations and the returns. Under the present model, that relationship is broken, since the asset price could depend on extraneous data. Hence, one cannot accept the conclusion, as in Cochrane (2007), that a limited degree of return forecastability rules out bubbles.

An evolutionary game theory mechanism describes how agents switch between three forecasting strategies. The *fundamental* forecast is based on the EMM. The *mystic* forecast is formed using a model of a rational bubble where extraneous information affects the asset price. The third *reflective* forecast is a weighted average of the two and represents the rational forecast in an environment with heterogeneity. Payoffs to these forecasting strategies are based on past forecast errors. A weighted replicator dynamic describes the evolution of the fractions of agents using the different strategies and allows for the parameterization of how fast agents switch to better performing strategies.

For some parameter choices, the fundamental forecast dominates, and the model behaves according to the EMM. For other plausible settings, however, mysticism can gain a temporary following, leading to deviations from the EMM, in that non-fundamental information affects the price. Such deviations are detected as excess variance in the simulated data. Mystic outbreaks require that the magnitude of the shock to the extraneous information are similar to that of the fundamental information and that agents are sufficiently aggressive about switching to better performing strategies. Mysticism cannot outperform reflectivism indefinitely so such mystic bubbles form and collapse endogenously, in contrast to other models of rational bubbles.

Many studies of IE regress asset returns on a lagged predictor variable such as the price-dividend ratio and conduct a standard  $t$ -test to determine if the coefficient on that variable is significantly different than zero. The coefficients are usually different than zero, though the level of significance and  $R^2$  are often unimpressive (Cochrane 2008). Campbell and Yogo (2006) point out that such tests do not properly account for the persistence of the predictor variable, and they construct a test that does. Their results on IE are mixed in that they depend on the sample, a reflection of the related literature.

Applying the test of Campbell and Yogo (2006) on simulated data from the model that allows for endogenous rational bubbles shows that returns are not predictable using both dividends and the price-

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<sup>1</sup>This terminology is used by Shiller (1981). Others refer to the model as representing the strong efficient markets hypothesis.

<sup>2</sup>Shiller (1981), LeRoy and Porter (1981) show evidence of excess volatility.

dividend ratio as predictors. For a range of parameter choices, the simulated data shows excess variance but not return predictability. Hence, even if the null of IE cannot be rejected, endogenous rational bubbles could be present, violating the EMM.

The behavior of all agents in the present work satisfies the *cognitive consistency principle*, described in Evans and Honkapohja (2011), which says that agents in a model are as smart as economists. More precisely, agents form expectations using reasonable models according to economic theory. The model also explains observed features of financial markets data such as excess kurtosis and GARCH effects in the returns, see Parke and Waters (2007) for details. Parke and Waters (2014) conduct a formal stability analysis on a related model. The primary aim of the present work is to study the implications for the interpretation of econometric tests of IE. Another contribution is the formal analysis of the endogenous collapse of the bubbles.

There are a number interesting alternative approaches to asset pricing that involve deviations from the EMM, though the implications for return predictability need to be examined in detail. Adam, Marcet and Nicolini (2008) and Lansing (2010) are able to match a number of the features of the U.S. stock market data. In the model in the former paper, a representative agent updates its estimate of the long run growth rate of the asset price, which is used for forecasting. In Lansing (2010), the forecasting model (perceived law of motion) includes a geometric random walk, making bubbles a possibility, and agents update a parameter in the forecasting model that determines the impact of the bubble. The agents in Branch and Evans' (2011) model of bubbles update an estimate of the conditional variance of the return using a linear model. The time series implications of this approach have yet to be explored in detail<sup>3</sup>.

There are a number of asset pricing models with heterogeneous forecasting strategies. In LeBaron (2010), some agents use a "buy and hold" strategy, which has intuitive appeal but may not satisfy cognitive consistency. The cognitive consistency of agents in the asset pricing models of Brock and Hommes (1998) and Branch and Evans (2007) is open to interpretation<sup>4</sup>. In the former paper, some agents have perfect foresight but must pay a cost. In contrast, the reflective forecast in the present work is constructed using all information available to the agents, and does not require a cost. In Branch and Evans (2007), some agents use underparameterized models, which exclude information that affects the asset price.

The paper is organized as follows. The asset pricing model, forecasting strategies, and their payoffs are specified in section 2. Section three has a discussion of the intuition behind the formation and collapse of mystic bubbles and the implications for return predictability. Section 4 describes the evolutionary game theory mechanism and the requirements for bubbles to arise. Section 5 gives details about the econometric tests on the simulated data and section 6 concludes.

## 2 Asset Pricing

This section specifies the three forecasts and the resulting realization of the asset price, which thereby determine the forecast errors for each strategy. The underlying motivation is the standard asset pricing equation

$$p_t = \alpha p_{t+1}^e + d_t, \tag{1}$$

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<sup>3</sup>Similarly, the noise trader model of DeLong et. al (1991) has deviations from the strong EMH, but the relationship to return predictability is discussed informally.

<sup>4</sup>These papers are part of a large literature using the multinomial logit dynamic to describe the evolution of heterogeneous forecasts. See Hommes (2006) for a survey.

where the asset price is  $p_t$ , the dividend is  $d_t$  and the parameter  $\alpha$  is the discount factor. This model is not fully sufficient for our purpose, since there is a representative forecast of the price. Brock and Hommes (1998) develop a model with mean-variance optimization where risk-neutral investors choose between a riskless and risky asset in constant supply. With risk neutral agents who have a common belief about the variance of the returns, the model with heterogeneous forecasts can be written as

$$p_t = \alpha \sum_{h=1}^n x_{h,t} e_{h,t} + d_t + C \quad (2)$$

where the vectors  $e_t = (e_{1,t}, \dots, e_{n,t})$  and  $x_t = (x_{1,t}, \dots, x_{n,t})$  are the different forecasts of  $p_{t+1}$  and the fractions of agents using the forecasts, respectively. The constant  $C$  is a risk premium, which is set to zero in the following to simplify the presentation.

The forecasts considered are motivated by the multiplicity of solutions to the model (1) in the homogeneous case. According to the efficient markets model (EMM), the price is given by the discounted expected future dividends as in the following solution to the model (1).

$$p_t^* = d_t + \sum_{j=1}^{\infty} \alpha^j E_t(d_{t+j}) \quad (3)$$

Agents referred to as *fundamentalists* adopt the forecast  $e_{2,t}$  determined by the above solution.

$$e_{2,t} = E_t(p_{t+1}^*) = \sum_{j=1}^{\infty} \alpha^{j-1} E_t(d_{t+j}) \quad (4)$$

However, this solution is not unique. As discussed in the rational bubbles literature, Evans (1991) for example, there is a continuum of solutions to (1) of the form

$$p_t^m = p_t^* + \alpha^{-t} m_t$$

where the stochastic variable  $m_t$  is a martingale such that  $m_t = m_{t-1} + \eta_t$ , for *iid*, mean zero shocks  $\eta_t$ . Though the information contained in the martingale  $m_t$  may be extraneous, if agents believe that information is important, it does affect the asset price. Agents that adopt the forecast  $e_{3,t}$  based on the rational bubble solution above are called *mystics*, and their forecast is as follows.

$$e_{3,t} = E_t(p_{t+1}^m) = E_t(p_{t+1}^*) + \alpha^{-t-1} m_t \quad (5)$$

A primary objection to such a solution is that it violates a transversality condition, see Lundqvist and Sargent (2004, section 13.6). As discussed in Lansing (2010), an agent could profitably short the risky asset if the prices follows such a path. However, this hypothetical agent would need to be infinitely lived with unlimited resources or ability to borrow. Furthermore, agents in the present model can adopt or abandon the forecast at any time so this objection to the mystic forecast is not a concern.

