

## Financial Calculator Solutions to Selected Time Value of Money Problems (copyright 2018)

Financial calculators are very poor teaching/learning tools. This instructor believes that you should first learn the underlying time value ideas/relationships, and only then begin computing answers by hitting the time value function keys on a financial calculator's keypad. Students too often expect the calculator to do the thinking for them. Indeed, I often observe senior-level finance students who think they truly understand time value until confronted with more complex cash flow situations (e.g., graduated payment mortgage loans). Then their beloved financial calculators fail them, and they are unable to structure the problems and create their own cell formulas in spreadsheets.

At the same time, financial calculators are very useful computational devices, especially for solving for unknown rates of return in cases of multiple cash flows (which involve trial-and-error computations that financial calculators are nicely programmed to solve). And it is frustrating to me when students complain that "your solutions don't show us how to do it on the calculator" (actually they do in a few cases), and for accounting and finance students in particular it becomes important ultimately to know how to navigate the standard financial calculator's basic functions. So below, for those who insist on using the function keys (please make sure you really know the supporting concepts first), are solutions to some key FIL 240 and 404 Topic 4, 6, and 10 problems, plus mortgage loan amortization (from FIL 360), based on the Texas Instruments BA II Plus financial calculator.

If you truly understand the underlying ideas, you should be able to master basic financial calculator key sequences in a few minutes. After all, what we call PMT the calculator calls PMT, what we call  $n$  the calculator calls  $N$ , what we call rate ( $r$ ) the calculator calls rate ( $I/Y$ ), and then what we call TOT the calculator calls either a big PV total amount or a big FV total amount (in the case of bonds there are actually one of each).

### **Basic time value problems that do not involve a series of equal or related cash flows ("non-annuities")**

Topic 4 #1. Bank pays a 4.45% annual interest rate on savings accounts. If you deposit \$2,000 today, and then make no further out-of-pocket deposits, what will your account balance be at the end of six years?

*Type: Non-Annuity; Ending Amount Unknown. My approach:*

$$\text{Beginning Amount } (1 + r)^n = \text{Ending Amount, or}$$

$$BAMT (1 + r)^n = EAMT$$

$$\$2,000 (1.0445)^6 = EAMT = \$2,597.05 .$$

Financial calculator: 6 N; 4.45 I/Y; 2,000 +/- PV; 0 PMT; CPT FV  $\Rightarrow$  2,597.05 .

There is no series of payments PMT in this example, so enter 0 PMT to make sure there is no value hiding in the PMT register from an earlier exercise that would mess up your answer. Also note that you enter the 2,000 as a negative value. It's good to get into the habit of always entering one (but only one) of the dollar figures as a negative value. The calculator is programmed with the logic that in every transaction money comes out of someone's pocket and goes into someone else's. If you are solving for an unknown dollar value and fail to enter a dollar figure as negative, the calculator actually will straighten out the "error" by assigning a negative value to the answer. Try it:

$$6 N; 4.45 I/Y; 2,000 PV; 0 PMT; CPT FV \Rightarrow - 2,597.05 .$$

But if you are solving for the rate of return or number of periods and enter all dollar figures as positive (or as negative), the calculator shows an error message. This idiosyncrasy has been known to cause students great frustration during exams (no, I can't tell them how to straighten it out).

Topic 4 #2. What would you pay for an investment that provides no cash flows in years 1 through 9, but then provides a payment of \$18,750 in year 10, if an investment of similar risk would provide a 9% annual rate of return?

Type: Non-Annuity; Beginning Amount Unknown. My approach:

$$\begin{aligned} \text{BAMT} (1 + r)^n &= \text{EAMT} \\ \text{BAMT} (1.09)^{10} &= \$18,750 \\ \text{So BAMT} &= \$18,750 \div (1.09)^{10} \text{ OR} \\ \text{BAMT} &= \$18,750 \left( \frac{1}{1.09} \right)^{10} \Rightarrow \text{BAMT} = \$7,920.20 . \end{aligned}$$

Financial calculator: 10 N; 9 I/Y; 0 PMT; 18,750 +/- FV; CPT PV  $\Rightarrow$  7,920.20 .

Topic 4 #3. What annual rate of return must you earn if \$100,000 deposited today is to grow to \$3,000,000 by the end of year 50?

Type: Non-Annuity; Rate of Return Unknown. My approach:

$$\begin{aligned} \text{BAMT} (1 + r)^n &= \text{EAMT} \\ \$100,000 (1 + r)^{50} &= \$3,000,000 \\ (1 + r)^{50} &= 30 \\ \sqrt[50]{(1 + r)^{50}} &= \sqrt[50]{30} \\ 1 + r &= \sqrt[50]{30} = 30^{1/50} = 30^{0.02} = 1.070391 \Rightarrow r = .070391 \text{ or } 7.0391\% . \end{aligned}$$

Financial calculator: 50 N; 100,000 +/- PV; 0 PMT; 3,000,000 FV; CPT I/Y  $\Rightarrow$  7.0391 .

When you solve for an unknown that is not a dollar amount it is essential that you enter one of the dollar figures as a negative, with the +/- key (you could have entered either the 100,000 PV or the 3,000,000 FV above as a negative value, but not both; only one of the dollar figures should have a negative sign or else you re-create the very problem you were addressing with the +/- key). If you enter both the 100,000 PV and 3,000,000 FV figures as positive (or negative) values, the calculator result is:

50 N; 100,000 PV; 0 PMT; 3,000,000 FV; CPT I/Y  $\Rightarrow$  Error 5

Topic 4 #4. You make a \$10,000 investment today. How long will it take for the money to grow to \$40,000 if you can earn a 5% average annual after-tax compounded rate of return?

Type: Non-Annuity; Number of Periods Unknown. My approach:

$$\begin{aligned} \text{BAMT} (1 + r)^n &= \text{EAMT} \\ \$10,000 (1.05)^n &= \$40,000 \\ (1.05)^n &= 4 . \\ \ln [(1.05)^n] &= \ln 4 \\ n \times \ln 1.05 &= \ln 4 \\ n (.048790) &= 1.386294 \Rightarrow n = 28.413398 . \end{aligned}$$

Financial calculator: 5 I/Y; 10,000 +/- PV; 0 PMT; 40,000 FV; CPT N;  $\Rightarrow$  28.413398 . Again here you are solving for an unknown that is not a dollar amount, so you have to enter either the 10,000 PV figure or the 40,000 FV figure (but not both!!) as a negative, with the +/- key. If you enter both the 10,000 and the 40,000 as positive (or as negative) values, the calculator result is:

5 I/Y; 10,000 PV; 0 PMT; 40,000 FV; CPT N  $\Rightarrow$  Error 5 .

**Basic time value problems that DO involve a series of equal or related cash flows (“annuities”)**

Topic 4 #7. If you invest \$750 into a mutual fund account at the end of each year, and the account’s growing balance earns a 6.5% compounded annual rate of return, how much money will you have after five years? What if you make your \$750 investment at the beginning of each year?

Type: *FV of Annuity; Total Unknown*. My approach:

$$\begin{aligned} \text{Payment} \times \text{Factor} &= \text{Total, or} \\ \text{PMT} \times \text{FAC} &= \text{TOT} \\ \$750 \left( \frac{(1.065)^5 - 1}{.065} \right) &= \text{TOT} \\ \$750 \times 5.693641 &= \$4,270.23 \text{ for end-of-period cash flows (“ordinary annuity”);} \end{aligned}$$

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \$750 \left[ \left( \frac{(1.065)^5 - 1}{.065} \right) (1.065) \right] &= \text{TOT} \\ \$750 \times 6.063728 &= \$4,547.80 \text{ for beginning-of-period cash flows (“annuity due”).} \end{aligned}$$

Financial calculator: 5 N; 6.5 I/Y; 0 PV; 750 +/- PMT; CPT FV  $\Rightarrow$  4,270.23 for end-of-period.

Now type in 2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT (sets calculator in beginning-of-period cash flow mode, as indicated by the letters BGN showing above the numbers on the calculator screen). Then, as before,

5 N; 6.5 I/Y; 0 PV; 750 +/- PMT; CPT FV  $\Rightarrow$  4,547.80 for beginning-of-period.

In fact, once you put the calculator in BGN mode, all you actually have to do is to re-enter CPT FV, because any value you already have entered into any of the time value registers remains in place until you replace it with something else. (That’s why I always encourage you to enter a 0 for the variable that does not affect the computations, like 0 PMT in the earlier non-annuity cases or 0 PV in this future value of an annuity case, just so that if a non-zero value is still in that register from an earlier set of computations you will clear it out.)

Note that here you are solving for an unknown dollar amount, so if you forget to enter the 750 PMT as a negative the calculator will solve the “error” just by assigning a negative value to the answer:

5 N; 6.5 I/Y; 0 PV; 750 PMT; CPT FV  $\Rightarrow$  - 4,270.23 for end-of-period  
5 N; 6.5 I/Y; 0 PV; 750 PMT; CPT FV  $\Rightarrow$  - 4,547.80 for beginning-of-period

Finally, note that typing in 2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT also takes the calculator FROM beginning-of-period cash flow mode back TO “normal” end-of-period (ordinary annuity) mode, which you can tell because the BGN letters no longer show above the numbers on the screen. Go ahead and do that now so you will be ready to handle an end-of-period application in the next question.

Topic 4 #8. How much money must you have on deposit today to be able to withdraw \$9,000 at the end of each year for the next 5 years if your account’s declining balance earns a 7% annual interest rate? What if you would instead want to take the \$9,000 out at the start of each year?

Type: *PV of Annuity; Total Unknown*. My approach:

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$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \$9,000 \left( \frac{1 - \left(\frac{1}{1.07}\right)^5}{.07} \right) &= \text{TOT} \end{aligned}$$

$\$9,000 \times 4.100197 = \$36,901.78$  for end-of-period cash flows ("ordinary annuity");

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \$9,000 \left[ \left( \frac{1 - \left(\frac{1}{1.07}\right)^5}{.07} \right) (1.07) \right] &= \text{TOT} \end{aligned}$$

$\$9,000 \times 4.387211 = \$39,484.90$  for beginning-of-period cash flows ("annuity due").

