The Algebra Behind Annuity Computations: Simple Proofs and the Distributive Property

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Even before hand-held technology eliminated the need to do much computing or thinking in answering time value of money questions, students struggled to understand this key component of financial knowledge. Textbooks tend to introduce concepts and provide good examples toward getting correct solutions, and earlier pedagogical articles have offered insights on intuitive explanations and clearer methods for identifying problem types. But these sources generally do not examine the simple yet compelling mathematical foundations on which all TVM analysis rests. It is easy to show how the future/present value of any related or unrelated payment series can be computed as the sum of the future/present values of the individual payments, while if payments are equal they can be grouped for computing because of the distributive property. Building from that important algebraic foundation shows how payments changing by a constant periodic percentage constitute a simple extension that also allows for grouping with the distributive property, and how perpetuities are mere extensions of finite annuity situations in which an exponent approaching infinity causes part of the finite annuity factor to zero out. Pre- and post-test results provide evidence that even a quick introduction to the algebraic proofs of TVM mechanics enhances student understanding.

Key words: level annuity, changing annuity, distributive property

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Abstract: Finance textbooks tend to provide good time value of money examples toward getting correct solutions, and earlier pedagogical articles have offered intuitive explanations and clearer methods for identifying problem types. But these sources generally do not examine the foundations on which all annuity computations rest. It is easily shown that equal payments can be grouped because of the distributive property. Building from that foundation shows how payments changing by a constant periodic percentage constitute a simple distributive property extension. Pre- and post-test results provide evidence that even a quick introduction to TVM mechanics' algebraic proofs enhances student understanding.

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Introduction

Introductory finance textbooks provide effective explanations of time value of money (TVM) ideas, accompanied by helpful examples. But they do not delve into the algebra behind the annuity factors used for grouping payments in TVM computations.[1] That algebra, involving simple mathematical proofs largely based on the distributive property, is not difficult to follow, and can help students acquire a deeper and more meaningful understanding of, and commitment to, time value mechanics than they get by working standard problems alone.

Prior Contributions

Convincing students to appreciate the richness of TVM analysis, and not just memorize calculator key sequences or steps to use with tables, is an ongoing problem. Finance instructors long have expressed concern that students ignore TVM mechanics, especially as newer cohorts increasingly believe that the technology at their fingertips is there to compute the answers for them. Textbooks explain the various types of time value problems and generally provide good supporting numerical examples, but they can tend to emphasize getting correct answers more than illuminating the underlying process. Many earlier articles have offered ideas to improve student understanding, based on intuitive explanations, breaking TVM computations into smaller discrete steps, or other methods.

An intuitive approach is seen in Eddy and Swanson (1996), who create more vivid mental pictures by reconfiguring traditional time lines. Stuebs (2011) proposes exercises that build progressively from intuitive TVM principles to bond valuation and retirement planning applications. Joining intuition with a computational perspective, Rosenstein and Reed (1988) contrast the ease of multiplying an ordinary annuity factor by $(1 + r)$, which gives correct results in all applicable cases, with the confusing approach of using tables to identify an annuity due factor as the ordinary annuity factor for an adjacent number of periods, with 1 added or subtracted. Bagamery (1991) provides an annuity computation technique that does not require using exponents, convenient for use with relatively short payment streams. McCarty (1995) finds that knowing the computational foundations helps students

[1] See, for example, the annuity coverages in Block, Hirt, and Danielsen (2017, p. 264 – 275); Brigham and Houston (2019, p. 161 – 171); Gallagher (2019, p. 173 – 186), and Ross, Westerfield, and Jordan (2019, p. 150 – 167).

create their own spreadsheet templates and better understand what their calculators do. Walker and Kramer (2018) explain details, generally not addressed in undergraduate finance textbooks, of annuity situations in which solving for *n* involves fractional time periods. Fortin (1997) provides an algebraic breakdown of some more advanced applications of level and changing annuity factors, based on an assumption that basic annuity relationships already are understood. Dempsey (2003) cites favorable experimental results; in stressing students' need to work with formulas toward understanding TVM he notes that tables can be used only for limited values, make spreadsheet and calculator operations harder to grasp, and do not save computing time. Zhang (2016) offers insights on TVM mechanics by showing how financial computations such as security valuations spring from basic time value concepts. Ovaska and Sumell (2017) demonstrate, with graphs and equations, how understanding compounded interest ties to economics majors' overall financial literacy.

Another computational issue is identifying the type of TVM problem at hand. Gardner's (2004) approach to distinguishing ordinary annuity from annuity due situations includes focusing on the number of cash flows rather than the number of time periods. Jalbert (2002), and Jalbert, Jalbert, and Chan (2004), offer a flowchart approach based on four or five questions for identifying time value problems. Martinez (2013) also offers a flowchart with five questions for identifying problems, along with factor-based computing that develops skills useful with financial calculators. Newfeld (2012) also uses a flowchart, with five questions that break problems into smaller steps, for dyslexic students' special needs, finding encouraging results in a small sample experiment. Bagamery (2011) offers an analysis of using the basic TVM keys on a financial calculator, specifically for computing values in changing annuity situations.

