

An Improved Way to Model the Evolution of Heterogeneous Forecasts

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September 28, 2010

Abstract

The α -BNN dynamic retains the desirable features of the BNN dynamic with a continuous derivative the lack of which complicates the analysis of the cobweb model under the latter. Both dynamics satisfy positive correlation and inventiveness, and there is an intuitively appealing steady state where one strategy dominates, which is not possible with the multinomial logit dynamic. Under the α -BNN dynamic, there is an unstable 2-cycle, as with the BNN dynamic, but it is not asymptotically unstable.

Keywords: Chaos, Cobweb Model, Heterogeneous Forecasts, α -BNN

JEL codes: C62, C73, D83, D84

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While the assumption that there is a unique representative forecast is common in macroeconomics, researchers should recognize that changing forecasting strategies is quite possible and could be an important determinant of economic dynamics. Much work in this area has focused on the cobweb model where the multinomial logit (MNL) dynamic describes switching between rational and naive forecasting strategies¹. However, this dynamic and imitative dynamics such as the replicator are problematic when some strategies are at or close to extinction. Waters (2009a) introduces the Brown, von Neumann and Nash (1950), henceforth BNN, dynamic in this context and shows that it does not have such problems and that there is a more natural interpretation of the steady states, compared to other dynamics. The major drawback is that the BNN dynamic does not have a continuous derivative where the payoffs are equal, a situation that often arises in non-trivial applications. Fortunately, the α -BNN dynamic, an extension suggested by Weibull (Nobel Seminar, 1994), is continuous in the payoff while retaining the desirable features of the BNN dynamics such as *positive correlation*, a weak monotonicity condition, and *inventiveness*, which means that strategies with no followers can gain some if they perform well.

Let $q_{k,t}$ be the fraction of followers and $\pi_{k,t}$ be the payoff of strategy k in time t . A general dynamic Φ_k describes the evolution of $q_{k,t}$ within the simplex $\bar{q} = \left\{ (q_{1,t}, \dots, q_{H,t}) \mid \sum_{h=1}^H q_{h,t} = 1 \right\}$ according to the vector of payoffs $\pi_t = (\pi_{1,t}, \dots, \pi_{H,t})$ such that $q_{k,t+1} = \Phi_k(q_t, \pi_t)$ where $q_t \in \bar{q}$. The population average payoff $\bar{\pi}_t = \sum_{h=1}^H q_{h,t} \pi_{h,t}$ determines the excess payoff $\hat{\pi}_{k,t} = \pi_{k,t} - \bar{\pi}_t$ for strategy k in time t .

The α -BNN dynamic depends on the positive excess payoffs in the population where the α parameterizes the speed of adjustment of the fractions of followers of the different strategies.

$$q_{k,t+1} = \frac{q_{k,t} + [\hat{\pi}_{k,t}]_+^\alpha}{1 + \sum_{h=1}^H [\hat{\pi}_{h,t}]_+^\alpha}, \quad (1)$$

This dynamic² is a special case of the class of excess payoff dynamics (Sandholm 2006), which satisfy positive correlation and inventiveness.

Definition 1 *The dynamic Φ_k satisfies **positive correlation** iff $\sum_{h=1}^H (q_{h,t+1} - q_{h,t}) \hat{\pi}_{h,t} > 0$ unless $\hat{\pi}_{h,t} = 0$ for all h .*

Positive correlation requires that the change of the fractions of the population using different strategies is correlated with their payoffs, so strategies with higher payoffs gain adherents on average, ensuring out-of-equilibrium dynamic paths reflect strategic incentives. Though this is the weakest monotonicity condition

¹Sethi and Franke (1995) develop the cobweb model with naive and rational forecasts. Brock and Hommes (1997) discuss the nature of the chaotic dynamics arising under MNL. Hommes (2006) surveys the succeeding literature.

²This particular form is found in Nash's (1951) proof of the stability of the Nash equilibrium.

in the literature, the multinomial logit models commonly used to describe switching between forecasting strategies does not satisfy positive correlation, see Waters (2009a).

Definition 2 *The dynamic Φ_k satisfies **inventiveness** if strategies with positive excess payoff in time t have a positive fraction of followers in time $t + 1$.*

Inventiveness (Weibull, Nobel Seminar, 1994) ensures that strategies needn't be permanently extinct if they perform well, which is the case under imitative dynamics such as the replicator³. Inspection shows that the α -BNN dynamic satisfies inventiveness.