Financial calculator: 5 N; 7 I/Y; 9,000 +/- PMT; 0 FV; CPT PV  $\Rightarrow$  36,901.78 for end-of-period.

Type in 2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT (sets calculator in beginning-of-period cash flow mode, as indicated by the letters BGN showing above the numbers on the calculator screen). Then, as before,

5 N; 7 I/Y; 9,000 +/- PMT; 0 FV; CPT PV  $\Rightarrow$  39,484.90 for beginning-of-period.

Again, once you put the calculator in BGN mode, you really only have to re-enter CPT PV, because any values already entered into the time value registers remain in place. Of course, if you know that you will be computing for both the ordinary annuity and annuity due cases, you can as easily start in BGN mode and solve for the part with beginning-of-period cash flows, and then type in 2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT to get OUT of BGN mode and compute for the end-of-period case. Always remember to look above the numbers showing on the calculator screen to verify whether you are in BGN mode or not. And here once again, if you did not enter the payment value as negative the calculator would sort things out by simply assigning a negative value to the computed PV total.

Topic 4 #9. How much must you deposit, at the end of each year, into a savings account that pays a 5.25% annual interest rate if you want to reach a \$65,000 balance by the end of year 4? What if you instead make the deposits at the beginning of each year?

Type: *FV of Annuity; Payment Unknown. My approach:*

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \text{PMT} \left( \frac{(1.0525)^4 - 1}{.0525} \right) &= \$65,000 \\ \text{PMT} \times 4.326170 &= \$65,000 \end{aligned}$$

$\text{PMT} = \$65,000 \div 4.326170 = \$15,024.84$  for end-of-period cash flows ("ordinary annuity");

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \text{PMT} \left[ \left( \frac{(1.0525)^4 - 1}{.0525} \right) (1.0525) \right] &= \$65,000 \\ \text{PMT} \times 4.553294 &= \$65,000 \end{aligned}$$

$\text{PMT} = \$65,000 \div 4.553294 = \underline{\underline{\$14,275.38}}$  for beginning-of-period cash flows ("annuity due").

Financial calculator: 4 N; 5.25 I/Y; 0 PV; 65,000 +/- FV; CPT PMT  $\Rightarrow$  15,024.84 for end-of-period.

Then type in 2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT (sets calculator in beginning-of-period cash flow mode, as indicated by the letters BGN showing above the numbers on the calculator screen). Then, as before,

4 N; 5.25 I/Y; 0 PV; 65,000 +/- FV; CPT PMT  $\Rightarrow$  14,275.38 for beginning-of-period.

As before, to compute for the annuity due case after computing the ordinary annuity answer (or to compute the ordinary annuity case after finding the annuity due answer) all you would actually have to do is to re-enter CPT PV, because whatever values you already entered will remain in the time value registers. Remember always to check to see whether the calculator is in BGN mode. And again in this problem, with an unknown dollar amount, forgetting to enter the 65,000 future value as a negative is not a big deal, since the calculator can react just by making the answer negative.

Topic 4 #10. What payment must you make at the end of each year to repay a \$14,000 loan with an 11.5% annual interest rate over 6 years? What if beginning-of-year payments instead were to be made?

Type: PV of Annuity; Payment Unknown. My approach:

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \text{PMT} \left( \frac{1 - \left(\frac{1}{1.115}\right)^6}{.115} \right) &= \$14,000 \end{aligned}$$

$$\text{PMT} \times 4.170294 = \$14,000$$

$\text{PMT} = \$14,000 \div 4.170294 = \$3,357.08$  for end-of-period cash flows ("ordinary annuity");

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \text{PMT} \left[ \left( \frac{1 - \left(\frac{1}{1.115}\right)^6}{.115} \right) (1.115) \right] &= \$14,000 \end{aligned}$$

$$\text{PMT} \times 4.649878 = \$14,000$$

$\text{PMT} = \$14,000 \div 4.649878 = \$3,010.83$  for beginning-of-period cash flows ("annuity due").

Let's assume that here the calculator already is in beginning-of-period mode (with BGN showing above the numbers on the calculator screen). The financial calculator solution therefore is

6 N; 11.5 I/Y; 14,000 +/- PV; 0 FV; CPT PMT  $\Rightarrow$  3,010.83 for beginning-of-period.

Now type in 2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT to set the calculator to end of-period cash flow mode (the letters BGN should no longer show above the numbers on the calculator screen). Then just enter

6 N; 11.5 I/Y; 14,000 +/- PV; 0 FV; CPT PMT  $\Rightarrow$  3,357.08 for end-of-period.

Or really, just take the calculator out of BGN mode and re-enter CPT PMT to get the answer for the ordinary annuity case. And here again, as in so many earlier examples, because the unknown is a dollar amount the calculator would merely assign a negative value to the answer if you neglected to assign a negative value to the lone dollar figure you entered (here, the amount borrowed). But in the next example the unknown to solve for will not be a dollar figure, so forgetting to assign a negative value to one of the dollar figures entered will cause the calculator to generate an error message. Thus I encourage you always to enter one of the dollar values as a negative with the +/- key, just so your habit will be always to enter numbers that the calculator will be able to work with. Trefzger FIL 240, 360, 404

Topic 4 # 11. What annual rate of return would you have to earn for a series of \$18,500 year-end deposits to grow to a \$1 million balance by the end of 15 years? What if the \$18,500 were instead invested at the start of each year?

Type: *FV of Annuity; Rate of Return Unknown*. My approach:

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \$18,500 \left( \frac{(1+r)^{15}-1}{r} \right) &= \$1,000,000; \end{aligned}$$

solve with trial-and-error as 16.5488% for end-of-period cash flows ("ordinary annuity");

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \$18,500 \left[ \left( \frac{(1+r)^{15}-1}{r} \right) (1+r) \right] &= \$1,000,000; \end{aligned}$$

solve with trial-and-error as 14.8655% for beginning-of-period cash flows ("annuity due").

This type of problem illustrates the beauty of the financial calculator as a computational device. Whereas in each of the earlier problems you could directly solve for the unknown value (albeit with a lot of algebraic steps in some cases), here both  $r$  and  $r$ -to-a-power are unknowns within the same equation, so you can not solve for  $r$  by putting  $r$  on one side of the = sign and having only terms that do not contain  $r$  on the other side of the = sign. Solving requires an iterative, or trial-and-error, process, which you or I could do only with a lot of tedious plugging and chugging - but which a financial calculator is nicely programmed to do quickly. [In problem 3 above the unknown to solve for also is the rate of return that relates to some given dollar figures, but  $r$  appears in the equation only once - in the  $(1+r)^{50}$  factor - so you can solve directly just by un-compounding the compounded  $r$ , which you do by taking the 50<sup>th</sup> root of each side of the equation:  ${}^{50}\sqrt{(1+r)^{50}} = {}^{50}\sqrt{30}$  such that  $1+r = 1.07039$ , meaning that  $r = 7.039\%$ .]

But again, here the financial calculator does the hard work for us:

15 N; 0 PV; 18,500 +/- PMT; 1,000,000 FV; CPT I/Y  $\Rightarrow$  16.5488% for end-of-period;

2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT CPT I/Y  $\Rightarrow$  14.8655% for beginning-of-period.

(Now that you have worked several annuity problems with the calculator, it should be evident that you do not have to re-enter time, rate, or dollar figures that have not changed from the previous computation; just switch to or from BGN mode and re-compute the desired unknown value.) Yes, despite my lack of affection for the financial calculator as a tool to use in the learning phase of our work with time value of money principles, I agree that it is nice not to have to do trial and error by hand (notice that after you hit the CPT and I/Y keys the calculator screen goes blank for a few seconds, while the processor does the trial and error work).

And as in so many earlier examples, note that if you neglect to enter either the 18,500 annual deposit (PMT) or the 1,000,000 you hope to have in 15 years (FV) as a negative [but not both!!], the calculator will generate an error message:

15 N; 0 PV; 18,500 PMT; 1,000,000 FV; CPT I/Y  $\Rightarrow$  Error 5 .

Topic 4 #12. What is your expected annual rate of return if a \$19,799 investment made today generates a \$3,750 cash flow at the end of each of the next 8 years? What if the \$3,750 is instead received at the start of each year?

Type: *PV of Annuity; Rate of Return Unknown*. My approach:

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \$3,750 \left( \frac{1 - \left(\frac{1}{1+r}\right)^8}{r} \right) &= \$19,799; \end{aligned}$$

solve with trial-and-error as 10.2865% for end-of-period cash flows ("ordinary annuity").

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \$3,750 \left[ \left( \frac{1 - \left(\frac{1}{1+r}\right)^8}{r} \right) (1+r) \right] &= \$19,799; \end{aligned}$$

solve with trial-and-error as 14.0655% for beginning-of-period cash flows ("annuity due").

Again you see the beauty of the financial calculator for doing certain types of computations. If you have a series of cash flows, then in adjusting for time value effects you will end up with both  $r$  and  $r$  to one or more powers in the same equation. The only way to solve for an unknown rate of return under such conditions is with trial & error - and either you do it by hand, or else you use a device that is programmed to do the trial-and-error quickly, which financial calculators (and spreadsheets, of course) are. So here again the financial calculator does the hard work for us. If the calculator is already in annuity due mode, with BGN showing at the top of the display screen, then enter

8 N; 19,799 +/- PV; 3,750 PMT; 0 FV; CPT I/Y  $\Rightarrow$  14.0655% for beginning-of-period;

2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT CPT I/Y  $\Rightarrow$  10.2865% for end-of-period.

