Below is a quick overview of where the FV and PV of annuity factors come from.

A. Deriving the Future Value of a Level Ordinary Annuity Factor

The future value of a level ordinary annuity factor is the sum of the future value of $1 factors over the same number of periods. For example, if we add the future value of $1 factors at a 7% periodic discount rate for payments to be made at the ends of each of periods 1 through 3, their sum is $(1.07)^2 + (1.07)^1 + (1.07)^0 = 1.1449 + 1.07 + 1.00 = 3.2149$, the same as future value of a level ordinary annuity factor $\left(\frac{(1.07)^3-1}{.07}\right) = 3.2149$. So we can see that it works.

But how can we prove *why* this relationship works, in an easily understood way? Let's say we plan to deposit PMT dollars into an account at the end of each of the next three years, and that accumulations will earn an annual interest rate of *r*%. Then by the end of year 3 we should have a total TOT of

$$\text{TOT} = \text{PMT}\,(1+r)^2 + \text{PMT}\,(1+r)^1 + \text{PMT}\,(1+r)^0$$

$$= \text{PMT}\,[(1+r)^2 + (1+r)^1 + (1+r)^0] \qquad \{\text{Equation 1}\}$$

(Because we put money in at the end of each year, the first deposit earns interest two times and the last one zero times.) Recall that we can perform any operation we choose on both sides of an equation, and the equivalency remains intact. If we multiply each side by $(1 + r)$, we have

$$\text{TOT}\,(1+r) = \text{PMT}\,(1+r)\,[(1+r)^2 + (1+r)^1 + (1+r)^0], \text{ or}$$
$$\text{TOT} + \text{TOT}(r) = \text{PMT}\,[(1+r)^3 + (1+r)^2 + (1+r)^1] \qquad \{\text{Equation 2}\}$$

Now subtract Equation 1 from Equation 2 (subtracting equal amounts from both sides of Equation 2):

$$\text{TOT} + \text{TOT}(r) - \text{TOT} = \text{PMT}\,[(1+r)^3 + (1+r)^2 + (1+r)^1] - \text{PMT}\,[(1+r)^2 + (1+r)^1 + (1+r)^0], \text{ or}$$

$$\text{TOT}(r) = \text{PMT}\,[(1+r)^3 + (1+r)^2 - (1+r)^2 + (1+r)^1 - (1+r)^1 - (1+r)^0], \text{ or}$$

$$\text{TOT}(r) = \text{PMT}\,[(1+r)^3 - (1+r)^0], \text{ or } \text{TOT}(r) = \text{PMT}\,(1+r)^3 - 1 \text{ [anything to the zero power is 1],}$$

$$\text{so TOT} = \text{PMT}\left(\frac{(1+r)^3-1}{r}\right),$$

with $\left(\frac{(1+r)^n-1}{r}\right)$ representing FAC for FV of a level ordinary annuity in our PMT x FAC = TOT structure.

Then note that for payments made at the beginning of each year, the individual factors would total to

$$[(1.07)^3 + (1.07)^2 + (1.07)^1] = 1.225043 + 1.1449 + 1.07 = 3.439943.$$

The terms in brackets can be factored as $[(1.07)^2 + (1.07)^1 + (1.07)^0]\,(1.07)$.

If $[(1.07)^2 + (1.07)^1 + (1.07)^0] = \left(\frac{(1.07)^3-1}{.07}\right)$, $[(1.07)^2 + (1.07)^1 + (1.07)^0]\,(1.07) = \left(\frac{(1.07)^3-1}{.07}\right)(1.07) = 3.439943$. So with payments at the start of each period, the only difference relative to the end-of-period payments case is that interest is applied one differential number of times (one more) over the plan's life.

So $\left(\frac{(1+r)^n-1}{r}\right)$ is the FV of a level ordinary annuity factor (consistent with end-of-period payments),

and $\left(\frac{(1+r)^n-1}{r}\right)(1+r)$ is the FV of a level annuity due factor (consistent with start-of-period payments).

B. Deriving the Present Value of a Level Ordinary Annuity Factor

The present value of a level annuity factor is the sum of the present value of $1 factors over the same number of periods. For example, if we add the present value of $1 factors at 7% for years 1 through 3, their sum is $\left(\frac{1}{1.07}\right)^1 + \left(\frac{1}{1.07}\right)^2 + \left(\frac{1}{1.07}\right)^3$ = .934579 + .873439 + .816298 = 2.624316, the same as the present value of a level ordinary annuity factor $\left(\frac{1-\left(\frac{1}{1.07}\right)^3}{.07}\right)$ = 2.624316. So we can see that it works.