While all of these earlier works yield helpful perspectives, both for general application and for use with particular learning needs, none goes into detail on the compelling, yet understandable, algebra that underpins the various future and present value of level and changing annuity factors that hold such prominent roles in TVM computations. Johnston, Hatem, and Woods (2016) do show algebraically how $[(1 + r)/(1 + g)] - 1$is the rate to enter as I/Y in solving for changing annuity values on financial calculators, but they do not cover the broader algebra steps that so convincingly demonstrate how the annuity factors originate. The contribution of this paper is presenting distributive property-based algebraic proofs that can help students comprehend the breakdown of annuity factors and see the important identification and computational aspects of TVM mechanics with greater clarity. Better academic performance results if students are taught with tools that comport with their learning styles, and a survey

(Shoemaker and Kelly, 2015) found the most common learning styles among business students to be "visual-numerical" (preference for examples that include numbers, particularly prevalent among finance majors) and "kinesthetic" (active involvement, including working problems). Pedagogical literature in Mathematics further suggests that understanding proofs allows students to master more complex material and positively relate what they learn to prior knowledge and other applications (Hanna & Jahnke, 1993). Enhancing students' ability to work confidently and productively with some essential building blocks of financial knowledge therefore should be rewarded with improved learning and comprehension among those willing to dig beneath the surface.

Future Value of Annuity Factors

Level Annuity

Any future value (FV) of an annuity factor is merely an extension of the FV of $1 factor, $(1 + r)^n$, that students generally seem to comprehend. Think of the expected balance at the end of period 3 for an account into which three end-of-period deposits are to be made. If the growing balance earns a 4% periodic rate of return, then the first deposit will earn the 4% return for only two periods by the end of period 3 and the third deposit will earn no interest, such that the applicable FV of $1 factors would be $(1.04)^2$, $(1.04)^1$, and $(1.04)^0$. If the deposits were different, unrelated amounts there would be no way to group them for computational purposes; $300 $(1.04)^2$ + $600 $(1.04)^1$ + $700 $(1.04)^0$ = $324.48 + $624 + $700 = $1,648.48. Of course, if the periodic deposit amounts were to be unchanging the same approach would yield the expected future total: $600 $(1.04)^2$ + $600 $(1.04)^1$ + $600 $(1.04)^0$ = $648.96 + $624 + $600 = $1,872.96. However, cash flows (CFs) that are equal can be grouped for computing with the *distributive property*, here as $600 $[(1.04)^2 + (1.04)^1 + (1.04)^0]$. The sum of those three FV of $1 factors is $[(1.04)^2 + (1.04)^1 + (1.04)^0] = 1.0816 + 1.04 + 1.00 = 3.1216$, the same value computed with the FV of a level *ordinary annuity* factor: $\left(\frac{(1.04)^3-1}{.04}\right) = 3.1216$. So the grouping in this example works correctly; $600 (3.1216) = $1,872.96.

The reason this relationship works can be proven in a generalizable and easily understood manner. Say that PMT dollars are to be deposited at the ends of the next *n* periods, and that accumulations will earn a periodic interest rate[2] of *r*%. By the end of period *n* the account balance should be a total TOT of

[2] We cite time "periods" rather than the years usually used in basic TVM coverage, since computations work the same way for, *e.g.*, quarterly periods, as long as *r* is a quarterly return and *n* is the number of quarters over the plan's life. For convenience, we identify the periodic return

$$\text{TOT} = \text{PMT}\,(1+r)^{n-1} + \cdots + \text{PMT}\,(1+r)^2 + \text{PMT}\,(1+r)^1 + \text{PMT}\,(1+r)^0$$

which, by the distributive property – the mathematical basis for all annuity computations in TVM analysis – can be restated as

$$\text{TOT} = \text{PMT}\,[(1+r)^{n-1} + \cdots + (1+r)^2 + (1+r)^1 + (1+r)^0] \qquad \text{[Equation 1]}$$

Any operation, if performed on both sides of an equation, leaves the equivalency intact, albeit with different values. If each side is multiplied by $(1 + r)$, a *geometric transformation* that creates a different but related equation, the result is

$$\text{TOT}\,(1+r) = \text{PMT}\,(1+r)\,[(1+r)^{n-1} + \cdots + (1+r)^2 + (1+r)^1 + (1+r)^0]$$

$$\text{TOT} + \text{TOT}\,(r) = \text{PMT}\,[(1+r)^n + (1+r)^{n-1} + \cdots + (1+r)^2 + (1+r)^1] \qquad \text{[Equation 2]}$$

Using a simultaneous equations approach with these two related equations,[3] we can subtract Equation 1 from Equation 2 (this step involves subtracting equal amounts from both sides of Equation 2); here it is easy to see how most terms cancel out in the subtracting, leaving:

$$\text{TOT} + \text{TOT}\,(r) - \text{TOT} = \text{PMT}\,[(1+r)^n + (1+r)^{n-1} + \cdots + (1+r)^2 + (1+r)^1] - \text{PMT}\,[(1+r)^{n-1} + \cdots + (1+r)^1 + (1+r)^0]$$

$$\text{TOT}\,(r) = \text{PMT}\,[(1+r)^n + \cancel{(1+r)^{n-1} - (1+r)^{n-1}} + \cdots + \cancel{(1+r)^2 - (1+r)^2} + \cancel{(1+r)^1 - (1+r)^1} - (1+r)^0]$$

$$\text{TOT}\,(r) = \text{PMT}\,[(1+r)^n - (1+r)^0] = \text{PMT}\,[(1+r)^n - 1] \quad \text{so} \quad \text{TOT} = \text{PMT}\left(\frac{(1+r)^n - 1}{r}\right)$$

$$\therefore \qquad (1+r)^{n-1} + \cdots + (1+r)^1 + (1+r)^0 = \left(\frac{(1+r)^n - 1}{r}\right).$$