Both the mystic and fundamental forecasts satisfy rational expectations in that they are unbiased in the homogeneous case. However, our goal is to allow for possible heterogeneity in forecasting strategies, so we introduce the *reflective* forecast, which satisfies rational expectations even in the presence of heterogeneity. Inclusion of the reflective forecast also ensures the endogenous collapse of any bubble, see Proposition 3.

The reflective forecast  $e_{1,t}$  is an average of the alternative forecasts used in the population weighted according to the relative popularity, such that

$$e_{1,t} = (1 - n_t) e_{2,t} + n_t e_{3,t} \quad (6)$$

where

$$n_t = \frac{x_{3,t}}{x_{2,t} + x_{3,t}}$$

The variable  $n_t$  is the relative popularity of mysticism among agents using mysticism or fundamentalism.

Reflectivism depends on alternative strategies, so to ensure its existence, we make the following assumption.

*Assumption:* The fraction of fundamentalists  $x_{2,t}$  never falls below some minimum  $\delta_2 > 0$ .

This assumption is not particularly restrictive, considering that in most asset pricing models, all investors are fundamentalists.

Given these three forecasting strategies (4), (5) and (6) and the asset pricing model allowing for heterogeneity (2), the realization of the asset price is

$$p_t = p_t^* + \alpha^{-t} n_t m_t. \quad (7)$$

One can verify that the reflective forecast has the same form as the realization of the price such that  $e_{1,t} = E_t p_{t+1}$ . The reflective forecast embodies the "beauty contest" characterization (Keynes 1935) of asset markets in that agents use the martingale in their forecast only to the extent that other agents use it, not because they regard it as inherently important. See Parke and Waters (2014) for a detailed discussion.

**Remark 1** *If mysticism is present in the population, the efficient markets model is violated.*

If the fraction  $n_t$  is greater than zero, then the extraneous martingale affects the asset price. Whether such an outbreak of agents adopting the mystic forecast is possible and quantitatively significant is a primary issue in the simulation results.

An evolutionary game theory mechanism describes how agents choose from the above forecasting strategies. Agents evaluate the performance of the forecasting strategies by comparing payoffs based on squared forecast errors. Hommes (2001) shows that the mean-variance optimization underpinning the model (2) is equivalent to minimizing squared forecast errors. Payoffs are defined as follows.

$$\pi_{i,t} = -(p_t - e_{i,t-1})^2 \quad (8)$$

The reflective forecast error  $U_t$  plays an important role in the payoffs to all three forecasting strategies, and is comprised of two terms.

$$U_t = (p_t^* - E_{t-1}(p_t^*)) + \alpha^{-t} (n_t m_t - n_{t-1} m_{t-1}) \quad (9)$$

The first term is the innovation to the current period dividend payment, which is the new fundamental information. The second term embodies the new information about the martingale's impact on the asset price.

In a model with a representative forecast, the fundamental and mystic forecasts are unbiased, but their forecast errors are affected by the level of the martingale in the presence of heterogeneity. A key term in

the payoffs is the weighted martingale  $A_{t-1} = \alpha^{-t} m_{t-1}$ . The reflective forecast depends only on  $U_t$  and, using (8) and (9), has payoff

$$\pi_{1,t} = -U_t^2. \quad (10)$$

Fundamentalism has forecast error  $U_t + n_{t-1}A_{t-1}$ , so its payoff is

$$\pi_{2,t} = -U_t^2 - 2n_{t-1}U_tA_{t-1} - n_{t-1}^2A_{t-1}^2. \quad (11)$$

Similarly, the payoff to mysticism is as follows.

$$\pi_{3,t} = -U_t^2 + 2(1 - n_{t-1})U_tA_{t-1} - (1 - n_{t-1})^2A_{t-1}^2 \quad (12)$$

### 3 Intuition

Much of the intuition behind the potential for endogenous formation and collapse of rational bubbles and the implications for IE can be observed in the above three payoffs, assuming a reasonable evolutionary dynamic. The third terms in the payoffs to mysticism (12) and fundamentalism (11) are unambiguously damaging to those payoffs in comparison with the payoff to reflectivism (10). If there is heterogeneity in the choice of forecasting strategies ( $0 < n_{t-1} < 1$ ), then mysticism and fundamentalism over- and under-react to the martingale. The unconditional expectation of the "covariance" term  $U_tA_{t-1}$  is zero, so reflectivism outperforms the other two strategies<sup>5</sup>.

However, mysticism can outperform the other strategies in a given period. If the realization of the covariance  $U_tA_{t-1}$  is positive and sufficiently large, the second term in (12) may outweigh the third term so that  $\pi_{3,t} > \pi_{1,t} > \pi_{2,t}$ . Such a positive covariance corresponds to a fortunate (for the mystic) correlation between the martingale and the innovations in the model. In distribution, dividend innovations are uncorrelated with the martingale, but over a number of periods, such correlations are likely to occur. For mysticism to have a chance of success, the level of  $A_t$  must be large enough so that the covariance is significant, but not so large that the martingale terms dominate. Intuitively, a forecast like "Dow 36 thousand" might attract a significant following (as it did, see Glassman and Hassett (1999)), but an absurdly large forecast such as "Dow 36 billion" would be dismissed.

Some formal results clarify the implications for the endogeneity of bubbles and return predictability. Saying that expected excess returns are constant is an alternative way of saying that returns are not forecastable, as in Ohlson (1977) for one example. Excess returns given by

$$Z_t = p_t + d_t - \alpha^{-1}p_{t-1} \quad (13)$$

are equivalent to the reflective forecast error up to a constant  $\bar{Z}$  such that  $Z_t = \bar{Z} + U_t$ , given the underlying model based on mean-variance optimization. This point is verified using the price realization (7).

Given reasonable assumptions for the information structure, the reflective forecast is unbiased and returns are unpredictable. The following proposition is not conclusive but does shed light on the underlying issues.

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<sup>5</sup>The term  $UA$  is not a covariance strictly speaking. The word is used descriptively, since the term depends on the covariances between the shocks and the level of martingale.

