In our time value of money coverage we note that the future or present value of a stream of unrelated payments must be computed as the sum of the FVs or PVs of the individual payments; there is no way to combine unrelated payments into one computation with the distributive property using an annuity factor. But the FV or PV of a series of equal payments can be computed with the FV or PV of a level annuity factor. And the FV or PV of a series of payments that are related, in that they change by a constant percentage amount from period to period, also can be computed with an annuity factor, albeit a factor slightly amended from its level annuity counterpart. Sections H and I of the appendix to the long Topic 4 outline go into considerable detail on FVs and PVs of changing annuities, for those who want to see such details. But the brief discussion below should suffice for the purpose of changing annuity problems that will appear on the honors section exam. (And for extra practice see part c in questions 5 and 6, in either the main practice problem set A or problem set B.)

<u>Future Value of Level Annuity *vs.* Future Value of Annuity Changing by Constant Percentage</u>
If you deposit $6,000 in an account at the end (beginning) of each year for 9 years, and can earn a 5% average annual return on the account's growing balance, how much will you have by the end of year 9? The respective answers for end-of-year and beginning-of-year deposits on these level annuities are

$$\$6{,}000\left(\frac{(1.05)^9-1}{.05}\right)=\$\underline{66{,}159.39} \quad \text{and} \quad \$6{,}000\left[\left(\frac{(1.05)^9-1}{.05}\right)(1.05)\right]=\$\underline{69{,}467.36}$$

If you make 9 years' worth of end-of-year (beginning-of-year) deposits that start at $6,000 in year 1 and then increase by a constant rate of 2% per year, and you can earn a 5% average annual return on the account's growing balance, how much will you have by the end of year 9? The increasing nature of the deposit stream leads to higher respective totals, for end-of-year and beginning-of-year deposits, on these changing annuities of

$$\$6{,}000\left(\frac{(1.05)^9-(1.02)^9}{.05-.02}\right)=\$\underline{71{,}247.13} \quad \text{and} \quad \$6{,}000\left[\left(\frac{(1.05)^9-(1.02)^9}{.05-.02}\right)(1.05)\right]=\$\underline{74{,}809.49}$$

<u>Present Value of Level Annuity *vs.* Present Value of Annuity Changing by Constant Percentage</u>
If a bank charges a 7% annual interest rate on loans, and you can commit to making payments of $15,000 at the end (beginning) of each year for 8 years, how much money should the bank be willing to lend you today? The respective answers for end-of-year and beginning-of-year payments on these level annuities are

$$\$15{,}000\left(\frac{1-\left(\frac{1}{1.07}\right)^8}{.07}\right)=\$\underline{89{,}569.48} \quad \text{and} \quad \$15{,}000\left[\left(\frac{1-\left(\frac{1}{1.07}\right)^8}{.07}\right)(1.07)\right]=\$\underline{95{,}839.34}$$

If a bank charges a 7% annual interest rate on loans, and you are able to commit to making 8 years' worth of payments that start at $15,000 and then increase by a constant rate of 3% per year, how much should the bank be willing to lend you today? The increasing nature of the payment stream allows for higher respective loan principal amounts, for end-of-year and beginning-of-year payments, on these changing annuities of

$$\$15{,}000\left(\frac{1-\left(\frac{1.03}{1.07}\right)^8}{.07-.03}\right)=\$\underline{98{,}523.10} \quad \text{and} \quad \$15{,}000\left[\left(\frac{1-\left(\frac{1.03}{1.07}\right)^8}{.07-.03}\right)(1.07)\right]=\$\underline{105{,}419.72}$$

On the exam you will be given the following four skeleton factors in no particular order, and they will not be labeled (g is the change or "growth" percentage); you should be familiar with them from working problems:

$$\left(\frac{(1+r)^n-1}{r}\right) \qquad \left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right) \qquad \left(\frac{(1+r)^n-(1+g)^n}{r-g}\right) \qquad \left(\frac{1-\left(\frac{1+g}{1+r}\right)^n}{r-g}\right)$$

As shown just above they are the FV of a level ordinary, PV of a level ordinary, FV of a changing ordinary, and PV of a changing ordinary annuity factors, relating to end-of-period payment streams. Each respective annuity due factor, relating to beginning-of-period payments, is just the ordinary factor multiplied by $(1 + r)$; beginning-of-period payments cause the total that relates to the payment stream to change by one application of the rate of return, whether the payments are unrelated, equal, or changing by a constant percentage. •