The FV of a level ordinary annuity factor is consistent with n payments, but only $n - 1$ applications of interest, as shown by the 0 exponent on the last of the terms that sum to the annuity factor. The FV of a level *annuity due* factor, relating to beginning-of-period payments, is simply the level ordinary annuity factor, computed above, multiplied by $(1 + r)$, since in any annuity situation with beginning-of-period payments interest is merely paid or earned one differential number of times over the account's life,[4] relative to the same numerical inputs but with end-

primarily as an interest rate, although TVM analysis could as easily involve a return on equity rather than the return on a debt investment that "interest" would suggest.

[3] Briefer presentations of this procedure sometimes are shown in higher-level undergraduate corporate finance texts; see, for example, Rao (1995), p. 80, 85 and Brealey, Myers, and Allen (2020), p. 28, 35. The paper by Walker and Kramer (2018), p. 60, shows a similar breakdown for the FV of a level ordinary annuity factor. But even these works do not highlight the distributive property foundations of the annuity factors.

[4] The FV of an annuity due has one more application of interest than does the otherwise similar FV of an ordinary annuity, while the PV of an annuity due has one less interest application than does its corresponding PV of an ordinary annuity, as shown in a later section. But the algebraic adjustment is the same in both cases: multiply the ordinary annuity factor by $(1 + r)$ to compute the annuity due factor, and let algebra guide further steps.

of-period CFs. With payments at the beginning of each period the exponents would be n to 1 rather than $n - 1$ to 0,[5] and the applicable FV of $1 factors would sum to

$(1 + r)^n + (1 + r)^{n-1} + \cdots + (1 + r)^2 + (1 + r)^1$, which can be factored as

$$[(1 + r)^{n-1} + \cdots + (1 + r)^2 + (1 + r)^1 + (1 + r)^0]\,(1 + r)$$

$$= \left[\left(\frac{(1+r)^n - 1}{r}\right)(1 + r)\right]$$

(the distributive property comes into play again, as $1 + r$ is distributed over all terms in the brackets[6]).

Changing Annuity

The math is more complicated when the payment stream changes at a consistent periodic rate, but the proof follows the same steps seen with level payments. Consider collecting deposits at the ends of the next three periods, with a first deposit of PMT dollars and each subsequent deposit exceeding its predecessor by a constant percentage g. Deposits that start at \$300 at the end of period 1 and then grow by 2% per period will be \$300 $(1.02)^0$, \$300 $(1.02)^1$, and \$300 $(1.02)^2$. If the account's growing balance is paid a 6% periodic return, the total at the end of period 3 will be [\$300 $(1.02)^0$] $(1.06)^2$ + [\$300 $(1.02)^1$] $(1.06)^1$ + [\$300 $(1.02)^2$] $(1.06)^0$ = \$337.08 + \$324.36 + \$312.12 = \$973.56. This changing annuity case, with a payment series changing by a steady periodic percentage, also allows for computational grouping with the distributive property, seen here as \$300 [$(1.02)^0 (1.06)^2 + (1.02)^1 (1.06)^1 + (1.02)^2 (1.06)^0$]. The sum of the three terms in brackets is 1.1236 + 1.0812 + 1.0404 = 3.2452, the same figure computed with the FV of a changing ordinary annuity factor: $\left(\frac{(1.06)^3 - (1.02)^3}{.06 - .02}\right)$ = 3.2452. So again a specific example works; \$300 (3.2452) = \$973.56.

But this result also can be generalized. PMT, here the first of multiple deposits, is an unchanging piece of every CF in the changing series, which lets us use the distributive property, on which all annuity computations rely. Therefore n deposits will be PMT, which also can be represented as PMT $(1 + g)^0$, through PMT $(1 + g)^{n-1}$. By the

[5] To find the FV of a level annuity due factor for, *e.g.*, a 4% periodic return and eight periods, students using a standard FV of a level ordinary annuity table are told to subtract 1 from the factor for 4% and *nine* periods. The logic is that the 8-period FV of a level ordinary annuity factor is consistent with eight cash flows and seven applications of interest, while the 9-period ordinary annuity factor relates to eight applications of interest but also to nine CFs, the last of which would occur at the end of period 9. Subtracting 1.0, representing that final phantom CF that would accrue no interest, leaves eight CFs and eight applications of interest. This convoluted process seems more difficult and confusing than merely multiplying the ordinary annuity factor by $(1 + r)$.