**Proposition 2** *Given the following assumptions:*

- i) the innovations to the dividend process  $d_t$  are iid,*
  - ii) the innovation to the martingale  $\eta_t$  and  $n_t$  are uncorrelated,*
  - iii) the martingale  $m_t$  and the change  $\Delta n_t$  are uncorrelated,*
  - iv) the fraction  $n_t$  is unpredictable,  $E_{t-1}\Delta n_t = 0$ ,*
- the conditional forecast of excess returns is constant,  $E_{t-1}Z_t = \bar{Z}$ .*

**Proof.** *Using the expression for  $U_t$  (9), the term  $E_{t-1}Z_t$  can be written as the sum of three expectations*

$$E_{t-1}Z_t = \bar{Z} + E_{t-1}(p_t^* - E_{t-1}(p_t^*)) + \alpha^{-t}E_{t-1}(n_t\eta_t) - \alpha^{-t}E_{t-1}(\Delta n_t m_t).$$

*The four assumptions in Proposition 2 guarantee that the three expectations are zero, since the martingale innovation is mean zero  $E_{t-1}\eta_t = 0$ . Hence,  $E_{t-1}Z_t = \bar{Z}$ . ■*

The first two assumptions in Proposition 2 are innocuous as *i)* is satisfied for most specifications of dividends in the finance literature<sup>6</sup> and the martingale innovation in *ii)* is independently distributed. Assumptions *iii)* and *iv)* are plausible, but potentially unjustified. For any level of the martingale  $m_t$ , it is equally likely that the fraction  $n_t$  rises or falls, since it depends on the covariance  $U_t A_{t-1}$ . Furthermore, agents do not know the value of  $n_t$  when they make their forecast of  $p_{t+1}$ . However, if there is persistence in  $n_t$ , the inherent persistence in the martingale  $m_t$  could lead to correlations between the two.

Proposition 2 implies that the forecastability of returns depends on the forecastability of the populations' choices of forecasting strategies. If agents are unable to forecast the fraction  $n_t$ , the model with mysticism satisfies IE but not the EMM. If there is a way to use the potential persistence in  $n_t$ , which depends on how agents update their choices of forecasting strategy, returns could be predictable. In practice, where there are many sources of extraneous data, forecasting forecasting strategy choices would be difficult, but in a model with three strategies, cannot be ruled out. Whether this phenomenon quantitatively impacts the IE of the market is a question to be addressed with the simulation exercises.

Mysticism cannot maintain a following indefinitely. Given the presence of reflectivists and the existence of a minimum fraction of fundamentalists  $\delta_2$ , bubbles collapse endogenously. If fundamentalism could be eliminated from the population, then the fraction  $n_t$  is one, the payoff to mysticism (12) is identical to the payoff to reflectivism (10), and the model collapses to a representative agent rational bubble model. However, the presence of a minimum fraction of fundamentalists implies that  $n_t < 1$  and that the reflective and mystic forecasts are not identical.

**Proposition 3** *Given that  $n_t$  is fixed at its maximum  $1 - \delta_2$ , for  $\delta_2 > 0$ , and assumptions *i)* and *ii)* from Proposition 2, the expectation of the reflective payoff is strictly greater than the mystic payoffs, i.e.  $E_{t-1}\pi_{1,t} > E_{t-1}\pi_{3,t}$  with probability one.*

**Proof.** The difference in the payoffs (10) and (12) is

$$\pi_{1,t} - \pi_{3,t} = -2(1 - n_{t-1})U_t A_{t-1} + (1 - n_{t-1})^2 A_{t-1}^2$$

Given a constant  $n_t$  and  $\Delta n_t = 0$ , then  $E_{t-1}U_t A_t = E_{t-1}[(p_t^* - E_{t-1}(p_t^*))\alpha^{-t}m_t] + \alpha^{-2t}E_{t-1}(n_t\eta_t m_t)$  using the expression for  $U_t$  in equation (9) and  $A_t = \alpha^{-t}m_{t-1}$  by definition. Since the innovations  $d_t$  and  $\eta_t$  are

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<sup>6</sup>For model with drift, dividends would be defined as deviations from the deterministic model.

mean zero and uncorrelated with  $m_t$ , the expectation  $E_{t-1}U_tA_t = 0$ . Hence, the unconditional expectation

$$E_{t-1}(\pi_{1,t} - \pi_{3,t}) = (1 - n_{t-1})^2 A_{t-1}^2.$$

Since  $\delta_2 > 0$ , so is  $1 - n_{t-1}$ , so the right hand side is positive as long as  $A_{t-1} \neq 0$ , which only occurs if  $m_t = 0$ . This condition is met with probability 1, as is  $E\pi_{1,t} > E\pi_{3,t}$ . ■

Mysticism cannot dominate indefinitely. Since the expected value of the term  $U_tA_{t-1}$  in (12) is zero, reflectivism outperforms mysticism in the long run. Further, the magnitude of  $A_t$  (a submartingale) grows over time, so the third term in the payoff to mysticism (11) dominates and the performance of mysticism deteriorates over time. While mysticism can gain a following temporarily, whereby the martingale affects the asset price, eventually agents abandon mysticism in favor of reflectivism, so bubbles endogenously form and collapse. The goal of the simulations is to determine the quantitative importance of such outbreaks of mysticism.

Since it limits the life of bubbles, the minimum fraction of fundamentalists plays a similar role as the *projection facility* used with least squares learning as in Adam, Marcet and Niccolini (2016). Similarly, Lansing (2010) limits parameters so that agents focus on the one bubble of a continuum of solutions, that leads to stationarity in the first difference of the endogenous variable being forecast. In these models, a representative agent updates the estimate of the parameters in a forecasting rule, but the projection facility limits the acceptable estimates. In the present model, a small fraction of agents rejects extraneous information.

The present model represents a minimal departure from rationality when mystics are introduced into the population. Mysticism appears due to a disagreement about what constitutes fundamental information, but all agents form expectations with a reasonable economic model, i.e. agents meet the *cognitive consistency* principle described in Evans and Honkapohja (2011). Further, both mysticism and fundamentalism satisfy rationality in the homogeneous case, and reflectivism satisfies rationality when there is heterogeneity in the forecasting strategies, and this forecasting strategy is available to agents at all times. When mystics are eliminated from the population, the reflective and fundamental forecasts coincide. Only when mystics are introduced do the mystic and fundamental forecasts deviate from rationality, but mystics believe that the extraneous information in the martingale is relevant to the forecast of the asset price, and that other agents will eventually realize it. All agents believe that they are making efficient use of the available information.

## 4 Evolutionary Dynamics

A generalization of the replicator dynamic, a workhorse in the evolutionary game theory literature, describes the evolution of the vector  $x_t$  of the fractions of agents using the different forecasting strategies. This dynamic allows for the parameterization of agents' aggressiveness in switching to better performing strategies, which is a key determinant for the potential adoption of mysticism. This section also discusses the necessary conditions for the resulting emergence of rational bubbles.