But how can we prove *why* this relationship works, in an easily understood way? Let's say we can afford to apply PMT dollars toward paying back a loan at the end of each of the next three years, and that we must pay an annual interest rate of *r*% on the loan's declining principal balance. Then we can afford to borrow a total of

$$\text{TOT} = \text{PMT}\left(\frac{1}{1+r}\right)^3 + \text{PMT}\left(\frac{1}{1+r}\right)^2 + \text{PMT}\left(\frac{1}{1+r}\right)^1$$

$$= \text{PMT}\left[\left(\frac{1}{1+r}\right)^3 + \left(\frac{1}{1+r}\right)^2 + \left(\frac{1}{1+r}\right)^1\right] \qquad \text{\{Equation 1\}}$$

If we multiply each side by (1 + *r*), we have

$$\text{TOT}\,(1+r) = \text{PMT}\,(1+r)\left[\left(\frac{1}{1+r}\right)^3 + \left(\frac{1}{1+r}\right)^2 + \left(\frac{1}{1+r}\right)^1\right], \text{ or}$$

$$\text{TOT} + \text{TOT}(r) = \text{PMT}\left[\left(\frac{1}{1+r}\right)^2 + \left(\frac{1}{1+r}\right)^1 + \left(\frac{1}{1+r}\right)^0\right] \qquad \text{\{Equation 2\}}$$

Now subtract Equation 1 from Equation 2 (subtracting equal amounts from both sides of Equation 2):

$$\text{TOT} + \text{TOT}(r) - \text{TOT} = \text{PMT}\left[\left(\frac{1}{1+r}\right)^2 + \left(\frac{1}{1+r}\right)^1 + \left(\frac{1}{1+r}\right)^0\right] - \text{PMT}\left[\left(\frac{1}{1+r}\right)^3 + \left(\frac{1}{1+r}\right)^2 + \left(\frac{1}{1+r}\right)^1\right],$$

$$\text{or} \quad \text{TOT}(r) = \text{PMT}\left[-\left(\frac{1}{1+r}\right)^3 + \left(\frac{1}{1+r}\right)^2 - \left(\frac{1}{1+r}\right)^2 + \left(\frac{1}{1+r}\right)^1 - \left(\frac{1}{1+r}\right)^1 + \left(\frac{1}{1+r}\right)^0\right], \text{ or}$$

$$\text{TOT}(r) = \text{PMT}\left[\left(\frac{1}{1+r}\right)^0 - \left(\frac{1}{1+r}\right)^3\right], \text{ or } \text{TOT}(r) = \text{PMT}\left(1 - \left(\frac{1}{1+r}\right)^3\right)$$

$$\text{so TOT} = \text{PMT}\left(\frac{1-\left(\frac{1}{1+r}\right)^3}{r}\right),$$

with $\left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right)$ representing FAC for PV of a level ordinary annuity in our PMT x FAC = TOT structure.

Then note that for payments made at the beginning of each year, the individual factors would total to

$$\left[\left(\frac{1}{1.07}\right)^0 + \left(\frac{1}{1.07}\right)^1 + \left(\frac{1}{1.07}\right)^2\right] = 1.000 + .934579 + .873439 = 2.808018,$$

The terms in brackets can be factored as $\left[\left(\frac{1}{1.07}\right)^1 + \left(\frac{1}{1.07}\right)^2 + \left(\frac{1}{1.07}\right)^3\right](1.07)$.

If $$\left[\left(\frac{1}{1.07}\right)^1 + \left(\frac{1}{1.07}\right)^2 + \left(\frac{1}{1.07}\right)^3\right] = \left(\frac{1-\left(\frac{1}{1.07}\right)^3}{.07}\right),$$

then $$\left[\left(\frac{1}{1.07}\right)^1 + \left(\frac{1}{1.07}\right)^2 + \left(\frac{1}{1.07}\right)^3\right](1.07) = \left(\frac{1-\left(\frac{1}{1.07}\right)^3}{.07}\right)(1.07) = 2.808018.$$

So if payments occur at the beginning of each period, the only difference relative to the end-of-period payments case is that interest is applied one differential number of times (one less) over the plan's life.

So $\left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right)$, is the PV of a level ordinary annuity factor (consistent with end-of-period payments),

and $\left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right)$ $(1 + r)$ is the PV of a level annuity due factor (consistent with start-of-period payments).