[6] A future total with start-of-period CFs is the total found for end-of-period payments multiplied by $(1 + r)$, even for unrelated payments: \$300 $(1.04)^3$ + \$600 $(1.04)^2$ + \$700 $(1.04)^1$ = [\$300 $(1.04)^2$ + \$600 $(1.04)^1$ + \$700 $(1.04)^0$] (1.04) = (\$1,648.48) (1.04) = \$1,714.42. $(1 + r)$ is distributed over each term of the original equation.

end of period n, the total balance in an r% periodic return account receiving end-of-period deposits that grow successively by g% per period should be

$$\text{TOT} = [\text{PMT}\,(1+g)^0]\,(1+r)^{n-1} + [\text{PMT}\,(1+g)^1]\,(1+r)^{n-2} + \cdots + [\text{PMT}\,(1+g)^{n-2}]\,(1+r)^1 + [\text{PMT}\,(1+g)^{n-1}]\,(1+r)^0$$

$$\text{TOT} = \text{PMT}\,[(1+r)^{n-1} + (1+g)^1\,(1+r)^{n-2} + \cdots + (1+g)^{n-2}\,(1+r)^1 + (1+g)^{n-1}] \qquad \text{[Equation 3]}$$

(Because deposits occur at the end of each period, the first is awarded interest n – 1 times, the second-to-last just one time, and the final zero times.) The geometric transformation of multiplying each side of Equation 3 by $(1 + g)$ results in related Equation 4:

$$\text{TOT}\,(1+g) = \text{PMT}\,(1+g)\,[(1+r)^{n-1} + (1+g)^1\,(1+r)^{n-2} + \cdots + (1+g)^{n-2}\,(1+r)^1 + (1+g)^{n-1}]$$

$$\text{TOT} + \text{TOT}\,(g) = \text{PMT}\,[(1+g)^1\,(1+r)^{n-1} + (1+g)^2\,(1+r)^{n-2} + \cdots + (1+g)^{n-1}\,(1+r)^1 + (1+g)^n] \qquad \text{[Equation 4]}$$

Another geometric transformation, multiplying each side of Equation 3 by $(1 + r)$, gives Equation 5:

$$\text{TOT}\,(1+r) = \text{PMT}\,(1+r)\,[(1+r)^{n-1} + (1+g)^1\,(1+r)^{n-2} + \cdots + (1+g)^{n-2}\,(1+r)^1 + (1+g)^{n-1}]$$

$$\text{TOT} + \text{TOT}\,(r) = \text{PMT}\,[(1+r)^n + (1+g)^1\,(1+r)^{n-1} + \cdots + (1+g)^{n-2}\,(1+r)^2 + (1+g)^{n-1}\,(1+r)^1] \qquad \text{[Equation 5]}$$

Finally, subtract Equation 4 from related Equation 5 (both are transformations of Equation 3); most terms cancel out in the subtracting:

$$[\text{TOT} + \text{TOT}\,(r)] - [\text{TOT} + \text{TOT}\,(g)] = \text{PMT}\,[(1+r)^n + (1+g)^1\,(1+r)^{n-1} + \cdots + (1+g)^{n-2}\,(1+r)^2 + (1+g)^{n-1}\,(1+r)^1]$$
$$- \text{PMT}\,[(1+g)^1\,(1+r)^{n-1} + (1+g)^2\,(1+r)^{n-2} + \cdots + (1+g)^{n-1}\,(1+r)^1 + (1+g)^n]$$

$$\text{TOT}\,(r-g) = \text{PMT}\,[(1+r)^n - (1+g)^n] \quad \text{so} \quad \text{TOT} = \text{PMT}\left(\frac{(1+r)^n - (1+g)^n}{r-g}\right)$$

$$\therefore \quad (1+g)^0\,(1+r)^{n-1} + (1+g)^1\,(1+r)^{n-2} + \cdots + (1+g)^{n-2}\,(1+r)^1 + (1+g)^{n-1}\,(1+r)^0 = \left(\frac{(1+r)^n - (1+g)^n}{r-g}\right).$$

The FV of a changing annuity due factor is just the FV of a changing ordinary annuity factor, as found above, multiplied by $(1 + r)$, because start-of-period payments would be accompanied by one additional interest application distributed over the life of the plan. With CFs at the beginning of each period the number of interest applications would be n to 1 rather than n – 1 to 0, and the relevant factors for FV of \$1, adjusted for expected growth, would sum to

$$(1+r)^n + (1+g)^1\,(1+r)^{n-1} + \cdots + (1+g)^{n-2}\,(1+r)^2 + (1+g)^{n-1}\,(1+r)^1, \text{ which can be factored as}$$
$$[(1+g)^0\,(1+r)^{n-1} + (1+g)^1\,(1+r)^{n-2} + \cdots + (1+g)^{n-2}\,(1+r)^1 + (1+g)^{n-1}\,(1+r)^0]\,(1+r)$$
$$= \left[\left(\frac{(1+r)^n - (1+g)^n}{r-g}\right)(1+r)\right].$$

The rate by which an FV of a changing annuity's periodic CFs change can have interesting impacts. If g is set equal to 0 the changing annuity factor simplifies to the FV of a level ordinary annuity factor; the latter is a special case of the former. The periodic rate of change in an FV of a changing annuity's CFs can as easily be negative as positive, with someone making or receiving payments that decline over time. Ordinarily we just have to be careful to keep the negative signs straight; however, over a long enough series of periods with negative compounded periodic growth the payments would eventually become such small fractions of a cent that the account into which they flowed would be impractical to administer. Avoiding errors with negative signs also is a concern when rate of change g is expected to exceed the average rate of return r, such as deposits rising by 8% per period in an account that earns 6% per period. But logical concerns also can arise as high expected growth could compound ultimately to unrealistically high payment amounts; indeed, over the long run expected periodic growth should be a component of, and thus less than, the average expected periodic rate of return.