(Again because none of the time, rate, or dollar figures have changed you can just switch from or to BGN mode and re-solve for the missing value). Once again it is wonderful not to have to do trial and error by hand (and once again, notice that after you hit CPT & I/Y the screen goes blank for a few seconds, while the financial calculator's processor does the trial and error work). Recall that in our exams you will not be expected to solve for an unknown rate in cases involving multiple cash flows, but merely to recognize the equation you WOULD solve with trial & error to find the correct rate.

And again it is essential to enter one (not both) of the dollar figures as negative, since you are not solving for a dollar amount such that the calculator could create its own negative dollar answer to straighten out the oversight. Here if you fail to enter either the 19,799 investment (PV) or the 3,750 regular annual payment (PMT) as a negative [but not both!!], the calculator will generate an error message:

8 N; 19,799 PV; 3,750 PMT; 0 FV; CPT I/Y  $\Rightarrow$  Error 5

Topic 4 #13. How long will it take for a series of \$22,500 year-end deposits to grow to \$400,000 if the account's growing balance earns a 7.5% annual interest rate? What if you instead made deposits at the beginning of each year?

Type: *FV of Annuity; Number of Periods Unknown*. My approach:

$$\begin{aligned}
 \text{PMT} \times \text{FAC} &= \text{TOT} \\
 \$22,500 \left( \frac{(1.075)^n - 1}{.075} \right) &= \$400,000 \\
 \frac{(1.075)^n - 1}{.075} &= 17.777778 \\
 (1.075)^n - 1 &= 1.333333 \\
 (1.075)^n &= 2.333333 \\
 \ln [(1.075)^n] &= \ln 2.333333 \\
 n \times \ln 1.075 &= \ln 2.333333 \\
 n (.072321) &= .847298
 \end{aligned}$$

$n = 11.715848$ , or a little less than 12 years for end-of-period cash flows ("ordinary annuity");

$$\begin{aligned}
 \text{PMT} \times \text{FAC} &= \text{TOT} \\
 \$22,500 \left[ \left( \frac{(1.075)^n - 1}{.075} \right) (1.075) \right] &= \$400,000 \\
 \left[ \left( \frac{(1.075)^n - 1}{.075} \right) (1.075) \right] &= 17.777778 \\
 \frac{(1.075)^n - 1}{.075} &= 16.537468 \\
 (1.075)^n - 1 &= 1.240310 \\
 \ln [(1.075)^n] &= \ln 2.240310 \\
 n \times \ln 1.075 &= \ln 2.240310 \\
 n (.072321) &= .806614
 \end{aligned}$$

$n = 11.153303$  years for beginning-of-period cash flows ("annuity due").

You can solve directly, with logarithms, for an unknown number of time periods in the annuity case, just as in the non-annuity case (it is when the rate is unknown that you can directly solve only in the case of a single initial, and then a single ending, dollar amount, with no regular payments in between). Yet here again it can be nice - once you really understand the underlying relationships - to have a computational tool that saves you from having to do a lot of algebraic manipulation.

If the calculator is in ordinary annuity mode (without the BGN symbol showing at the top of the display screen), then enter

7.5 I/Y; 0 PV; 22,500 +/- PMT; 400,000 FV; CPT N  $\Rightarrow$  11.715848 for end-of-period;

2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT CPT N  $\Rightarrow$  11.153303 for beginning-of-period.

(Since none of the rate or dollar figures have changed you can just switch to or from BGN mode and re-solve for the unknown number of periods.) Once again it is essential to enter one (not both) of the dollar figures as negative, since you are not solving for a dollar amount and the calculator thus can not correct the "error" by assigning a negative dollar answer. Here if you fail to enter either the 22,500 yearly deposit (PMT) or the 400,000 you hope to have off in the future (FV) as a negative [but not both!!], the calculator will generate an error message:

7.5 I/Y; 0 PV; 22,500 PMT; 400,000 FV; CPT N  $\Rightarrow$  Error 5 .

Topic 4 #14. For how many years must you make year-end \$850 payments to fully repay a \$13,000 loan with a 6% annual interest rate? What if you instead make the \$850 payments at the beginning of each year?

Type: *PV of Annuity; Number of Periods Unknown*. My approach:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$850 \left( \frac{1 - \left(\frac{1}{1.06}\right)^n}{.06} \right) = \$13,000$$

$$\left( \frac{1 - \left(\frac{1}{1.06}\right)^n}{.06} \right) = 15.294118$$

$$1 - \left(\frac{1}{1.06}\right)^n = .917647$$

$$1 - (.943396)^n = .917647$$

$$- (.943396)^n = -.082353$$

$$(.943396)^n = .082353$$

$$\ln [(.943396)^n] = \ln .082353$$

$$n \times \ln .943396 = \ln .082353$$

$$n [-.058269] = -2.496740$$

$n = 42.848599$  years for end-of-period cash flows ("ordinary annuity");

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$850 \left[ \left( \frac{1 - \left(\frac{1}{1.06}\right)^n}{.06} \right) (1.06) \right] = \$13,000$$

$$\left[ \left( \frac{1 - \left(\frac{1}{1.06}\right)^n}{.06} \right) (1.06) \right] = 15.294118$$

$$\left( \frac{1 - \left(\frac{1}{1.06}\right)^n}{.06} \right) = 14.428413$$

$$1 - \left(\frac{1}{1.06}\right)^n = .865705$$

$$1 - (.943396)^n = .865705$$

$$- (.943396)^n = -.134295$$

$$(.943396)^n = .134295$$

$$\ln [(.943396)^n] = \ln .134295$$

$$n \times \ln .943396 = \ln .134295$$

$$n [-.058269] = -2.007716$$

$n = 34.456021$  years for beginning-of-period cash flows ("annuity due").

Again you are dealing with an unknown number of periods (not an unknown rate of return), and thus could solve directly with logarithms, though as shown above the steps can be time-consuming. If you truly understand the ideas at work here (caution: many students do not at this early stage), it may not be all that risky to take the shortcut computational approach that financial calculators provide.

If the calculator already is in annuity due mode (with BGN at the top of the display screen), enter

6 I/Y; 13,000 +/- PV; 850 PMT; 0 FV; CPT N  $\Rightarrow$  34.456021 for beginning-of-period;  
 2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT CPT N  $\Rightarrow$  42.848599 for beginning-of-period.

(With none of the rate or dollar figures changing you can switch from/to BGN mode and just re-solve for the unknown N). As always when solving for a missing rate or number of periods, forgetting to enter one (not both) of the dollar figures as negative causes a problem. Here, interestingly, the calculator does not generate an error message, but rather yields an implausible negative solution:

6 I/Y; 13,000 PV; 850 PMT; 0 FV; CPT N  $\Rightarrow$  - 11.174027 .

What you are essentially telling the calculator is that someone is giving you \$13,000 today and then giving you another \$850 every year; it can't figure out how that could occur in a standard financial transaction. This kind of difficulty is one of the many reasons I do not suggest using the financial calculator time value function keys until you really know what you are doing.

Topic 4 #15. What would you pay for an investment that is expected to pay you \$1,000 at the end of every year forever, if your required annual rate of return is 6.35%? What if the \$1,000 were instead to be received at the beginning of each year forever? What annual rate of return would you earn if you instead paid \$18,000?

Type: *PV of Perpetuity*. My approach:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$1,000 \left( \frac{1}{.0635} \right) \text{ OR } \frac{\$1,000}{.0635} = \text{TOT} = \$15,748.03 \text{ for end-of-period cash flows ("ord. perpetuity");}$$

$$\$1,000 \left[ \left( \frac{1}{.0635} \right) (1.0635) \right] = \text{TOT} = \$16,748.03 \text{ for beginning-of-period cash flows ("perp. due").}$$

An investor paying \$18,000 for this bond would earn an annual rate of return of only

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$1,000 \left( \frac{1}{r} \right) = \$18,000$$

$$\left( \frac{1}{r} \right) = 18 \Rightarrow r \left( \frac{1}{r} \right) = 18r \Rightarrow 1 = 18r$$

$r = 1/18 = 5.5556\%$  for end-of-period cash flows (ordinary perpetuity);

$$\$1,000 \left( \frac{1}{r} \right) (1 + r) = \$18,000$$

$$\left[ \left( \frac{1}{r} \right) (1 + r) \right] = 18 \Rightarrow \left( \frac{1}{r} \right) + 1 = 18 \Rightarrow \left( \frac{1}{r} \right) = 17 \Rightarrow 1 = 17r$$

$r = 1/17 = 5.8824\%$  for beginning-of-period cash flows (perpetuity due).

The financial calculator can not deal directly with an endless number of time periods; it can not manipulate the value "infinity." The present value of a perpetual annuity ("perpetuity") case is one in which it probably is easier to avoid the time value function keys and just use regular calculator capabilities, as shown above. But if you insisted on using the calculator's financial function keys, about all you could do would be to get an approximate answer by setting N at the highest value the calculator will allow you to enter (9,999,999,999 periods):

9,999,999,999 N; 6.35 I/Y; 1,000 +/- PMT; 0 FV; CPT PV  $\Rightarrow$  15,748.03 for end-of-period;

2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT CPT PV  $\Rightarrow$  16,748.03 for beginning-of-period.

Then keeping some of the values you already have entered and changing a couple:

18,000 PV; CPT I/Y  $\Rightarrow$  5.88235% for beginning-of-period (calculator is already in BGN mode);

2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT CPT I/Y  $\Rightarrow$  5.55556% for end-of-period.

(In theory all of the calculator-computed answers here are only estimates, though with using the huge 9,999,999,999 value for the number of periods you get exactly the same answers as with the formulas, which correctly incorporate a truly infinite number of periods.) All the cautions that normally apply to annuity computations still apply here; for example, remember to assign a negative value to one of the dollar figures if solving for the rate of return in a perpetuity case.

Topic 4 #16. How much can you borrow if you will make no repayments in years 1 through 8, but then will make payments of \$35,000 at the end of each of years 9 through 20, and if the lender charges a 10.5% annual interest rate? What if the \$35,000 payments would be made at the beginning of each of years 9 – 20?