Let the weighting function  $w(\pi)$  be a positive, increasing function of the payoffs. The general replicator dynamic is

$$x_{i,t+1} - x_{i,t} = x_{i,t} \frac{w(\pi_{i,t}) - \bar{w}_t}{\bar{w}_t}, \tag{14}$$

where the expression  $\bar{w}_t$  is the weighted population average  $\bar{w}_t = x_{1,t}w(\pi_{1,t}) + \dots + x_{n,t}w(\pi_{n,t})$  and  $\sum_{i=1}^n x_{i,t} = 1$ . A strategy gains followers if its weighted payoff above the weighted population average, i.e.



has positive fitness, in the language of evolutionary game theory. Such a dynamic is said to be *imitative* since strategies that are popular today, larger  $x_{i,t}$ , tend to gain more adherents if they are successful. Such dynamics have the potential to impart persistence to  $x_t$ , so assumptions *iii*) and *iv*) in Proposition 2 could be violated, which motivates the simulation exercises.

A general form for the dynamic (14) allows for a range of behavior of the agents. Compared to a linear weighting function  $w(\pi)$ , under a convex  $w(\pi)$ , agents switch to better performing strategies more quickly, see Hofbauer and Weibull (1996). A linear weighting function in the dynamic (14) corresponds to the special case of the replicator dynamic studied in Weibull (1998) and Samuelson (1997). Sandholm (2011) gives a thorough comparison of the features of a number of evolutionary dynamics. Waters (2009) discusses discrete time dynamics used in macroeconomic applications.

Using a version of the dynamic (14) with an alternate timing, Parke and Waters (2014) demonstrate that, for bounded dividends, the payoff to reflectivism is always above the population average<sup>7</sup>. Therefore, under the replicator (linear  $w(\pi)$ ), mysticism cannot take followers away from reflectivism. Under linear weighting, the covariance (second) terms in the payoffs to mysticism and fundamentalism, (12) and (11), cancel in the population average payoff<sup>8</sup>, but the third terms with  $A_{t-1}^2$  do not. Since the payoff to reflectivism is unaffected by the martingale, it is larger than the population average, so reflectivism gains followers.

Reflectivism's dominance is weaker in the case of a convex weighting function. Here, a positive covariance term  $U_t A_{t-1} > 0$  has greater benefit to mysticism than harm to fundamentalism, so it enters the population average payoff and, if it is large enough, mysticism can gain a following. The model used for simulations focuses on the exponential weighting function

$$w(\pi) = e^{\theta^2 \pi}, \tag{15}$$

so  $\theta$  parameterizes the aggressiveness of the agents. An increase in  $\theta$  means that agents are switching more quickly to the best strategy, but as  $\theta$  decreases the dynamic approaches the linear weighting case.

One drawback to imitative dynamics such as the generalized replicator (14) is their lack of inventiveness, see Waters (2009) for a discussion. If a strategy has no followers ( $x_i = 0$ ), then it cannot gain any. Hence, game theorists usually focus on equilibria that are evolutionarily stable, meaning they are robust to the introduction of a small fraction of deviating agents. Similarly, the focus of the present class of models is whether the fundamental forecast is robust to the introduction of a small fraction of mystics.

It is possible for mysticism to gain a following given the following conditions. *i*) Some agents believe that extraneous information may be important to the value of an asset. *ii*) In some periods, the extraneous martingale is correlated with fundamentals. *iii*) Agents must be sufficiently aggressive in switching to superior performing strategies. See Parke and Waters (2014) for a formal analysis.

## 5 Simulations

Simulation results show that the model satisfies IE but not the EMM. For some parameter choices, the data from the simulations is well represented by the EMM, but if agents are sufficiently aggressive about switching to better performing strategies and shocks to the martingale are of a similar magnitude to the dividend shocks, the data shows bubble-like behavior including excess variance in the asset price. However,

<sup>7</sup>The timing for the present work is chosen to avoid complications in the tests for return predictability.

<sup>8</sup>Given the timing of the present version of the model, the covariance terms may not cancel out to zero, but their impact is minimal.

returns are not significantly predictable under any circumstances.

The underlying dividend process is calibrated to the annual S&P 500 data for the sample 1871-2013 used by Shiller (2005) (updated here), using earnings as a proxy for dividends. Since not all firms pay dividends, earnings are a better measure of market fundamentals. Given the dividend  $d_t$  and the martingale  $m_t$ , the model is determined by the dynamic (14) along with the exponential weighting function (15), the payoffs (10), (11) and (12), and the realization of the asset price (7). The dividend process is specified as a stationary process with parameter choices below<sup>9</sup>.

$$d_t = \bar{d} + \rho_d (d_{t-1} - \bar{d}) + v_t \quad (16)$$

$\bar{d}$	$\rho_d$	$\sigma_v$
0.1166	0.465	0.203

The constant  $\bar{d}$  is chosen so that for  $\alpha = 0.95$ , the steady state price-dividend ratio (log difference) is 2.66, which is close to the long run average for the S&P 500 from the Shiller data. The persistence parameter  $\rho_d$  and shocks  $v_t \sim N(0, \sigma_v)$  are chosen to match values from the H-P detrended earnings series. They are also close to the linearly detrended series for postwar sample 1945-2013.

The free parameters  $\theta$ , which measures agent aggressiveness, and  $\sigma_\eta$ , the standard deviation of the martingale innovations, play a large role in determining the potential for outbreaks of mysticism and bubbles. For such events to occur, agents must be sufficiently aggressive, meaning  $\theta$  is sufficiently large, and the magnitude of the martingale innovations must be large enough to have a noticeable impact on the payoffs and the asset price, but not so large so that the third term in the payoff to mysticism (12) dominates.

Figures 1 and 2 demonstrate the effects of varying these two parameters. The simulations are initialized where the fraction of followers of fundamentalism and mysticism are at their minima. Furthermore the minimum fraction of mysticism 0.001 is much smaller than the minimum fraction of fundamentalism 0.1, so the fraction  $n_t$  is initially small, and the effect of mysticism on the asset price is minimal. If the fraction of followers of mysticism  $x_{3,t}$  falls below its minimum, it is reset to the minimum and the martingale  $m_t$  is restarted at zero.

In Figure 1, both parameters  $\theta$  and  $\sigma_\eta$  are set to low values. Here, mysticism never gains a following and the log price-dividend ratio and the excess returns are determined solely by the fundamental price  $p_t^*$ , which is determined by the exogenous dividend process under the EMM. However, for the larger values of these parameters in the simulation of Figure 2, there are occasional outbreaks where mysticism gains a following, and the price-dividend ratio deviates significantly from its steady state value. Note that the log  $(p_t - d_t)$  exceeding 3.23 is equivalent to the level of the price dividend ratio doubling its steady state value. The endogenous formation and collapse of the mystic bubbles and their effect on the asset price are evident in Figure 2.

## 5.1 Excess Variance

Studies such as Shiller (1981) demonstrate that asset prices fluctuate more than predicted by the EMM, and endogenous rational bubbles can explain such excess variance. Simulations determine a ratio of the realized variance and the predicted variance based on the variance of the dividends and the EMM embodied in (3).

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<sup>9</sup>LeBaron, Arthur and Palmer (1999) and Branch and Evans (2011) use stationary dividends. Adam, Marcat and Niccolini and Lansing (2010) both model dividends as a random walk with drift, which would complicate the present model and is left as a possibility for future work.