A seemingly perplexing situation arises when periodic growth is predicted to equal the periodic rate of return: $g = r$; the FV of a changing annuity factor's denominator is 0 in this case, and the rules of algebra do not allow for dividing by 0. One path around this obstacle is to compound the CFs to future values individually. If deposits that start with PMT and grow by 4.5% per period are received in an account generating a 4.5% periodic return, the balance at the end of period 3 will be

$$[\text{PMT } (1.045)^0]\,(1.045)^2 + [\text{PMT } (1.045)^1]\,(1.045)^1 + [\text{PMT } (1.045)^2]\,(1.045)^0$$

$$= \text{PMT } [(1.045)^2 + (1.045)^2 + (1.045)^2] = \text{PMT } [(3)\,(1.045)^2] = \text{PMT } (3.27607500).$$

We can generalize the FV of a changing ordinary annuity factor when $g = r$ as $n\,(1 + r)^{n-1}$. Another way to approach this situation would be to obtain an almost-correct answer by setting g to a value almost, but not exactly, equal to r, thereby replacing 0 in the computations with a very small magnitude we actually can compute with. In the above example, with three payment periods and $g = r = 4.5\%$, we can pretend that g is instead 4.49999%, and compute the essentially identical

$$\text{PMT}\left(\frac{(1.045)^3 - (1.0449999)^3}{.045 - .0449999}\right) = \text{PMT}\left(\frac{.00000033}{.00000010}\right) = \text{PMT } (3.27607000).$$

And, of course, if CFs were to occur at the start of each time period the FV of a changing annuity due factor, with $g = r$, would be $n\,(1 + r)^{n-1}$ multiplied by $(1 + r)$, for a product of $n\,(1 + r)^n$: the n deposits grow by the same rate at which interest is paid or earned on the account's growing balance.

Present Value of Annuity Factors

Level Annuity: Finite Time Periods

As is true in the FV case, any present value (PV) of an annuity factor is just an extension of the PV of a single dollar amount factor $\left(\frac{1}{1+r}\right)^{n}$. Consider the balance needed today to provide a series of three end-of-period payouts. The account earns a 5% periodic return, so the entire initial endowment earns 5% during the first period, while declining remaining amounts earn returns during the second and third periods. The applicable present value of \$1 factors therefore are $\left(\frac{1}{1.05}\right)^{1}$, $\left(\frac{1}{1.05}\right)^{2}$, and $\left(\frac{1}{1.05}\right)^{3}$. If the payments made by the account manager and received by the beneficiary were different, unrelated amounts there would be no way to group them for computing; \$200 $\left(\frac{1}{1.05}\right)^{1}$ + \$300 $\left(\frac{1}{1.05}\right)^{2}$ + \$800 $\left(\frac{1}{1.05}\right)^{3}$ = \$190.48 + \$272.11 + \$691.07 = \$1,153.66. The same approach could be used in computing the present total relating to equal periodic payments: \$400 $\left(\frac{1}{1.05}\right)^{1}$ + \$400 $\left(\frac{1}{1.05}\right)^{2}$ + \$400 $\left(\frac{1}{1.05}\right)^{3}$ = \$380.95 + \$362.81 + \$345.54 = \$1,089.30. Of course, equal CFs can be grouped for computing through the distributive property: \$400 $\left[\left(\frac{1}{1.05}\right)^{1} + \left(\frac{1}{1.05}\right)^{2} + \left(\frac{1}{1.05}\right)^{3}\right]$. Summing the three PV of \$1 factors from the above examples yields $\left[\left(\frac{1}{1.05}\right)^{1} + \left(\frac{1}{1.05}\right)^{2} + \left(\frac{1}{1.05}\right)^{3}\right]$ = .952381 + .907029 + .863838 = 2.723248, which is the value determined with the PV of a level ordinary annuity factor: $\left(\frac{1-\left(\frac{1}{1.05}\right)^{3}}{.05}\right)$ = 2.723248. So the grouping done in this example works correctly; \$400 (2.723248) = \$1,089.30.