Type: *PV of a Deferred Annuity*. My approach (using the "compute a factor" technique):

$$\begin{aligned} & \text{PMT} \times \text{FAC} = \text{TOT} \\ \$35,000 \left[ \left( \frac{1 - \left(\frac{1}{1.105}\right)^{20}}{.105} \right) - \left( \frac{1 - \left(\frac{1}{1.105}\right)^8}{.105} \right) \right] &= \text{TOT} \end{aligned}$$

\$35,000  $\times$  2.991721 = \$104,710.24 for end-of-period cash flows;

$$\begin{aligned} & \text{PMT} \times \text{FAC} = \text{TOT} \\ \$35,000 \left( \left[ \left( \frac{1 - \left(\frac{1}{1.105}\right)^{20}}{.105} \right) - \left( \frac{1 - \left(\frac{1}{1.105}\right)^8}{.105} \right) \right] (1.105) \right) &= \text{TOT} \end{aligned}$$

\$35,000  $\times$  3.305852 = \$115,704.81 for beginning-of-period cash flows.

This situation involves cash flows that do not begin at the end or start of the current period, and thus is one that does not lend itself to easy computation with the automated financial calculator functions. But if you understand the underlying ideas and can create a good "mental picture" of the scenario, the calculator eliminates some tedious computations. First, for end-of-period cash flows:

8 N; 10.5 I/Y; 35,000 +/- PMT; 0 FV; CPT PV  $\Rightarrow$  183,371.58; STO 1 (stores PV of an 8-year annuity);  
20 N; CPT PV  $\Rightarrow$  288,081.81 (computes 20-year annuity's value); then type - RCL 1 =  $\Rightarrow$  104,710.24 .

Then for beginning-of-period cash flows:

2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT 8 N; CPT PV  $\Rightarrow$  202,625.59 STO 1 (stores PV of an 8-year annuity)  
20 N; CPT PV  $\Rightarrow$  318,330.40 (computes 20-year annuity's value); then type - RCL 1 =  $\Rightarrow$  115,704.81 .

The calculator saves some computing effort because so many important values remain the same from step to step, and thus remain correctly saved in the various registers. But without knowing how a problem like this one is structured, I don't see how the calculator will help. This is the type of case that leads me to say that the financial calculator's function keys, if used too early in your learning process, simply lead to making bigger mistakes faster.

Another, perhaps simpler, technique to use with the calculator is to treat this problem as a net present value computation. Net present value (NPV) usually is computed in an investment setting, with a negative cash flow in period 0 followed by positive, sometimes mixed with negative, cash flows expected in various single or multiple period combinations in periods 1 through n. But at its core NPV is just a netting, or adding together, of any group of present values. Computing NPV with a mix of different cash flows, using the down-arrow  $\downarrow$  key, is discussed more fully with problem Topic 6 #5 on page 18 below. But for now note that in the problem above no money changes hands today (a \$0 cash flow in period 0,) none changes hands for the next 8 years either, but then \$35,000 is to be received at the ends of all of the following 12 years (years 9 through 20). Type in

CF 2<sup>nd</sup> CLR WORK 0 ENTER  $\downarrow$  0 ENTER  $\downarrow$  8  $\downarrow$  ENTER  $\downarrow$  35,000 ENTER  $\downarrow$  12 ENTER  $\downarrow$  NPV 10.5  
ENTER  $\downarrow$  CPT (should show 104,710.24). Then just type  $\times$  1.105 = (should show 115,704.81). I'm not aware of a way to align the NPV function key with beginning-of-period payments (having calculator in BGN mode does not change the answer you get with the steps above). But once again I don't see how this series of calculator steps makes any sense if you do not have a good picture in your head of how the various dollar figures and the 10.5% annual discount rate relate to each other.

Topic 4 #17, part a. You plan to save \$12,000 at the end of each year for 4 years, and then deplete the account with withdrawals over the ensuing 3 years. You expect to earn a 4.25% annual rate of return on any money that remains in your savings account from year to year. How much can you take out at the end of each of the 3 withdrawal years? What if you instead take the withdrawals at the beginning of each of the 3 years?

Type: *FV of Annuity and PV of Annuity Combined*. My approach: first compute

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \$12,000 \left( \frac{(1.0425)^4 - 1}{.0425} \right) &= \text{TOT} \\ \$12,000 \times 4.262302 &= \$51,147.62 . \text{ Then} \end{aligned}$$

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \text{PMT} \left( \frac{1 - \left( \frac{1}{1.0425} \right)^3}{.0425} \right) &= \$51,147.62 \end{aligned}$$

$\text{PMT} = \$51,147.62 \div 2.761976 = \$18,518.49$  for end-of-period cash flows;

$$PMT \left[ \left( \frac{1 - \left( \frac{1}{1.0425} \right)^3}{.0425} \right) (1.0425) \right] = \$51,147.62$$

$PMT = \$51,147.62 \div 2.879360 = \$17,763.54$  for beginning-of-period cash flows.

Financial calculator (first for end-of-period withdrawals):

4 N; 4.25 I/Y; 0 PV; 12,000 +/- PMT; CPT FV  $\Rightarrow$  51,147.62 ;

then hit the +/- key and the PV key to make 51,147.62 the present value for next part of problem:

3N; 0 FV; CPT PMT  $\Rightarrow$  18,518.49 .

Then for beginning-of-period cash withdrawals just enter

2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT; CPT PMT  $\Rightarrow$  17,763.54 .

Once again the computations are greatly facilitated by the automated financial function keys, but I simply do not see how someone who does not understand the underlying relationships, based on the approach in which we work with the formulas, can use the calculator to get the correct answers.

Topic 4 #19, part b. How much money will you have after 13 years if you deposit \$700 per quarter for the first 3 years, \$750 per quarter for the next 6 years, and \$800 per quarter during the last 4 years, and you expect to earn a 10% APR with quarterly compounding.? What if you make beginning-of-quarter deposits?

Type: "Truncated" annuity, non-annual time periods. My approach:

Quarterly periodic interest rate is  $10\% \div 4 = 2.5\%$ , total number of periods is  $13 \times 4 = 52$  quarters.

$$\$700 \left[ \left( \frac{(1.025)^{12} - 1}{.025} \right) (1.025)^{40} \right] + \$750 \left[ \left( \frac{(1.025)^{24} - 1}{.025} \right) (1.025)^{16} \right] + \$800 \left( \frac{(1.025)^{16} - 1}{.025} \right)$$

= \$25,929.36 + \$36,016.75 + \$15,504.18 = \$77,450.29 with year-end deposits and

$$\begin{aligned} & \$700 \left[ \left( \frac{(1.025)^{12} - 1}{.025} \right) (1.025)^{40} (1.025) \right] + \$750 \left[ \left( \frac{(1.025)^{24} - 1}{.025} \right) (1.025)^{16} (1.025) \right] \\ & + \$800 \left[ \left( \frac{(1.025)^{16} - 1}{.025} \right) (1.025) \right] \end{aligned}$$

= \$26,577.59 + \$36,917.17 + \$15,891.78 = \$79,386.54 with beginning-of-year deposits.

Financial calculator:

40 N; 2.5 I/Y; 0 PV; 700 +/- PMT; CPT FV  $\Rightarrow$  47,181.79; STO 1 to store this value;

52 N; CPT FV  $\Rightarrow$  73,111.15; then type in - RCL 1 =  $\Rightarrow$  25,929.36; hit STO 2 to store this value.

16 N; 750 +/- PMT; CPT FV  $\Rightarrow$  14,535.17 STO 1 to store this value

40 N; CPT FV  $\Rightarrow$  50,551.92; then type in - RCL 1 =  $\Rightarrow$  36,016.75; hit STO 3 to store this value.

16 N; 800 +/- PMT; CPT FV  $\Rightarrow$  15,504.18. Then type in + RCL 2 = + RCL 3 =  $\Rightarrow$  77,450.28 for end-of-period cash flows. For beginning-of-period cash flows, simply repeat these steps while in BGN mode.

Topic 4 #20, part a. If you borrow \$8,000 at a 6.72% Annual Percentage Rate (APR) of interest for a five-year term, and the contract calls for you to pay at the end of each month, how much will each payment be? What if the payments are instead to be made at the beginning of each month?

Type: *PV of Annuity with Non-Annual Payments*. My approach: Number of periods is  $5 \times 12 = 60$  months, monthly periodic interest rate is  $6.72\% \div 12 = .56\%$ .

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \text{PMT} \left( \frac{1 - \left(\frac{1}{1.0056}\right)^{60}}{.0056} \right) &= \$8,000 \end{aligned}$$

$\text{PMT} = \$8,000 \div 50.840493 = \$157.35$  for end-of-period cash flows ("ordinary annuity");

$$\text{PMT} \left[ \left( \frac{1 - \left(\frac{1}{1.0056}\right)^{60}}{.0056} \right) (1.0056) \right] = \$8,000$$

$\text{PMT} = \$8,000 \div 51.125200 = \$156.48$  for beginning-of-period cash flows ("annuity due").

Financial calculator: students often make mistakes when there are multiple payment periods per year. My preference is to reset the calculator's defaults to 1 payment per period rather than to stick with the default of 12. Then you can think more clearly about periodic rates and numbers of periods. Here enter  $5 \times 12 = N$ ;  $6.72 \div 12 = I/Y$ ; 8,000 +/- PV; 0 FV; CPT PMT  $\Rightarrow$  157.35 for end-of-year. Then do 2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT CPT PMT  $\Rightarrow$  156.48 for beginning-of-year.

part d. If you borrow \$8,000 and pay back \$163.75 at the end of each month over a five-year period, what monthly periodic rate of interest are you paying? What corresponding APR would the lender be charging? What would be the corresponding Effective Annual Rate (EAR) of interest?