The most common specification of the cobweb model with heterogeneous forecasting strategies allows producers to forecast output prices using either a rational or naive forecast. Let q be the fraction using the rational forecast, which is equivalent to perfect foresight in the absence of stochastic terms in the model. While the rational forecast is more accurate than the naive forecast, which simply uses the price in the previous period, the producer must pay a cost $C > 0$ to use it. With linear demand and supply determined by these two forecasting strategies, price dynamics⁴ are given by the following, where $\hat{b} = b/B$ is the ratio of the supply elasticity b and the demand elasticity B .

$$p_{t+1} = - \left[\frac{\hat{b}(1 - q_t)}{\hat{b}q_t + 1} \right] p_t. \quad (2)$$

The variable p_t is the deviation of the price from a positive steady state. Clearly, $p_t = 0$ is a steady state of (2), though there is also the possibility of a 2-cycle if the bracketed term equals one.

The dynamics of q_t governing the choice of forecasting strategy are given by the α -BNN dynamic (1), which depend on the excess payoffs for the rational and naive strategies $\hat{\pi}_{R,t}$ and $\hat{\pi}_{N,t}$. The payoffs to the two strategies are profits for the firms, taking into account the cost of the rational forecast. With two strategies, the excess payoffs take the following form.

$$\hat{\pi}_{R,t} = (1 - q_t) (\pi_{R,t} - \pi_{N,t}) \quad (3)$$

$$\hat{\pi}_{N,t} = q_t (\pi_{N,t} - \pi_{R,t}). \quad (4)$$

The difference in the rational and naive payoffs $\pi_{R,t}$ and $\pi_{N,t}$ is important so let the payoff difference function be $\gamma(p_t, q_t) = \pi_{R,t} - \pi_{N,t}$. Computation (Waters, 2009a) yields the following expression for $\gamma(\cdot)$.

³Sandholm (2006) compares imitative and excess payoff dynamics.

⁴See Waters (2009a) for a detailed development of the model. The equation for price dynamics and the payoffs are equivalent to those found in Brock and Hommes (1997).

$$\gamma(p_t, q_t) = p_t^2 \left[\frac{B\hat{b}(\hat{b}+1)^2}{2(\hat{b}q_t+1)^2} \right] - C \quad (5)$$

The non-negativity restriction on excess payoffs in the α -BNN dynamic (1) complicates the analysis, so we express the dynamics for q_t as follows.

$$q_{t+1} = \left\{ \begin{array}{ll} \frac{q_t}{1 - q_t \gamma(p_t, q_t)^\alpha} & \text{for } \gamma(p_t, q_t) \leq 0 \\ 1 - \frac{1 - q_t}{1 + (1 - q_t) \gamma(p_t, q_t)^\alpha} & \text{for } \gamma(p_t, q_t) > 0 \end{array} \right\} \quad (6)$$

Whether q_t rises or falls depends on whether the payoff difference $\gamma(p_t, q_t)$ is positive or negative. The function F describe the motion of (p_t, q_t) .

Definition 3 *The evolution function F where $(p_{t+1}, q_{t+1}) = F(p_t, q_t)$ is determined by the price dynamics (2), and the evolution of q_t (6) along with the payoff difference function $\gamma(p_t, q_t)$ in (5).*

Now that the dynamics of the model are concisely described, consider first the existence of the steady state such that $(p_t, q_t) = F(p_t, q_t)$. As noted above, a zero price deviation is a steady state of the price dynamics equation (2). If $p_t = 0$, then $\gamma(p_t, q_t) = -C$ meaning the naive forecast payoff dominates, so $(p_t, q_t) = (0, 0)$ is a steady state. The stability of this steady state is determined by the Jacobian of the evolution function evaluated at the origin. Intuitively, the relevant stability concept implies that if (p_t, q_t) is sufficiently close to the steady state, then the path given by F will remain within a neighborhood of the steady state. This concept is weaker than asymptotic stability, which implies convergence to a steady state. See Lakshmikantham and Trigiante (2002) for a formal discussion of the stability of difference equations.

Proposition 4 *The origin is a stable steady state of F for $\hat{b} \leq 1$ but is not stable for $\hat{b} > 1$.*

Proof. The origin is a steady state of (2) and the $\gamma(\cdot) \leq 0$ case of (6) by inspection. The Jacobian for these two equations evaluated at the origin is $J_{(0,0)} = \begin{pmatrix} -\hat{b} & 0 \\ 0 & 1 \end{pmatrix}$. If $\hat{b} < 1$, one eigenvalue of $J_{(0,0)}$ are less one and the other equals one, so the time path of (p_t, q_t) will remain within a given neighborhood of the origin. However, if $\hat{b} > 1$, there is an eigenvalue larger than one and the origin is not stable. The Jacobian is the identity matrix in the special case of $\hat{b} = 1$, which yields stability. ■

When the price remains close to the steady state, the naive forecasting strategy performs well and there is no reason for agents to incur the cost of the rational forecast, so the presence of a stable steady state at

$(p_t, q_t) = (0, 0)$ is quite natural. There is a related result in Branch and McGough (2008) using a modified replicator dynamic, though the model of Brock and Hommes (1997) using the MNL dynamic cannot have steady states where one strategy dominates.