But again it is important to prove, with basic steps, why this relationship works in general. Someone able to apply PMT dollars at the ends of the next *n* periods toward paying back a loan, if *r*% in periodic interest is applied to the declining principal owed, can afford to borrow a total TOT today of

$$\text{TOT} = \text{PMT}\left(\frac{1}{1+r}\right)^{1} + \text{PMT}\left(\frac{1}{1+r}\right)^{2} + \cdots + \text{PMT}\left(\frac{1}{1+r}\right)^{n-1} + \text{PMT}\left(\frac{1}{1+r}\right)^{n},$$

grouped with the distributive property as

$$\text{TOT} = \text{PMT}\left[\left(\frac{1}{1+r}\right)^{1} + \left(\frac{1}{1+r}\right)^{2} + \cdots + \left(\frac{1}{1+r}\right)^{n-1} + \left(\frac{1}{1+r}\right)^{n}\right] \qquad \text{[Equation 6]}$$

A geometric transformation, multiplying each side by (1 + *r*), yields the different but related equation

$$\text{TOT}\,(1+r) = \text{PMT}\,(1+r)\left[\left(\frac{1}{1+r}\right)^{1} + \left(\frac{1}{1+r}\right)^{2} + \cdots + \left(\frac{1}{1+r}\right)^{n-1} + \left(\frac{1}{1+r}\right)^{n}\right]$$

$$\text{TOT} + \text{TOT}\,(r) = \text{PMT}\left[\left(\frac{1}{1+r}\right)^0 + \left(\frac{1}{1+r}\right)^1 + \cdots + \left(\frac{1}{1+r}\right)^{n-2} + \left(\frac{1}{1+r}\right)^{n-1}\right] \qquad \text{[Equation 7]}$$

Now subtract Equation 6 from Equation 7 (subtracting equal amounts from both sides of Equation 7); most terms cancel out in the subtracting:

$$\text{TOT} + \text{TOT}\,(r) - \text{TOT} = \text{PMT}\left[\left(\frac{1}{1+r}\right)^0 + \left(\frac{1}{1+r}\right)^1 + \cdots + \left(\frac{1}{1+r}\right)^{n-2} + \left(\frac{1}{1+r}\right)^{n-1}\right]$$

$$- \text{PMT}\left[\left(\frac{1}{1+r}\right)^1 + \left(\frac{1}{1+r}\right)^2 + \cdots + \left(\frac{1}{1+r}\right)^{n-1} + \left(\frac{1}{1+r}\right)^n\right]$$

$$\text{TOT}(r) = \text{PMT}\left[\left(\frac{1}{1+r}\right)^0 + \left(\frac{1}{1+r}\right)^1 - \left(\frac{1}{1+r}\right)^1 + \cdots + \left(\frac{1}{1+r}\right)^{n-1} - \left(\frac{1}{1+r}\right)^{n-1} - \left(\frac{1}{1+r}\right)^n\right]$$

$$\text{TOT}(r) = \text{PMT}\left[\left(\frac{1}{1+r}\right)^0 - \left(\frac{1}{1+r}\right)^n\right] = \text{PMT}\left[1 - \left(\frac{1}{1+r}\right)^n\right] \quad \text{so} \quad \text{TOT} = \text{PMT}\left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right)$$

$$\therefore \qquad \left(\frac{1}{1+r}\right)^1 + \left(\frac{1}{1+r}\right)^2 + \cdots + \left(\frac{1}{1+r}\right)^{n-1} + \left(\frac{1}{1+r}\right)^n = \left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right).$$

The PV of a level ordinary annuity factor is consistent with *n* payments and *n* applications of interest, as shown by exponent 1 on the first of the terms added in computing the annuity factor. With CFs at the start of each period the PV of a level annuity due factor's exponents are 0 to $n - 1$ rather than 1 to n.[7] Distributing $(1 + r)$ over the individual PV of $1 terms brings the applicable PV of $1 factors' sum to

$$\left(\frac{1}{1+r}\right)^0 + \left(\frac{1}{1+r}\right)^1 + \cdots + \left(\frac{1}{1+r}\right)^{n-2} + \left(\frac{1}{1+r}\right)^{n-1}, \text{ which can be factored as}$$

$$\left[\left(\frac{1}{1+r}\right)^1 + \left(\frac{1}{1+r}\right)^2 + \cdots + \left(\frac{1}{1+r}\right)^{n-1} + \left(\frac{1}{1+r}\right)^n\right](1 + r)$$

$$= \left[\left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right)(1+r)\right].$$

As noted earlier, any annuity that carries beginning-of-period payments merely involves one differential number of interest applications over the plan's life. However, in the PV case the annuity due has one *less* interest application than is seen with the otherwise equivalent ordinary annuity. Mechanically, multiplying $\left(\frac{1}{1+r}\right)^n$ by $(1 + r)$ yields $\left(\frac{1}{1+r}\right)^{n-1}$ (or think of it as $(1 + r)^{-n} (1 + r)^1 = (1 + r)^{1-n}$). Conceptually, the 0 exponent on the first payment in the

[7] To find the PV of a level annuity due factor for, *e.g.*, a 5% periodic return and seven periods, students using a standard PV of a level ordinary annuity table are told to add 1 to the factor for 5% and *six* periods. The logic is that the 6-period PV of a level ordinary annuity factor is consistent with six CFs and six applications of interest, while a 7-period PV of annuity due should have six applications of interest and seven CFs, the first occurring at the start of period 1. Adding 1.0, representing a new CF that would occur before any interest could accrue, to the 6-period factor leaves seven CFs and six interest applications. As noted in earlier footnote 5, finding the annuity due factor as the ordinary annuity factor multiplied by $(1 + r)$ is easier to implement, and easier to understand, than is adjusting an ordinary annuity factor for an adjacent number of periods with $\pm$ 1.