My approach:

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \$163.75 \left( \frac{1 - \left(\frac{1}{1+r}\right)^{60}}{.0056} \right) &= \$8,000 . \end{aligned}$$

With both  $r$  and  $r^{60}$  in the factor, must use trial-and-error approach that ultimately yields .007. Financial calculator: hit 2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT to return to end-of-period mode. Note that  $N$  is still 60,  $PV$  is still 8000 +/-, and  $FV$  is still 0, so you need not adjust those figures. Just type 163.75 PMT; CPT I/Y  $\Rightarrow$  .700. The financial calculator definitely helps here, preventing the need otherwise to engage in tedious trial-and-error computations. With a .7% periodic rate, the APR (which does not take into account intra-year compounding) would be  $.007 \times 12 = 8.4\%$ , and the EAR (which does take into account intra-year compounding) would be  $(1.007)^{12} - 1 = 8.7311\%$ .

Topic 4, #5. If Gladys can earn a 3.75% annual rate of return on her account's growing balance from year to year, how much will she have by the end of year 6 if she makes the series of beginning-of-year deposits described in each of the situations listed below? (Another way to word this problem is: what is the future value of each of the following cash flow streams, with beginning-of-year cash flows and a 3.75% annual compounding rate?)

part c. \$500 in year 1, \$506.25 in year 2, \$512.58 in year 3, \$518.99 in year 4, \$525.47 in year 5, and \$532.04 in year 6 (amounts that increase by 1.25% from year to year; FIL 404 students take special note).

My approach: the amounts deposited are not equal, but because they follow a convenient pattern (changing from period to period by a constant percentage) we can lump them together and use the distributive property and the FV of a changing annuity factor:

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \$500 \left[ \left( \frac{(1.0375)^6 - (1.0125)^6}{.0375 - .0125} \right) (1.0375) \right] &= \text{TOT} \\ \$500 \times 7.046508 &= \underline{\underline{\$3,523.25}} \end{aligned}$$

(here we multiply the ordinary changing annuity factor by  $1 + r$  because payments are made at the beginning of each period; a corresponding answer for end-of-year changing payments would be

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \$500 \left( \frac{(1.0375)^6 - (1.0125)^6}{.0375 - .0125} \right) &= \text{TOT} \\ \$500 \times 6.791815 &= \underline{\underline{\$3,395.91}} \end{aligned}$$

Financial calculator: one way to deal with unequal cash flows is to enter them individually, so the calculator solution is just a glorified version of doing "brute force" by hand. Enter 3.75 I/Y and 0 PMT (since there is not a series of equal payments); these values are common to all individual computations that must be done. Then 5 N; 500 +/- PV; CPT FV  $\Rightarrow$  601.05; STO 1. Then hit RCL PV (shows 500)  $\times$  1.0125 = (shows - 506.25) and hit PV to enter it as the new cash flow expected in year 2. Then hit 4 N; CPT FV  $\Rightarrow$  586.57; STO 2. Then complete the sequence as:

RCL PV  $\times$  1.0125 =; PV; 3 N; CPT FV (should show 572.43); STO 3  
 RCL PV  $\times$  1.0125 =; PV; 2 N; CPT FV (should show 558.64); STO 4  
 RCL PV  $\times$  1.0125 =; PV; 1 N; CPT FV (should show 545.18); STO 5  
 RCL PV  $\times$  1.0125 =; PV; 0 N; CPT FV (should show 532.04); STO 6

Then hit CE/C to clear the screen, and start adding together everything in the memory registers: RCL 1 + RCL 2 + RCL 3 + RCL 4 + RCL 5 + RCL 6 = 3,395.91 for end-of-period cash flows. Then you'd just have to multiply that total by  $(1 + r)$ , which here is 1.0375, to get the answer for the beginning-of-period cash flows case:  $3,395.91 \times 1.0375 = 3,523.25$ .

In a case like this one knowing, and working with, the formula is likely to be much easier to understand than would be an approach using the calculator's automated financial function keys. There actually is a shortcut calculator solution, but it may not be intuitively easy to follow. Consider the case above with end-of-period payments. The amount we want to enter as PMT for computing the FV of this changing series is the final expected payment in the series, here it would be  $\$500 (1.0125)^5 = \$532.04$ , while for I/Y we do not use the periodic rate of return 3.75% but instead substitute the quantity  $[(1 + r)/(1 + g)] - 1$ , or  $(1.0375/1.0125) - 1 = .024691$ , which in full percentage terms for calculator entry would be 2.4691%. So to solve we enter 6 N;  $1.0375 \div 1.0125 - 1 = \times 100 =$  (should show 2.469136) I/Y;  $1.0125 y^x 5 = \times 500 = +/-$  (should show - 532.04) PMT; 0 PV; CPT FV  $\Rightarrow$  3395.91. (This technique is explained in a Summer 2016 *Journal of Economics and Finance Education* article by Ken Johnston, John Hatem, and Benjamin Woods.) Then for beginning-of-period payments we could just multiply the end-of-period payment answer by  $(1 + r)$ , but if we want to use the calculator's automated function keys we must enter BGN mode and also make another change that may not seem intuitive; change PMT to a phantom final payment that would occur in a 7<sup>th</sup> year:  $\$500 (1.0125)^6 = \$538.69$ . Once in BGN mode enter  $1.0125 y^x 6 = \times 500 = +/-$  (should show - 538.69) PMT CPT FV  $\Rightarrow$  3,523.25.

Topic 4 #22, part a. Over the next 9 years, you plan to withdraw amounts that start at \$25,000 and then increase by the expected inflation rate of 1.6% per year. If you expect to earn a 4.8% annual rate of return on any money remaining in your account from year to year, how much must you have on deposit today? What if you instead want to make beginning-of-year withdrawals?

Type: Present Value of a Changing Annuity. My approach:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$25,000 \left( \frac{1 - \left( \frac{1.016}{1.048} \right)^9}{.048 - .016} \right) = \text{TOT}$$

\$25,000 × 7.610306 = \$190,257.65 for end-of-period cash flows;

$$\$25,000 \left[ \left( \frac{1 - \left( \frac{1.016}{1.048} \right)^9}{.048 - .016} \right) (1.048) \right] = \text{TOT}$$

\$25,000 × 7.975601 = \$199,390.02 for beginning-of-period cash flows.

Financial calculator: one way to deal with unequal cash flows is to enter them individually, so the calculator solution is just a glorified version of doing "brute force" by hand. Enter 4.8 I/Y and 0 PMT (since there is not a series of equal payments); these values are common to all individual computations that must be done. Then 1 N; 25,000 +/- FV; CPT PV ⇒ 23,854.96; STO 1. Then hit RCL FV (shows 25,000) × 1.016 = (shows - 25,400) and hit FV to enter it as the new cash flow expected in year 2. Then hit 2 N; CPT PV ⇒ 23,126.57; STO 2. Then complete the sequence as:

RCL FV × 1.016 =; FV; 3 N; CPT PV (should show 22,420.41); STO 3  
 RCL FV × 1.016 =; FV; 4 N; CPT PV (should show 21,735.82); STO 4  
 RCL FV × 1.016 =; FV; 5 N; CPT PV (should show 21,072.13); STO 5  
 RCL FV × 1.016 =; FV; 6 N; CPT PV (should show 20,428.71); STO 6  
 RCL FV × 1.016 =; FV; 7 N; CPT PV (should show 19,804.93); STO 7  
 RCL FV × 1.016 =; FV; 8 N; CPT PV (should show 19,200.20); STO 8  
 RCL FV × 1.016 =; FV; 9 N; CPT PV (should show 18,613.93); STO 9

Then hit CE/C to clear the screen, and start adding together everything in the memory registers: RCL 1 + RCL 2 + RCL 3 + RCL 4 + RCL 5 + RCL 6 + RCL 7 + RCL 8 + RCL 9 = 190,257.65 for end-of-period cash flows. Then you'd just have to multiply that total by (1 + r), which here is 1.048, to get the answer for the beginning-of-period cash flows case: 190,257.65 × 1.048 = 199,390.02.

Here you can see a case where knowing, and working with, the formula is likely to be much easier to understand than would be an approach using the calculator's automated financial function keys.

There actually is a shortcut calculator solution, but it may not be intuitively easy to follow. Consider the case above with end-of-period payments. The amount we want to enter as PMT for computing the PV of this changing series is the phantom payment that would have occurred before the first payment; here it would be \$25,000 ÷ (1.016) = \$24,606.30, while for I/Y we do not use the periodic rate of return 4.8% but instead substitute the quantity [(1 + r)/(1 + g)] - 1, or (1.048/1.016) - 1 = .031496, which in full percentage terms would be 3.1496%. So to solve we enter 9 N; 1.048 ÷ 1.016 = - 1 = × 100 = (should show 3.149606) I/Y; 25000 ÷ 1.016 = +/- (should show - 24606.30) PMT; 0 FV; CPT PV ⇒ 190,257.65. Then for beginning-of-period payments we could just multiply the answer computed for end-of-period payments by (1 + r), but if we want to use the calculator's automated function keys we must make another change that may not seem intuitive; in addition to Trefzger FIL 240, 360, 404

setting the calculator to BGN mode we change PMT to the initial period's actual \$25,000 payment. Once in BGN mode enter 25000 PMT CPT PV  $\Rightarrow$  199,390.02.

### Capital budgeting problems

Topic 6 #6. A company invests \$5,750,000 in a project that has a 9.5% annual cost of capital and expected net cash flows of \$1,500,000 at the end of each year for the following six years. Compute the project's net present value (NPV) and internal rate of return (IRR). What would the NPV be if the annual cost of capital were instead 17%?