Both models have a 2-cycle related to the one described next. To simplify the analysis, let the evolution function \widehat{F} be such that $\widehat{F}(p_t, q_t) = F(-p_t, q_t)$ for $p_t < 0$, which has the effect of reflecting negative price deviations to positive values, for a positive initial $p_0 > 0$. To study the stability of a two cycle, one must focus on the second iterate of the evolution function, and the second iterates of F and \widehat{F} are identical.

Proposition 5 *For $\alpha \neq 1, \widehat{b} > 1$, there exists a 2-cycle of F (and steady state of \widehat{F}) given by $(\pm p^*, q^*) = \left(\pm \sqrt{\frac{C}{2b}}, \frac{\widehat{b}-1}{2\widehat{b}} \right)$. The evolution function \widehat{F} has a continuous derivative at (p^*, q^*) , i.e. the Jacobians are identical for the cases of $\gamma(p_t, q_t) \leq 0$. The steady state (p^*, q^*) of \widehat{F} is not stable.*

Proof. The 2-cycle requires that the bracketed term $[\cdot]$ in the price dynamics equation (2) must be one, which determines q^* . The payoff difference function $\gamma(p_t, q_t)$ must be zero given q^* , which determines p^* .

The stability analysis makes use of the Jacobian $\widehat{J}_{(p^*, q^*)}$ of \widehat{F} at its steady state $(p^*, q^*) = \left(\sqrt{\frac{C}{2b}}, \frac{\widehat{b}-1}{2\widehat{b}} \right)$. The equations describing \widehat{F} are identical to those for F in Definition 3, except that the negative sign in the price dynamics equation (2) does not appear. Using the $\gamma(p_t, q_t) \leq 0$ case of the function for the evolution of q_t (6), the Jacobian is

$$\widehat{J}_{(p^*, q^*)} = \begin{pmatrix} 1 & -\sqrt{\frac{2C}{b}} \left(\frac{2\widehat{b}}{\widehat{b}+1} \right) \\ 0 & 1 \end{pmatrix}.$$

Furthermore, the Jacobian using the $\gamma(p_t, q_t) \leq 0$ of (6) is identical. With ones on the diagonal of \widehat{J} , and non-zero elements off the diagonal imply that for any points off the p - and q -axes, time paths governed by \widehat{F} move away from (p^*, q^*) , so it is not stable. ■

That \widehat{F} has a single Jacobian governing the neighborhood around (p^*, q^*) is the primary difference with the analysis of the case where $\alpha = 1$. In that case, there are distinct Jacobians for $\gamma(p_t, q_t) \leq 0$, and the steady state is asymptotically unstable⁵, but we cannot make such a conclusion for $\alpha \neq 1$.

As mathematical analysis, the present work is not particularly sophisticated, but that is precisely the point. The α -BNN dynamic has desirable characteristics not found in other dynamics and avoids the difficulties that can arise from the lack of a global continuous derivative for $\alpha = 1$, see Waters (2009a, b).

The cobweb model with the α -BNN dynamic has chaotic dynamics in the presence of the 2-cycle described in Proposition 2. There are bifurcations in both the ratio of supply and demand elasticities \widehat{b} and the speed

⁵Asymptotic stability implies that under $\widehat{F}^{-1}(p_t, q_t)$, then $(p_t, q_t) \rightarrow (p^*, q^*)$ as $t \rightarrow \infty$. Waters (2009b) proves exponential instability, an even stronger stability concept, of the 2-cycle under the BNN dynamic.

of adjustment parameter α . For varying parameter choices there are both periodic and strange attractors⁶ as in other studies. One novel aspect of the α -BNN dynamic is the behavior for large values of α . Unlike the MNL approach with high search intensity, the the variables in the cobweb model show little variation for sufficiently large α .

The α -BNN dynamic has all of the positive attributes of excess payoff dynamics along with a continuous derivative that alleviates much of the difficulty in analyzing behavior under the BNN dynamic. Excess payoff dynamics satisfy positive correlation and inventiveness and admit naturally interpretable steady states where one strategy dominates. The α -BNN dynamic has great promise for modeling the evolution of heterogeneous forecasting strategies.

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⁶Waters (2009a) shows bifurcation figure for $\alpha = 1$ with a varying \hat{b} , and for $\hat{b} = 1$ with varying α . Results are similar for other choices of α and \hat{b} .