series shows that no interest accrues for the beneficiary prior to the first payment, which is made in the present, as soon as the account is funded.[8]

Level Annuity: Infinite Time Periods

This PV of a level annuity factor structure works for both finite, as shown above, and infinite numbers of time periods. An annuity whose CFs are projected never to end is called a *perpetuity*.[9] The level perpetuity idea is quite simple: just take out the interest generated every period, leaving the principal intact to have the same amount of interest applied to it in each subsequent period forever. (Perpetuity is a PV of an annuity concept: the amount needed today to provide for an indefinite stream of payments. The perpetuity idea is not sensible in an FV of annuity case, because if the same amount were deposited or received in an account every period forever the total would be infinite, and there would not be much more to analyze.) If \$100 is placed today in an account that earns a 7% periodic rate of return, and at the end of each period the \$100 (.07) = \$7 in generated interest is taken out, then the \$100 principal remains intact to keep producing \$7 in each ensuing period perpetually. Reverse that reasoning to show that if an account is expected to generate \$7 in interest every period, and the risk of the situation calls for a 7% expected periodic return, then the account's total initial balance today must be \$7 ÷ .07 = \$7 $\left(\frac{1}{.07}\right)$ = \$100. It makes intuitive sense that the PV of a level ordinary perpetuity factor is $\left(\frac{1}{r}\right)$.

This intuitive result can be proven more formally with straightforward algebra. Think of the PV of a level ordinary annuity factor structure outlined above. If r is a percentage value greater than 0 (as an expected periodic rate of return almost always would be), then as the exponent on the fraction that constitutes the upper-right term approaches infinity that term's magnitude approaches 0, and in the perpetuity case the PV of a level ordinary annuity factor simplifies to

$$\left(\frac{1-\left(\frac{1}{1+r}\right)^{\infty}}{r}\right) = \left(\frac{1-0}{r}\right) = \left(\frac{1}{r}\right).$$

[8] A present total with unrelated start-of-period CFs, as with its future total counterpart, again is the total for end-of-period payments multiplied by $(1 + r)$: \$200 $\left(\frac{1}{1.05}\right)^0$ + \$300 $\left(\frac{1}{1.05}\right)^1$ + \$800 $\left(\frac{1}{1.05}\right)^2$ = $\left[\$200\left(\frac{1}{1.05}\right)^1 + \$300\left(\frac{1}{1.05}\right)^2 + \$800\left(\frac{1}{1.05}\right)^3\right]$ (1.05) = (\$1,153.66) (1.05) = \$1,211.34; again quantity $(1 + r)$ is distributed over each term of the original equation.

[9] While unending payments may be explicitly specified, as with some German and Japanese government bonds, British consuls issued in the 1800s, or Bank of China perpetual bonds issued early in 2019, more typically we use level or changing perpetuity computations in cases involving long, uncertain time periods, such as establishing a charitable endowment (the charity may remain in existence for a long time, but likely not forever). This practical convenience tends not to sacrifice much in accuracy, as a perpetuity's PV is very close to the PV of a long, finite annuity based on the same periodic payment and r figures.

As in all annuity cases, the factor for a level perpetuity due, with an infinite series of equal beginning-of-period payments, is but the PV of a level ordinary perpetuity factor multiplied by (1 + *r*):

$$\left[\left(\frac{1}{r}\right)(1+r)\right] = \left[\left(\frac{1}{r}\right)+1\right]$$

(1/*r* is distributed over each of the terms 1 and *r* in parentheses). Based on the example above, the total needed today to fund this payment stream would be \$7 $\left[\left(\frac{1}{.07}\right)(1.07)\right]$ = \$7 $\left[\left(\frac{1}{.07}\right)+1\right]$ = \$7 $\left(\frac{1}{.07}\right)$ + \$7 (1) = \$107. Payments at the beginning of each period mean the first occurs as soon as the plan is set up, with no interest buildup before that first payment is taken, such that any initial balance must be sufficient to allow the desired immediate withdrawal and leave enough principal to fund later receipts. Thus the initial balance must be bigger by the amount of one withdrawal, which is taken immediately; then the remainder is exactly the figure needed to generate the interest that constitutes the unchanging payment in each successive period.

Changing Annuity: Finite Time Periods

The same approach applies in showing what happens when payments corresponding to a large present value are projected to grow or decline at a constant periodic rate, as often is hypothesized in dividend discount models. Think of a plan funded today to allow three end-of-period payments to be made to a beneficiary, starting with the amount PMT dollars and then growing with each subsequent period by the constant percentage g. If the first payout is to be \$900 and the amounts are to grow by 3% per period, the figures will be \$900 $(1.03)^0$, \$900 $(1.03)^1$, and \$900 $(1.03)^2$. If the endowment's declining balance earns 8% per period, the initial total balance must be [\$900 $(1.03)^0$] $\left(\frac{1}{1.08}\right)^1$ + [\$900 $(1.03)^1$] $\left(\frac{1}{1.08}\right)^2$ + [\$900 $(1.03)^2$] $\left(\frac{1}{1.08}\right)^3$ = \$833.33 + \$794.75 + \$757.96 = \$2,386.04. Again a series of payments changing by the same percentage from period to period lets us use the distributive property to group values together in computing, here \$900 $\left[(1.03)^0\left(\frac{1}{1.08}\right)^1+(1.03)^1\left(\frac{1}{1.08}\right)^2+(1.03)^2\left(\frac{1}{1.08}\right)^3\right]$. The three terms in brackets sum to .925926 + .883059 + .842177 = 2.651162, the same value found with the PV of a changing ordinary annuity factor: $\left(\frac{1-\left(\frac{1.03}{1.08}\right)^3}{.08-.03}\right)$ = 2.651162. Again a specific case shows the equivalence; \$900 (2.651162) = \$2,386.04.