Type: *Evaluating Expansion Project with Equal Expected Cash Flows.* My approach:

$$\text{NPV} = \frac{-\$5,750,000}{(1.095)^0} + \frac{\$1,500,000}{(1.095)^1} + \frac{\$1,500,000}{(1.095)^2} + \frac{\$1,500,000}{(1.095)^3} + \frac{\$1,500,000}{(1.095)^4} + \frac{\$1,500,000}{(1.095)^5} + \frac{\$1,500,000}{(1.095)^6}$$

which can be restated in a more convenient form (by grouping the \$1,500,000 payments together) as

$$\begin{aligned} \text{NPV} &= \frac{-\$5,750,000}{(1.095)^0} + \$1,500,000 \left( \frac{1 - \left(\frac{1}{1.095}\right)^6}{.095} \right) \\ &= -\$5,750,000 + \$1,500,000 (4.419825) \\ &= -\$5,750,000 + \$6,629,738 = \$879,738 \text{ for NPV at a 9.5\% annual cost of capital;} \end{aligned}$$

$$\begin{aligned} \text{NPV} &= \frac{-\$5,750,000}{(1.17)^0} + \$1,500,000 \left( \frac{1 - \left(\frac{1}{1.17}\right)^6}{.17} \right) \\ &= -\$5,750,000 + \$1,500,000 (3.589185) \\ &= -\$5,750,000 + \$5,383,777 = -\$366,223 \text{ for NPV at a 17\% annual cost of capital;} \end{aligned}$$

$$\$0 = \frac{-\$5,750,000}{(1+r)^0} + \frac{\$1,500,000}{(1+r)^1} + \frac{\$1,500,000}{(1+r)^2} + \frac{\$1,500,000}{(1+r)^3} + \frac{\$1,500,000}{(1+r)^4} + \frac{\$1,500,000}{(1+r)^5} + \frac{\$1,500,000}{(1+r)^6}$$

$$\$0 = \frac{-\$5,750,000}{(1+r)^0} + \$1,500,000 \left( \frac{1 - \left(\frac{1}{1+r}\right)^6}{r} \right)$$

Solve with trial-and-error to find 14.5257% for IRR.

Financial calculator: for this problem, with a single up-front investment and then a series of equal expected cash flows, you can just use the time value function keys the way you would for any other type of annuity problem. To compute NPV at a 9.5% annual cost of capital, first type in 5,750,000 STO 1 (to store the original investment amount so you can later subtract it out). Then:

6 N; 9.5 I/Y; 1,500,000 +/-PMT; 0 FV; CPT PV  $\Rightarrow$  6,629,738; type in - RCL 1 =  $\Rightarrow$  879,738 NPV for the 9.5% cost of capital case, and then 17 I/Y; CPT PV  $\Rightarrow$  5,383,777; type in - RCL 1 =  $\Rightarrow$  - 366,223 NPV for the 17% cost of capital assumption. Since the time period and cash flow figures remain the same as in the earlier computation, you need only change I/Y and recompute PV (and then subtract out the original investment) to compute NPV for a different cost of capital (you could plot out an NPV profile in just a couple of minutes).

Then to compute IRR:

5,750,000 PV; CPT I/Y  $\Rightarrow$  14.5257. Because you just got done computing NPV values based on 6 N, 1,500,000 +/-PMT, and 0 FV values, those amounts remain intact in the time value registers (they remain intact until you take steps to change them by entering in new values or totally clearing out the time value registers). As always when dealing with an unknown other than a dollar amount, be sure that you enter a dollar value as a negative, or the answer you get will be an error message.

The analysis here is limited to computing the NPV and IRR with the financial calculator's automated function keys. Whereas in the original problem 6 we computed the profitability index and modified internal rate of return as well, those computations are not facilitated much by the financial calculator's time value keys or cash flow registers.

Topic 6 #5. A company invests \$3,200,000 in a project that has a 10.75% cost of capital and expected net cash flows of \$650,000 in year 1; \$750,000 in year 2; \$850,000 in year 3; \$950,000 in year 4; \$1,100,000 in year 5; and \$750,000 in year 6. Compute the project's net present value (NPV) and internal rate of return (IRR).

Type: *Evaluating Expansion Project with Uneven Expected Cash Flows.* My approach:

$$\begin{aligned} \text{NPV} &= \frac{-\$3,200,000}{(1.1075)^0} + \frac{\$650,000}{(1.1075)^1} + \frac{\$750,000}{(1.1075)^2} + \frac{\$850,000}{(1.1075)^3} + \frac{\$950,000}{(1.1075)^4} + \frac{\$1,100,000}{(1.1075)^5} + \frac{\$750,000}{(1.1075)^6} \\ &= -\$3,200,000 + \$586,907 + \$611,468 + \$625,731 + \$631,464 + \$660,198 + \$406,442 \\ &= -\$3,200,000 + \$3,522,210 = \$322,210 \text{ for NPV;} \end{aligned}$$

$$\$0 = \frac{-\$3,200,000}{(1+r)^0} + \frac{\$650,000}{(1+r)^1} + \frac{\$750,000}{(1+r)^2} + \frac{\$850,000}{(1+r)^3} + \frac{\$950,000}{(1+r)^4} + \frac{\$1,100,000}{(1+r)^5} + \frac{\$750,000}{(1+r)^6}$$

solving with trial and error, you would find 13.9621% for IRR.

Financial calculator: now things get a bit more complicated, because with different cash flows each period you have to abandon the nice, clean time value registers and start using the cash flow registers and the  $\downarrow$  key. It's hard to remember the sequence of steps to use, but understanding the underlying ideas makes things a lot easier. Type in

CF 2<sup>nd</sup> CLR WORK 3,200,000 +/- ENTER  $\downarrow$  650,000 ENTER  $\downarrow\downarrow$  750,000 ENTER  $\downarrow\downarrow$  850,000 ENTER  $\downarrow\downarrow$  950,000 ENTER  $\downarrow\downarrow$  1,100,000 ENTER  $\downarrow\downarrow$  750,000 ENTER  $\downarrow$  NPV 10.75 ENTER  $\downarrow$  CPT (should show 322,210); IRR CPT (should go blank while doing trial & error, and then show 13.9621).

The  $\downarrow\downarrow$  step shown repeatedly above enters the cash flow into the calculator's memory, and then tells the calculator that the cash flow in question occurs only once (if a cash flow is expected to occur for more than one consecutive period, you type the cash flow amount and hit ENTER  $\downarrow$ , and then type in the number of periods the cash flow will occur and again type in ENTER  $\downarrow$ ).

The financial calculator is especially nice to have in a capital budgeting situation like this one, with a complex cash flow pattern, because you can compute NPVs for different cost of capital assumptions just by going back and entering a new cost of capital (since the dollar figures in the cash flow registers remain intact once they have been entered). For example if you wanted to know the NPV

in this problem if the cost of capital were instead 9% or 12%, you would just type in NPV 9 ENTER ↓ CPT (should show 519,075); NPV 12 ENTER ↓ CPT (should show 191,151).

Yet despite the beauty of the financial calculator as a computational tool here, I don't see how anyone could comprehend NPV and IRR in an example like this one just by following the correct series of financial calculator steps, i.e. without first understanding the underlying relationships. As in the prior example, the analysis here is limited to computing the NPV and IRR with the financial calculator's automated function keys (computing profitability index or modified internal rate of return is not facilitated much by the financial calculator's time value keys or cash flow registers).

### Bond problems

Topic 10 #3. A bond with a \$1,000 par value and a 7% annual coupon interest rate (with interest paid semiannually) will mature in 4 years. At what price should this bond sell today if a rational investor would require a 10% effective annual rate (EAR) of return, also called the bond's yield to maturity?

*Type: Bond Valuation; Semiannual Coupon Payments. My approach:*

Here  $n = 4 \text{ years} \times 2 \text{ periods per year} = 8 \text{ total periods for } n$ , and  $\sqrt[2]{1.10} - 1 = .048809$  for  $r$  (you have to un-compound the 10% compounded EAR to get the "working" periodic rate). Plugging these figures into the bond valuation equation results in

$$\begin{aligned} V_B &= \$35 \left( \frac{1 - \left( \frac{1}{1.048809} \right)^8}{.048809} \right) + \$1,000 \left( \frac{1}{1.048809} \right)^8 \\ &= \$35 (6.494448) + \$1,000 (.683013) \\ &= \$227.31 + \$683.01 = \$910.32 . \end{aligned}$$

Financial calculator: when you buy a bond with regular "coupon" interest payments, you pay a big amount today in return for the expectation that you will receive both a series of payments (annuity) and a single big amount when the bond matures. So now you will use all of the financial calculator's time value function keys; there will not be one for which you enter a zero value. Also recall that here you have to convert the number of years and the annual rate of return measure to semiannual figures (I recommend having the calculator set for 1 payment and 1 compounding period per year, and then you can make the adjustments shown above and be in better control of what the calculator is doing). type in:

$4 \times 2 = N$ ;  $1.10 \text{ } y^x .5 = - 1 = \times 100 = I/Y$ ;  $.07 \times 1,000 = \div 2 = +/- \text{ PMT}$ ;  $1,000 +/- \text{ FV}$ ; CPT PV  $\Rightarrow 910.32$  .

In other words, the interesting thing about using the financial calculator to work with a coupon-paying bond is that you will ultimately have dollar values into all three of the PV, PMT, and FV registers. A complication to watch for is that the PMT and FV figures (the amounts you expect to get back after buying the bond) must have the same signs, and then PV must have the opposite sign (the calculator will give it the opposite sign if you are solving for the value, or else you enter it with the opposite sign if you are solving for an unknown rate). In this problem, if you did not enter the PMT and FV values as negatives the calculator would simply assign a negative sign to the computed value. But in keeping with our habit of always entering a cash flow value (here two of the three) as negatives, here I suggest hitting the +/- key in connection with both the PMT and FV entries.

Topic 10 #4. A bond with a \$1,000 par value and no annual coupon interest payments will mature in 4 years. At what price should this bond sell today if a rational investor would require a 10% effective annual rate (EAR) of return, also called the bond's yield to maturity?