A statistical test of the variance of the price-dividend ratio provides more definitive evidence.

In the absence of mysticism ( $n_t = 0$ ), the asset price behaves according to the efficient markets model and depends only on the dividend process.

$$p_t^* = \bar{d} \left( \frac{\alpha}{1 - \alpha} - \frac{\alpha \rho_d}{1 - \alpha \rho_d} \right) + d_t \left( \frac{1}{1 - \alpha \rho_d} \right) \quad (17)$$

Hence, the variance of the asset price determined by fundamentals is  $\sigma_{p^*}^2 = (1 - \alpha \rho_d)^{-2} \sigma_d^2$ . Table 6 reports the ratio  $\sigma_p^2 / \sigma_{p^*}^2$  of the variance of the simulated asset prices  $\sigma_p^2$  and the predicted variance  $\sigma_{p^*}^2$  using the sample variance  $\hat{\sigma}_d^2$  of the simulated dividends. For the EMM, the ratio is unity, which occurs for very low levels of  $\theta$  and  $\sigma_\eta$ . For higher levels, the ratio rises above one, and, for some pairs of parameter values over two. This level is smaller than Shiller's (1981) initial estimate, but other research<sup>10</sup> has found smaller estimated values as well.

Table 1 reports results on a test of the significance of the excess variance<sup>11</sup>. Given normal errors in the dividend innovations  $v_t$  for the EMM, the ratio  $\sigma_p^2 / \sigma_{p^*}^2$  has the distribution  $\chi^2(n) / n$  where  $n$  is the number of periods. Table 2 shows the fraction of runs that exhibit excess variance beyond a 5% significance level. For small shocks to the martingale, represented by  $\sigma_\eta$ , and low levels of aggressiveness, shown by  $\theta$ , there is little or no excess variance, but for moderate to high values, large fractions, often well over half, show excess variance, corresponding to mystic outbreaks that lead to bubble-like behavior. The pattern in these results on excess variance qualitatively match the occurrence of excess kurtosis and GARCH effects found in Parke and Waters (2007). Mysticism can produce significant deviations from the predictions of the EMM.

## 5.2 Return Predictability

Proper tests give evidence that returns are not forecastable, so IE holds, a necessary but not sufficient condition of the efficient markets model.

Excess returns  $Z_t$  are given by eq. (13) which is the reflective forecast error (9) plus a constant. Proposition 2 gives shows that excess returns are constant if the fractions of followers of the different strategies are unpredictable. Whether those fractions impart persistence on the excess returns is a question for the simulation exercises.

To test predictability, the following equation is estimated to test whether lagged data contains information about current returns  $Z_t$ , similar to those used in Fama and French (1988), among many others.

$$Z_t = \beta_0 + \beta_1 x_{t-1} + \varepsilon_t, \quad (18)$$

The error term  $\varepsilon_t$  has standard deviation  $\sigma_\varepsilon$ , and there are multiple candidates for the predictor variable  $x_t$ . Results are reported for the price-dividend ratio, a common choice for the predictor, and dividends, which is particularly relevant in the present context. A standard approach is to test the null  $\widehat{\beta}_1 = 0$  and least squares estimates on market data are often significantly different than zero, though the economic significance is often questionable (Cochrane 2008). Least squares estimates of  $\widehat{\beta}_1$  on the simulated data show predictability in many cases, roughly following the pattern in Tables 1 and 2, meaning that for choices of the parameters  $\theta$  and  $\sigma_\eta$  that have frequent occurrences of excess variance also have tend to have estimates of  $\widehat{\beta}_1$  that are

<sup>10</sup>Some examples are LeRoy and Porter (1981), Campbell and Shiller (1989) and LeRoy and Parke (1992). The issue is complicated since some of these models account for a time varying interest rate or discount factor.

<sup>11</sup>For all the tests, the simulations are initialized with 50 periods followed by a run of 100, which is similar to the samples used for calibration. The table report results over 10,000 runs.

significantly different than zero. However, the persistence in both choices for the predictor variable means that standard least squares  $t$  statistics are biased away from zero.

Campbell and Yogo (2006) discuss this issue in detail and develop a consistent statistic  $Q$  where the predictor variable  $x_t$  could be persistent such that

$$x_t = \chi + \rho x_{t-1} + \xi_t. \quad (19)$$

The covariance  $\sigma_{\xi\varepsilon}$  between the innovations  $\zeta_t$  and  $\varepsilon_t$  from (18) is used to define the following parameters:

$$\gamma_{\xi\varepsilon} = \frac{\sigma_{\xi\varepsilon}}{\sigma_\xi^2}; \delta = \frac{\sigma_{\xi\varepsilon}}{\sigma_\varepsilon\sigma_\xi}.$$

The test statistic  $Q$  for the null  $\widehat{\beta}_1 = 0$  follows.

$$Q = \frac{\widehat{\beta}_1 - \gamma_{\xi\varepsilon}(\widehat{\rho} - \rho)}{\sigma_\varepsilon(1 - \delta^2)^{\frac{1}{2}} \left( \sum_{t=1}^T \widehat{x}_t^2 \right)^{-\frac{1}{2}}} \quad (20)$$

The value of  $\widehat{\rho}$  is the least squares estimate of the true predictor persistence parameter  $\rho$  in (19). Campbell and Yogo (2006) do not have knowledge of  $\rho$  for their applied work, and they develop a method for estimating Bonferroni bounds for this parameter, but that is unnecessary with simulated data.

Note that if the persistence parameter equals the true value  $\widehat{\rho} = \rho$  and errors  $v_t$  and  $\varepsilon_t$  are uncorrelated so that  $\delta = 0$ , the  $Q$  statistic collapses to the standard least squares  $t$  statistic. Lewellen (2004) refers to the  $\gamma_{ev}(\widehat{\rho} - \rho)$  term as the "finite sample bias" correction, see the discussion in Campbell and Yogo (2006).

These considerations mean that dividends  $d_t$  is a natural choice for the predictor variable, since the true value of the persistence parameter is known from the specification (16) so that  $\rho = \rho_d$ . The most common choice for the predictor variable in applied work is the price-dividend ratio. Using the price-dividend ratio  $x_t = p_t - d_t$ , the maintained hypothesis that the EMM holds implies that the persistence parameter  $\rho$  in (19) is the same as the case with dividends so that  $\rho = \rho_d$ . So the chosen value of  $\rho_d$  is the appropriate choice for  $\rho$  in the  $Q$  statistic (20) for both choices of the predictor.

Tables 3 and 4 report the percentage of runs that show predictability of returns with a significance of 5%. Critical values for  $Q$  are determined by a Monte Carlo simulation of  $10^6$  runs of the model without mysticism, meaning the martingale  $m_t$  is fixed at zero. Table 3 shows results using dividends as a predictor, and the fractions are all close to 0.05, the significance level of the test. Therefore, excess returns are not predictable at all. Table 4 shows analogous results using the price-dividend ratio. There is some deviation from 0.05, but the largest fraction is 0.112 which is not nearly as large as 0.762, the fraction showing significant excess variance for that choice of  $\theta$  and  $\sigma_\eta$ . In cases where excess variance occurs more than would be predicted under the EMM, the fraction of runs with predictable returns is far smaller.