*Type: Zero-Coupon Bond Valuation. My approach:*

$$V_B = \$0 \left( \frac{1 - \left(\frac{1}{1.10}\right)^4}{.10} \right) + \$1,000 \left( \frac{1}{1.10} \right)^4$$

$$= \$1,000 (.683013) = \$683.01 .$$

Financial calculator:  $4 \times 2 = N$ ;  $1.10 \text{ y}^\times .5 = -1 = \times 100 = I/Y$ ;  $0 \text{ PMT}$ ;  $1,000 \text{ +/- FV}$ ;  $\text{CPT PV} \Rightarrow 683.01$ .  
A zero-coupon bond has only an initial investment and a single return cash flow at the end; analyzing such a bond is a simple non-annuity problem.

Topic 10 # 13. You buy a \$1,000 par value bond for \$1,224.33. What is the yield to maturity on this investment if the coupon interest rate is 8.25% annually (with semiannual payments) and the maturity date is 28 years from today?

*Type: Yield to Maturity; Semiannual Coupon Payments. My approach:*

With interest paid semiannually the Coupon Payment should be  $(8.25\% \div 2) \times \$1,000 = \$41.25$  every six months, and with 28 years of semiannual interest payments remaining  $n$  is  $28 \times 2 = 56$ . Plugging into our bond valuation equation, we find

$$\$1,224.33 = \$41.25 \left( \frac{1 - \left(\frac{1}{1+r}\right)^{56}}{r} \right) + \$1,000 \left( \frac{1}{1+r} \right)^{56} ;$$

solving for  $r$  with trial-and-error (with both  $r$  and  $r^{56}$  in the above equation you can not solve for  $r$  directly) you find  $r = .0325$  as the semiannual rate of return. Converting that semiannual working  $r$  into an EAR gives  $(1.0325)^2 - 1 = 6.6056\%$ .

Financial calculator:  $28 \times 2 = N$ ;  $1,224.33 \text{ PV}$ ;  $.0825 \times 1,000 = \div 2 = \text{+/- PMT}$ ;  $1,000 \text{ +/- FV}$ ;  $\text{CPT I/Y} \Rightarrow 3.25$ . To convert this figure to an EAR (yield to maturity, in bond terminology):  $\div 100 = +1 = \text{y}^\times 2 = -1 \Rightarrow .066056$ , or 6.6056%.

Topic 10 #14. Now assume the bond described in the previous question contains a call provision, under which the issuer can call the bond in eight years at a price of \$1,082.50 (par value plus extra year's worth of interest). If you buy the bond today for \$1,224.33, and then it is called at the earliest possible date, what will be the yield to first call?

*Type: Yield to First Call; Semiannual Coupon Payments. My approach:*

$$\$1,224.33 = \$41.25 \left( \frac{1 - \left(\frac{1}{1+r}\right)^{16}}{r} \right) + \$1,082.50 \left( \frac{1}{1+r} \right)^{16} ;$$

solving for  $r$  with trial-and-error (with both  $r$  and  $r^{16}$  in the above equation you can not solve for  $r$  directly) you find  $r = .0278465$  as the semiannual rate of return. Converting that semiannual working  $r$  into an EAR gives  $(1.0278465)^2 - 1 = 5.6468\%$ .

Financial calculator:  $8 \times 2 = N$ ;  $1,082.50$  +/- FV; CPT I/Y  $\Rightarrow 2.78465$ . (You do not have to re-enter the values that should stay the same as in the yield to maturity computation:  $1,224.33$  PV;  $.0825 \times 1,000 = \div 2 = +/-$  PMT). To convert the computed semiannual figure to an EAR (yield to first call in this case):  $\div 100 = + 1 = y \times 2 = - 1 = \Rightarrow .056468$ , or  $5.6468\%$ .

Topic 10 #15. Assume that the bond discussed in the prior two problems is not called after 8 years, but you also do not keep it until the maturity date in 28 years. Instead, after holding the bond for 9 years you sell it for  $\$952.84$ . What will your holding period yield have been?

Type: *Holding Period Yield; Semiannual Coupon Payments*. My approach:

$$\$1,224.33 = \$41.25 \left( \frac{1 - \left( \frac{1}{1+r} \right)^{18}}{r} \right) + \$952.84 \left( \frac{1}{1+r} \right)^{18} ;$$

solving for  $r$  with trial-and-error you find  $r = .023668$  as the semiannual rate of return. Converting that semiannual working  $r$  into an EAR gives  $(1.023668)^2 - 1 = 4.7896\%$ .

Financial calculator:  $9 \times 2 = N$ ;  $952.84$  +/- FV; CPT I/Y  $\Rightarrow 2.3668$ . (You do not have to re-enter the values that should stay the same as in the yield to maturity or yield to first call computations:  $1,224.33$  PV;  $.0825 \times 1,000 = \div 2 = +/-$  PMT). To convert the computed semiannual figure to an EAR (holding period yield in this case):  $\div 100 = + 1 = y \times 2 = - 1 = \Rightarrow .047896$ , or  $4.7896\%$ .

### Mortgage Loan Payments and Amortization

Finally, this problem comes from my FIL 360 course; I've been told that Accounting majors also need to deal with loan amortization issues, so here is an example. If someone gets a  $\$120,000$  mortgage loan at a  $9\%$  stated annual percentage rate (APR) of interest, with monthly payments to be made over 30 years, how much is each payment?

With a  $9\%$  stated APR and monthly payments, monthly periodic interest rate is  $.09 \div 12 = .0075$ . With 30 years of monthly payments, there are  $30 \times 12 = 360$  payment periods. With  $\$120,000$  borrowed, my approach to computing monthly payment is:

$$\begin{aligned} & \text{PMT} \times \text{FAC} = \text{TOT} \\ & \text{PMT} \left( \frac{1 - \left( \frac{1}{1.0075} \right)^{360}}{.0075} \right) = \$120,000 \end{aligned}$$

$$\text{So PMT} = \$120,000 \div 124.281866 = \$965.55 .$$

Financial calculator:  $30 \times 12 = N$ ;  $9 \div 12 = I/Y$ ;  $120,000$  +/- PV;  $0$  FV; CPT PMT  $\Rightarrow \$965.55$ .

How much interest does the borrower pay during the first *month* of the loan's life?

Easy. Interest paid in any period is just periodic (here monthly) rate times principal owed at start of period. In first month the entire principal borrowed is still owed, so interest component of first month's payment is just

$$.0075 \times \$120,000 = \$900.00$$

(not really a financial calculator issue here).

How much interest and principal does the borrower pay during the first *year* of the loan's life?

Much harder. My approach: the remaining principal balance, at any point during life of a long-term loan, is PV of remaining payment stream when discounted at original contract's periodic interest rate. Here, at end of year 1, with 29 years (348 months) of payments left to make, PV of remaining payment stream is

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \$965.55 \left( \frac{1 - \left( \frac{1}{1.0075} \right)^{348}}{.0075} \right) &= \text{TOT} \\ \$965.55 \times 123.432776 &= \$119,180.52 \end{aligned}$$

If \$119,180.52 of original \$120,000 principal remains unpaid, the remaining \$120,000 - \$119,180.52 = \$819.48 has been repaid. Everything else paid during year 1 has been interest. So here note that

$$\begin{array}{r} \text{Total payments during year 1: } 12 \times \$965.55 = \$11,586.60 \\ \text{Minus principal portion} \qquad \qquad \qquad \underline{\qquad 819.48} \\ \text{Equals interest paid during year 1} \qquad \qquad \qquad \underline{\qquad \$10,767.12} \end{array}$$

Before using financial calculator, you might want to do a couple of things. First, make sure the defaults are reset so you have one payment and one compounding period "per year" (per period) by entering 2<sup>nd</sup> P/Y 1 ENTER ↓ 1 ENTER 2<sup>nd</sup> QUIT. Otherwise, with the default set at monthly, knowing what to enter for r and I/Y can become confusing (especially if you have, say, quarterly or semi-annual payments).

The other helpful preliminary step might be to look at a full amortization over the first 3 years (based on the "brute force" method; thank heaven for spreadsheets):

Month	Beginning Principal Balance	Plus Interest	Owed at End of Month Prior to Payment	Minus End of Month Payment	Principal Portion of Payment (pmt minus int)	Ending Principal Balance		
1	\$120,000.00	\$900.00	\$120,900.00	\$965.55	\$65.55	\$119,934.45		
2	\$119,934.45	\$899.51	\$120,833.96	\$965.55	\$66.04	\$119,868.41		
3	\$119,868.41	\$899.01	\$120,767.43	\$965.55	\$66.53	\$119,801.88		
4	\$119,801.88	\$898.51	\$120,700.39	\$965.55	\$67.03	\$119,734.85		
5	\$119,734.85	\$898.01	\$120,632.86	\$965.55	\$67.54	\$119,667.31		
6	\$119,667.31	\$897.50	\$120,564.82	\$965.55	\$68.04	\$119,599.27		
7	\$119,599.27	\$896.99	\$120,496.26	\$965.55	\$68.55	\$119,530.72		
8	\$119,530.72	\$896.48	\$120,427.20	\$965.55	\$69.07	\$119,461.65		
9	\$119,461.65	\$895.96	\$120,357.61	\$965.55	\$69.58	\$119,392.06	Sum of	Sum of
10	\$119,392.06	\$895.44	\$120,287.51	\$965.55	\$70.11	\$119,321.96	Principal	Interest
11	\$119,321.96	\$894.91	\$120,216.87	\$965.55	\$70.63	\$119,251.33	<u>Mo. 1 -- 12</u>	<u>Mo. 1 -- 12</u>
12	\$119,251.33	\$894.38	\$120,145.71	\$965.55	\$71.16	\$119,180.16	\$819.84	\$10,766.73
13	\$119,180.16	\$893.85	\$120,074.01	\$965.55	\$71.70	\$119,108.47		
14	\$119,108.47	\$893.31	\$120,001.78	\$965.55	\$72.23	\$119,036.23		
15	\$119,036.23	\$892.77	\$119,929.01	\$965.55	\$72.78	\$118,963.46		
16	\$118,963.46	\$892.23	\$119,855.68	\$965.55	\$73.32	\$118,890.14		