The model with mysticism can produce excess variance in returns but returns are not predictable. There are clear deviations from the EMM, but they rarely appear in the tests for IE. Therefore, excess variance does not necessarily correspond to predictable returns. Hence, tests for predictability of returns have very little power to reject the deviations from the EMH caused by mystic bubbles.

### 5.3 Excess Variance Robustness

The robustness of the results is examined using alternative tests for excess variance and a different parameterization. In the spirit of the construction of the  $Q$ -statistic, it might be more appropriate to use the least squares estimate for the persistence parameter  $\hat{\rho}$ . Table 5 reports results about the fraction of runs showing excess variance when the variance of the fundamental price  $\sigma_{p^*}^2 = (1 - \alpha\rho_d)^{-2} \sigma_d^2$  is estimated using  $\frac{\sigma_\varepsilon^2}{1 - \hat{\rho}^2}$  for the variance of the dividends  $\sigma_d^2$ . This test statistic has the proper level of type 1 error in cases without mysticism, and shows the same pattern, in that there are many cases with excess variance for parameter values that allow for mystic outbreaks.

In a study of the yield curve, Flavin (1983) demonstrates that excess variance results can arise due to small sample bias. For the present model, tests on longer samples only strengthen the evidence for excess variance, as was found in the applied work of Shiller (1990).

Tables 6-10 show the same results for a different parameterization of the model, and the conclusions are unchanged. Using a linear trend for the earnings series over the full sample 1871-2013, gives the calibrated values,  $\rho = 0.69$  and  $\sigma_v = 0.228$  for the detrended series  $d_t$  given in eq. (16). Simulations using these parameters are reported in tables 5-10. The results for the excess variance are not as dramatic, for only one choice of  $\theta$  and  $\sigma_\eta$  do more than half of the runs show significant excess variance, for example. However, the overall message is the same. Excess variance is a common occurrence, while return predictability is not. In fact, using dividends as the predictor, there is no predictability beyond that at the expected level of type 1 error (Table 8).

## 6 Conclusion

The model of mysticism is a specific example that satisfies informational efficiency but does not conform to the efficient markets model. Returns are unpredictable, but asset prices depend on more than fundamentals. Proper tests of return predictability have very little power to reject mystic bubbles.

The information structure of the model satisfies the cognitive consistency principle, in that agents' forecasts are based on reasonable economic models. Mystic bubbles form endogenously due to fortuitous correlations between extraneous and fundamental data, but they cannot last since the reflectivist forecast, which takes into account the behavior of the other agents, outperforms the mystic forecast in the long run. The model explains multiple stylized facts about asset markets such as excess variance and GARCH effects.

One might argue that mystic bubbles are not plausible, but heterogeneous forecasts are an observed fact. Since agents do not adopt mysticism indefinitely, any transversality condition argument does not apply. Further, there are many candidates for extraneous information represented by the martingale such as exchange rates, commodity prices or "expert" forecasts.

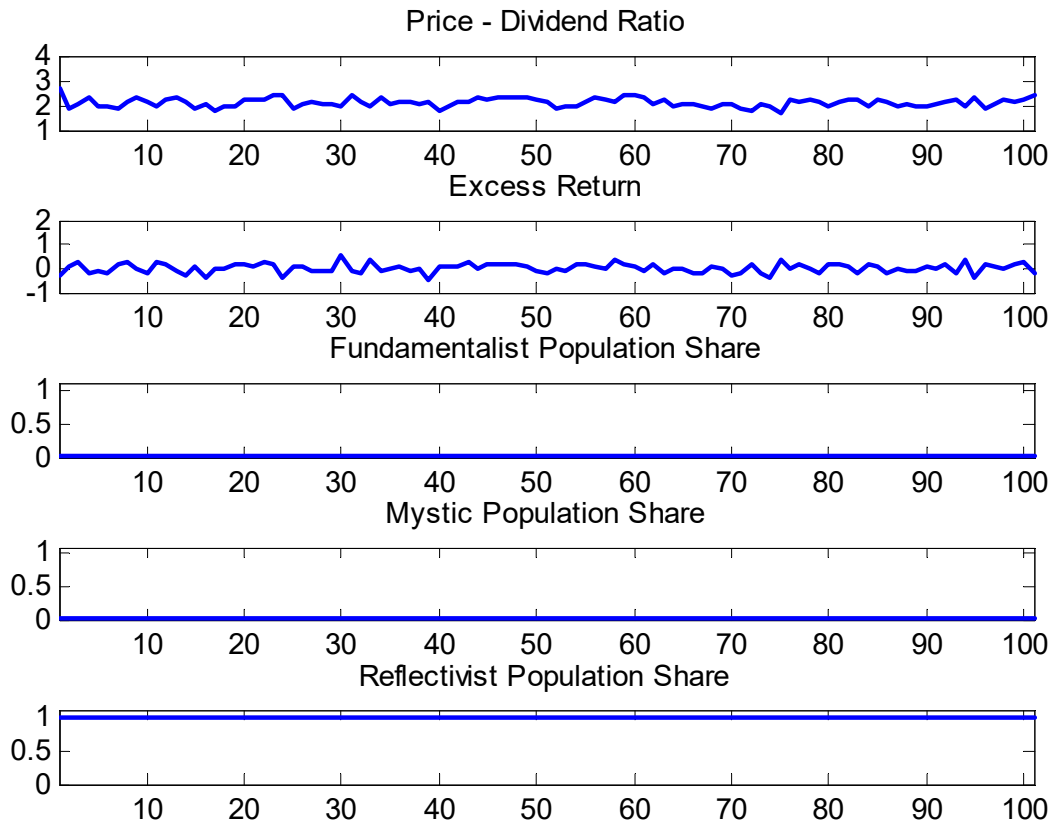
The weak power of the tests of return predictability shows the limitations of the interpretations of all such empirical results. The inability to reject unpredictability is not evidence against the presence of extraneous information nor excess variance in the asset price. The exercise could be conducted in a more sophisticated environment, explicitly modeling trends or including behavioral forecasting strategies, for example. However, if the interpretation of tests for return predictability is problematic in a simple environment, they will not be more meaningful with added complications.