17	\$118,890.14	\$891.68	\$119,781.81	\$965.55	\$73.87	\$118,816.27		
18	\$118,816.27	\$891.12	\$119,707.39	\$965.55	\$74.43	\$118,741.84		
19	\$118,741.84	\$890.56	\$119,632.40	\$965.55	\$74.98	\$118,666.86		
20	\$118,666.86	\$890.00	\$119,556.86	\$965.55	\$75.55	\$118,591.31		
21	\$118,591.31	\$889.43	\$119,480.75	\$965.55	\$76.11	\$118,515.20	Sum of	Sum of
22	\$118,515.20	\$888.86	\$119,404.06	\$965.55	\$76.68	\$118,438.52	Principal	Interest
23	\$118,438.52	\$888.29	\$119,326.81	\$965.55	\$77.26	\$118,361.26	<u>Mo. 13 -- 24</u>	<u>Mo. 13 -- 24</u>
24	\$118,361.26	\$887.71	\$119,248.97	\$965.55	\$77.84	\$118,283.42	\$896.74	\$10,689.82
25	\$118,283.42	\$887.13	\$119,170.55	\$965.55	\$78.42	\$118,205.00		
26	\$118,205.00	\$886.54	\$119,091.54	\$965.55	\$79.01	\$118,125.99		
27	\$118,125.99	\$885.94	\$119,011.93	\$965.55	\$79.60	\$118,046.39		
28	\$118,046.39	\$885.35	\$118,931.74	\$965.55	\$80.20	\$117,966.19		
29	\$117,966.19	\$884.75	\$118,850.93	\$965.55	\$80.80	\$117,885.39		
30	\$117,885.39	\$884.14	\$118,769.53	\$965.55	\$81.41	\$117,803.98		
31	\$117,803.98	\$883.53	\$118,687.51	\$965.55	\$82.02	\$117,721.96		
32	\$117,721.96	\$882.91	\$118,604.88	\$965.55	\$82.63	\$117,639.33		
33	\$117,639.33	\$882.29	\$118,521.63	\$965.55	\$83.25	\$117,556.08	Sum of	Sum of
34	\$117,556.08	\$881.67	\$118,437.75	\$965.55	\$83.88	\$117,472.20	Principal	Interest
35	\$117,472.20	\$881.04	\$118,353.24	\$965.55	\$84.51	\$117,387.70	<u>Mo. 25 -- 36</u>	<u>Mo. 25 -- 36</u>
36	\$117,387.70	\$880.41	\$118,268.10	\$965.55	\$85.14	\$117,302.56	\$980.86	\$10,605.70

Now for the financial calculator. You already have entered  $30 \times 12 = N$ ;  $9 \div 12 = I/Y$ ;  $120,000 +/-$  PV;  $0$  FV; CPT PMT  $\Rightarrow$   $\$965.55$  (or re-do if need be). Now, to figure out what has happened during months 1 - 12, type 2<sup>nd</sup> AMORT 1 ENTER  $\downarrow$  12 ENTER (thereby asking for key amortization figures over months 1 through 12)  $\downarrow \Rightarrow$   $\$119,180.16$  is principal balance still owed 12 months into loan's 360-month life; then  $\downarrow$  again  $\Rightarrow$   $\$819.84$  is principal repaid over the first 12 months; then  $\downarrow$  again  $\Rightarrow$   $\$10,766.73$  is interest paid over those first 12 months.

(Note that the spreadsheet output and the financial calculator show identical values, while the figures computed with the equations are a bit different - less accurate rounded figures (because in the "hand" computations we do not carry the numbers out for many decimal places).

How much interest and principal does the borrower pay during the second year of the loan's life?

My approach: again, the remaining principal balance, at any point during life of a long-term loan, is PV of remaining payment stream when discounted at original contract's periodic interest rate. So at end of year 2, with 28 years (336 months) of payments left to make, PV of remaining payment stream is

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \$965.55 \left( \frac{1 - \left( \frac{1}{1.0075} \right)^{336}}{.0075} \right) &= \text{TOT} \\ \$965.55 \times 122.504035 &= \$118,283.77 \end{aligned}$$

If  $\$118,283.77$  of year's initial  $\$119,180.52$  principal remains unpaid, then remaining  $\$119,180.52 - \$118,283.77 = \$896.75$  has been repaid. Everything else paid during year 2 has been interest. So here note that

Total payments during year 2:	$12 \times \$965.55 = \$11,586.60$
Minus principal portion	<u>896.75</u>
Equals interest paid during year 2	<u>\$10,689.85</u>

Financial calculator: to figure out what happens during year 2, type 2<sup>nd</sup> AMORT 13 ENTER ↓ 24 ENTER (thereby asking for key amortization figures over months 13 through 24) ↓ ⇒ \$118,283.42 is principal balance still owed 24 months into loan's 360-month life; then ↓ again ⇒ \$896.74 is principal repaid over the months 13 through 24; then ↓ again ⇒ \$10,689.82 is interest paid over those months 13 through 24. Again, the more accurate calculator functions produce the exact same answers as the spreadsheet.

How much interest and principal does the borrower pay during the third year of the loan's life?

My approach: at end of year 2, with 27 years (324 months) of payments left to make, PV of remaining payment stream is

$$\begin{aligned} \text{PMT} \times \text{FAC} &= \text{TOT} \\ \$965.55 \left( \frac{1 - \left( \frac{1}{1.0075} \right)^{324}}{.0075} \right) &= \text{TOT} \\ \$965.55 \times 121.488172 &= \$117,302.90 \end{aligned}$$

If \$117,302.90 of year's initial \$118,283.77 principal remains unpaid, then remaining \$118,283.77 - \$117,302.90 = \$980.87 has been repaid. Everything else paid during year 3 has been interest. So here note that

Total payments during year 3:	12 × \$965.55 = \$11,586.60
Minus principal portion	<u>980.87</u>
Equals interest paid during year 3	\$10,605.73

Financial calculator: to figure out what happens during year 3, type 2<sup>nd</sup> AMORT 25 ENTER ↓ 36 ENTER (thereby asking for key amortization figures over months 25 through 36) ↓ ⇒ \$117,302.56 is principal balance still owed 36 months into loan's 360-month life; then ↓ again ⇒ \$980.86 is principal repaid over the months 25 through 36; then ↓ again ⇒ \$10,605.70 is interest paid over those months 25 through 36. As earlier, the more accurate calculator functions produce the exact same answers as the spreadsheet.

Finally, if the borrower makes regular monthly payments for three years and then decides to pay the loan off, how much principal is still owed at the end of year 3? How much in total interest has the borrower paid over this three-year period?

As we know from the computations above, three years into the loan's life, with 27 years (324 months) of payments to go, the remaining outstanding principal balance is the present value of the remaining payment stream: \$117,302.90.

Then with \$117,302.90 still owed, the principal that has been repaid over the first 36 months is \$120,000 - \$117,302.90 = \$2,697.10. So we can show

Total payments during years 1 - 3 :	36 × \$965.55	=	\$34,759.80
Minus principal portion		=	<u>2,697.10</u>
Equals interest paid during first 3 years			\$32,062.70

This figure is the "balloon payment" that the borrower would have to make to retire the loan after three years, at the end of month 36. Financial calculator: with all the information from above still

entered (or re-enter it if need be), enter 36 N (to change our time focus from 360 months to 36); Compute FV  $\Rightarrow$  \$117,302.56, and proceed as above. Alternatively, type 2<sup>nd</sup> AMORT 1 ENTER  $\downarrow$  36 ENTER (thereby asking for key amortization figures over months 1 through 36)  $\downarrow$   $\Rightarrow$  \$117,302.56 is principal balance still owed 36 months into loan's 360-month life; then  $\downarrow$  again  $\Rightarrow$  \$2,697.44 is principal repaid over the months 1 through 36; then  $\downarrow$  again  $\Rightarrow$  \$32,062.25 is interest paid over those first 36 months.

HOPE THIS TUTORIAL HAS HELPED. If you feel ready to do your computations without a net, you might cut out the following "quick reference" card and tape in your calculator cover. Enjoy.

*Trefzger FIL 240/360/404/440*  
*Quick Instructions for TI BAII Plus*  
*(takes CR2032 Battery)*

- Set 6 decimal places (default = 2)  
 2<sup>nd</sup> FORMAT 6 ENTER 2<sup>nd</sup> QUIT
- Set 1 Pmt & 1 Compounding Period per Year (default = 12 for both)  
 2<sup>nd</sup> P/Y 1 ENTER  $\downarrow$  1 ENTER 2<sup>nd</sup> QUIT
- Clear all memory and reset to defaults  
 2<sup>nd</sup> RESET ENTER
- Switch from Begin to End of Yr Pmts  
 2<sup>nd</sup> BGN 2<sup>nd</sup> SET 2<sup>nd</sup> QUIT
- Store 3.5 in Mem. 1 and 8.4 in Mem. 2  
 3.5 STO 1 8.4 STO 2
- Multiply contents of Memory 1 & 2  
 RCL 1 x RCL 2 = [should show 29.4]
- Amortize loan after computing payment  
 2<sup>nd</sup> AMORT 1 ENTER  $\downarrow$  36 ENTER  $\downarrow$   $\downarrow$   $\downarrow$   
 (shows balance, total prin. & int. paid, after first 36 periods of n-period loan)
- Compute NPV/IRR for \$100 initial cost, \$30 CF for 1 yr, then \$20/yr CF for 5 yrs, then \$10 CF for 1 yr, 8% WACC  
 CF 2<sup>nd</sup> CLR WORK 100 +/- ENTER  $\downarrow$   
 30 ENTER  $\downarrow$  20 ENTER  $\downarrow$  5 ENTER  
 $\downarrow$  10 ENTER  $\downarrow$  NPV 8 ENTER  $\downarrow$  CPT  
 IRR CPT [should show 7.55, 10.45]