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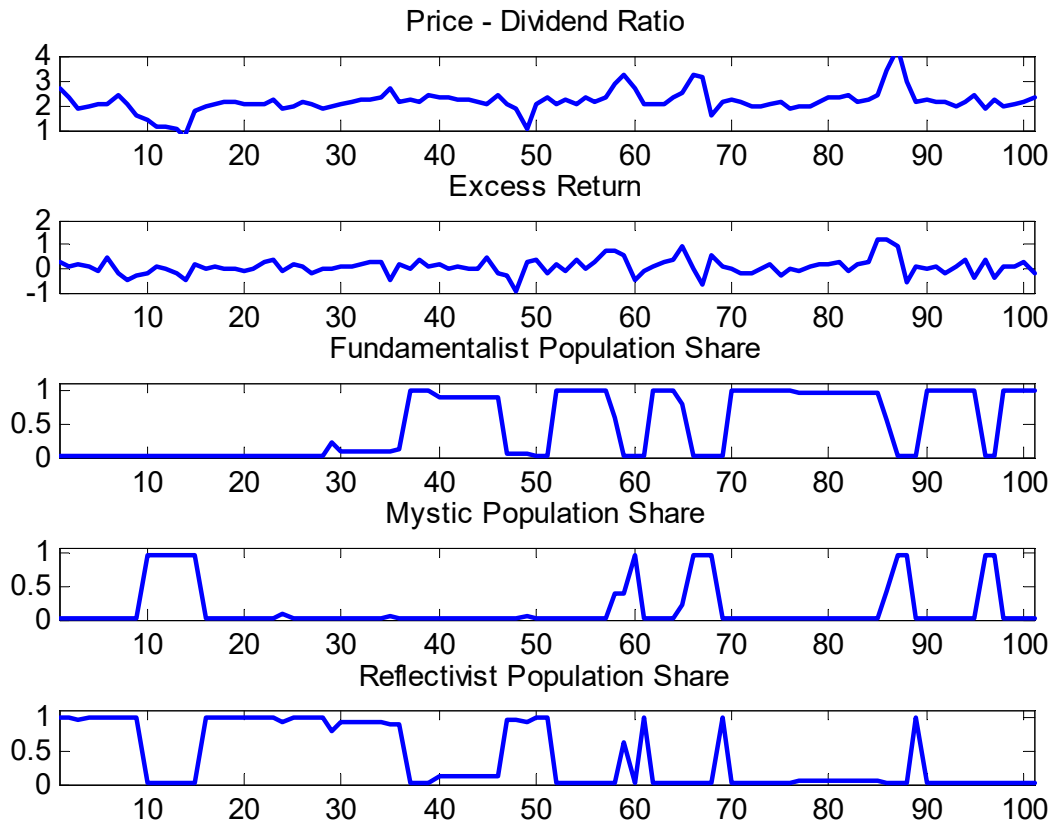
Figure 1



$$\theta = 1/4, \sigma_{\eta} = \sigma_v \times 1/4$$



Figure 2



$$\theta = 10.0, \sigma_{\eta} = \sigma_v \times 2.0$$

Table 1

		$\sigma_{\eta} = \sigma_v X$						
		<b>1/8</b>	<b>1/4</b>	<b>1/2</b>	<b>1</b>	<b>2</b>	<b>4</b>	<b>8</b>
$\theta$	<b>5/8</b>	1.009	1.016	1.025	1.034	1.043	1.0642	1.1689
	<b>5/4</b>	1.010	1.016	1.026	1.035	1.041	1.0583	1.148
	<b>5/2</b>	1.046	1.074	1.302	1.679	1.701	1.4557	1.2933
	<b>5</b>	1.713	2.118	2.413	1.904	1.420	1.1937	1.1356
	<b>10</b>	1.110	1.263	1.501	1.599	1.531	1.5396	1.8083
	<b>20</b>	1.014	1.043	1.116	1.221	1.320	1.5088	1.9944
	<b>40</b>	1.003	1.012	1.039	1.091	1.169	1.3216	1.6861

Each cell shows the ratio  $\sigma_p^2/\sigma_{p^*}^2$  of the variance of  $p_t$  to the variance of the fundamental price  $p_t^*$  for each choice of  $\theta$  and  $\sigma_{\eta}$ .

Table 2

		$\sigma_{\eta} = \sigma_v X$						
		<b>1/8</b>	<b>1/4</b>	<b>1/2</b>	<b>1</b>	<b>2</b>	<b>4</b>	<b>8</b>
$\theta$	<b>5/8</b>	0.000	0.000	0.000	0.000	0.000	0	0.195
	<b>5/4</b>	0.000	0.000	0.000	0.000	0.000	0.0002	0.1084
	<b>5/2</b>	0.004	0.009	0.025	0.059	0.088	0.087	0.0829
	<b>5</b>	0.145	0.248	0.326	0.346	0.269	0.1798	0.1269
	<b>10</b>	0.081	0.283	0.587	0.762	0.751	0.743	0.7496
	<b>20</b>	0.000	0.004	0.070	0.338	0.615	0.7922	0.8853
	<b>40</b>	0.000	0.000	0.001	0.026	0.222	0.5558	0.7803

Each cell shows the fraction of runs with significant excess variance for each choice of  $\theta$  and  $\sigma_{\eta}$ .

Table 3

		$\sigma_\eta = \sigma_v X$						
		<b>1/8</b>	<b>1/4</b>	<b>1/2</b>	<b>1</b>	<b>2</b>	<b>4</b>	<b>8</b>
$\theta$	<b>5/8</b>	0.048	0.048	0.051	0.049	0.049	0.043	0.038
	<b>5/4</b>	0.048	0.049	0.050	0.049	0.046	0.047	0.043
	<b>5/2</b>	0.046	0.051	0.046	0.052	0.046	0.043	0.051
	<b>5</b>	0.042	0.047	0.044	0.042	0.043	0.050	0.054
	<b>10</b>	0.053	0.042	0.044	0.041	0.050	0.052	0.049
	<b>20</b>	0.051	0.052	0.049	0.048	0.042	0.040	0.026
	<b>40</b>	0.050	0.046	0.050	0.053	0.043	0.038	0.024

Using  $d_t$  as the predictor variable  $x_t$ , each cell shows the fraction of runs where the estimate of  $\hat{\beta}_1$  (from (18)) is significantly different than zero, according to the test statistic  $Q$  (20), for each choice of  $\theta$  and  $\sigma_\eta$ .

Table 4

		$\sigma_\eta = \sigma_v X$						
		<b>1/8</b>	<b>1/4</b>	<b>1/2</b>	<b>1</b>	<b>2</b>	<b>4</b>	<b>8</b>
$\theta$	<b>5/8</b>	0.054	0.047	0.043	0.047	0.042	0.041	0.030
	<b>5/4</b>	0.050	0.051	0.048	0.052	0.041	0.042	0.034
	<b>5/2</b>	0.049	0.052	0.054	0.060	0.063	0.066	0.055
	<b>5</b>	0.064	0.086	0.095	0.093	0.089	0.068	0.051
	<b>10</b>	0.063	0.074	0.094	0.112	0.073	0.039	0.024
	<b>20</b>	0.050	0.054	0.064	0.058	0.036	0.026	0.020
	<b>40</b>	0.048	0.049	0.048	0.045	0.033	0.026	0.021

Using  $p_t - d_t$  as the predictor variable  $x_t$ , each cell shows the fraction of runs where the estimate of  $\hat{\beta}_1$  (from (18)) is significantly different than zero, according to the test statistic  $Q$  (20), for each choice of  $\theta$  and  $\sigma_\eta$ .

Table 5

		$\sigma_\eta = \sigma_v X$						
		<b>1/8</b>	<b>1/4</b>	<b>1/2</b>	<b>1</b>	<b>2</b>	<b>4</b>	<b>8</b>
$\theta$	<b>5/8</b>	0.044	0.049	0.063	0.081	0.114	0.2498	0.6874
	<b>5/4</b>	0.046	0.052	0.055	0.077	0.118	0.2253	0.6005
	<b>5/2</b>	0.046	0.046	0.056	0.088	0.146	0.1964	0.2601
	<b>5</b>	0.071	0.106	0.144	0.175	0.234	0.2607	0.2167
	<b>10</b>	0.039	0.041	0.098	0.406	0.759	0.8369	0.8527
	<b>20</b>	0.043	0.058	0.114	0.435	0.804	0.9169	0.9596
	<b>40</b>	0.044	0.051	0.096	0.281	0.611	0.8159	0.9206

Each cell shows the fraction of runs with significant excess variance for each choice of  $\theta$  and  $\sigma_\eta$ , using an estimated value of  $\hat{\rho}$  to compute  $\sigma_{p^*}^2$ .

Table 6

		$\sigma_\eta = \sigma_v X$						
		<b>1/8</b>	<b>1/4</b>	<b>1/2</b>	<b>1</b>	<b>2</b>	<b>4</b>	<b>8</b>
$\theta$	<b>5/8</b>	1.006	1.009	1.012	1.015	1.017	1.0206	1.0458
	<b>5/4</b>	1.006	1.009	1.013	1.016	1.016	1.0217	1.0517
	<b>5/2</b>	1.025	1.075	1.184	1.341	1.277	1.1169	1.0594
	<b>5</b>	1.196	1.280	1.321	1.257	1.136	1.068	1.0533
	<b>10</b>	1.034	1.080	1.143	1.171	1.158	1.1632	1.246
	<b>20</b>	1.006	1.015	1.035	1.062	1.086	1.1302	1.249
	<b>40</b>	1.001	1.004	1.011	1.025	1.043	1.079	1.1672

Each cell shows the ratio  $\sigma_p^2/\sigma_{p^*}^2$  of the variance of  $p_t$  to the variance of the fundamental price  $p_t^*$  for each choice of  $\theta$  and  $\sigma_\eta$

Tables 6-10 have results for the same exercises in Tables 1-5 for an alternative parameterization of the dividends.

Table 7

		$\sigma_\eta = \sigma_v \times$						
		<b>1/8</b>	<b>1/4</b>	<b>1/2</b>	<b>1</b>	<b>2</b>	<b>4</b>	<b>8</b>
$\theta$	<b>5/8</b>	0.046	0.051	0.055	0.057	0.059	0.0667	0.0944
	<b>5/4</b>	0.046	0.055	0.057	0.056	0.061	0.0609	0.0796
	<b>5/2</b>	0.062	0.082	0.111	0.173	0.188	0.1519	0.1079
	<b>5</b>	0.204	0.280	0.343	0.372	0.269	0.1697	0.1384
	<b>10</b>	0.076	0.150	0.291	0.369	0.341	0.3629	0.4951
	<b>20</b>	0.047	0.050	0.073	0.122	0.170	0.2752	0.5233
	<b>40</b>	0.040	0.047	0.057	0.067	0.092	0.1622	0.3664

Each cell shows the fraction of runs with significant excess variance for each choice of  $\theta$  and  $\sigma_\eta$ .

Table 8

		$\sigma_\eta = \sigma_v \times$						
		<b>1/8</b>	<b>1/4</b>	<b>1/2</b>	<b>1</b>	<b>2</b>	<b>4</b>	<b>8</b>
$\theta$	<b>5/8</b>	0.052	0.052	0.050	0.046	0.046	0.0447	0.0432
	<b>5/4</b>	0.047	0.046	0.053	0.046	0.049	0.0483	0.0412
	<b>5/2</b>	0.053	0.046	0.052	0.052	0.044	0.0483	0.0445
	<b>5</b>	0.050	0.046	0.046	0.042	0.046	0.0468	0.0505
	<b>10</b>	0.048	0.053	0.047	0.051	0.057	0.0495	0.0405
	<b>20</b>	0.051	0.048	0.049	0.051	0.051	0.0412	0.0291
	<b>40</b>	0.046	0.046	0.052	0.051	0.047	0.0417	0.0361

Using  $d_t$  as the predictor variable  $x_t$ , each cell shows the fraction of runs where the estimate of  $\widehat{\beta}_1$  (from (18)) is significantly different than zero, according to the test statistic  $Q$  (20), for each choice of  $\theta$  and  $\sigma_\eta$ .

Table 9

		$\sigma_\eta = \sigma_v X$						
		<b>1/8</b>	<b>1/4</b>	<b>1/2</b>	<b>1</b>	<b>2</b>	<b>4</b>	<b>8</b>
$\theta$	<b>5/8</b>	0.045	0.048	0.046	0.046	0.043	0.0415	0.0357
	<b>5/4</b>	0.050	0.044	0.050	0.046	0.046	0.0428	0.0389
	<b>5/2</b>	0.052	0.050	0.055	0.063	0.063	0.0719	0.0552
	<b>5</b>	0.058	0.066	0.073	0.073	0.077	0.0624	0.0478
	<b>10</b>	0.052	0.057	0.068	0.075	0.066	0.0393	0.0238
	<b>20</b>	0.048	0.045	0.051	0.052	0.038	0.029	0.0197
	<b>40</b>	0.050	0.047	0.050	0.043	0.039	0.0347	0.0227

Using  $p_t - d_t$  as the predictor variable  $x_t$ , each cell shows the fraction of runs where the estimate of  $\hat{\beta}_1$  (from (18)) is significantly different than zero, according to the test statistic  $Q$  (20), for each choice of  $\theta$  and  $\sigma_\eta$ .

Table 10

		$\sigma_\eta = \sigma_v X$						
		<b>1/8</b>	<b>1/4</b>	<b>1/2</b>	<b>1</b>	<b>2</b>	<b>4</b>	<b>8</b>
$\theta$	<b>5/8</b>	0.046	0.051	0.055	0.057	0.059	0.0667	0.0944
	<b>5/4</b>	0.046	0.055	0.057	0.056	0.061	0.0609	0.0796
	<b>5/2</b>	0.062	0.082	0.111	0.173	0.188	0.1519	0.1079
	<b>5</b>	0.204	0.280	0.343	0.372	0.269	0.1697	0.1384
	<b>10</b>	0.076	0.150	0.291	0.369	0.341	0.3629	0.4951
	<b>20</b>	0.047	0.050	0.073	0.122	0.170	0.2752	0.5233
	<b>40</b>	0.040	0.047	0.057	0.067	0.092	0.1622	0.3664

Each cell shows the fraction of runs with significant excess variance for each choice of  $\theta$  and  $\sigma_\eta$ , using and estimated value of  $\hat{\rho}$  to compute  $\sigma_{p^*}^2$ .