Yet as before we can prove why this relationship works in all PV of changing annuity situations. If a graduate wants to fund a scholarship that will make *n* end-of-period distributions that start in period 1 with PMT dollars and then increase each subsequent period by the estimated average periodic inflation rate of g%, and r% in interest can be earned each period on the remaining principal, then the total amount that should be given today is

$$\text{TOT} = [\text{PMT}\,(1+g)^0]\left(\frac{1}{1+r}\right)^1 + [\text{PMT}\,(1+g)^1]\left(\frac{1}{1+r}\right)^2 + \cdots + [\text{PMT}\,(1+g)^{n\text{-}2}]\left(\frac{1}{1+r}\right)^{n-1} + [\text{PMT}\,(1+g)^{n\text{-}1}]\left(\frac{1}{1+r}\right)^n]$$

$$\text{TOT} = \text{PMT}\left[(1+g)^0\left(\frac{1}{1+r}\right)^1 + (1+g)^1\left(\frac{1}{1+r}\right)^2 + \cdots + (1+g)^{n-2}\left(\frac{1}{1+r}\right)^{n-1} + (1+g)^{n-1}\left(\frac{1}{1+r}\right)^n\right]$$

[Equation 8]

Multiplying each side of Equation 8 by $(1 + g)$, a geometric transformation similar to that seen in so many earlier examples, produces the different but related Equation 9:

$$\text{TOT}\,(1+g) = \text{PMT}\,(1+g)\left[(1+g)^0\left(\frac{1}{1+r}\right)^1 + (1+g)^1\left(\frac{1}{1+r}\right)^2 + \cdots + (1+g)^{n-2}\left(\frac{1}{1+r}\right)^{n-1} + (1+g)^{n-1}\left(\frac{1}{1+r}\right)^n\right]$$

$$\text{TOT} + \text{TOT}\,(g) = \text{PMT}\left[(1+g)^1\left(\frac{1}{1+r}\right)^1 + (1+g)^2\left(\frac{1}{1+r}\right)^2 + \cdots + (1+g)^{n-1}\left(\frac{1}{1+r}\right)^{n-1} + (1+g)^n\left(\frac{1}{1+r}\right)^n\right]$$

[Equation 9]

Then multiplying each side of Equation 8 by $(1 + r)$ yields the different but related Equation 10:

$$\text{TOT}\,(1+r) = \text{PMT}\,(1+r)\left[(1+g)^0\left(\frac{1}{1+r}\right)^1 + (1+g)^1\left(\frac{1}{1+r}\right)^2 + \cdots + (1+g)^{n-2}\left(\frac{1}{1+r}\right)^{n-1} + (1+g)^{n-1}\left(\frac{1}{1+r}\right)^n\right]$$

$$\text{TOT} + \text{TOT}\,(r) = \text{PMT}\left[(1+g)^0 + (1+g)^1\left(\frac{1}{1+r}\right)^1 + \cdots + (1+g)^{n-2}\left(\frac{1}{1+r}\right)^{n-2} + (1+g)^{n-1}\left(\frac{1}{1+r}\right)^{n-1}\right]$$

[Equation 10]

Subtract Equation 9 from Equation 10 (both are geometric transformations of Equation 8, so the two are related); after most terms cancel out in the subtraction the result is

$$[\text{TOT} + \text{TOT}\,(r)] - [\text{TOT} + \text{TOT}\,(g)] = \text{PMT}\left[1 + (1+g)^1\left(\frac{1}{1+r}\right)^1 + \cdots + (1+g)^{n-2}\left(\frac{1}{1+r}\right)^{n-2} + (1+g)^{n-1}\left(\frac{1}{1+r}\right)^{n-1}\right]$$

$$- \text{PMT}\left[(1+g)^1\left(\frac{1}{1+r}\right)^1 + (1+g)^2\left(\frac{1}{1+r}\right)^2 + \cdots + (1+g)^{n-1}\left(\frac{1}{1+r}\right)^{n-1} + (1+g)^n\left(\frac{1}{1+r}\right)^n\right]$$

$$\text{TOT}\,(r-g) = \text{PMT}\left[1 - (1+g)^n\left(\frac{1}{1+r}\right)^n\right] = \text{PMT}\left[1 - \left(\frac{1+g}{1+r}\right)^n\right]$$

$$\text{so} \quad \text{TOT} = \text{PMT}\left(\frac{1-\left(\frac{1+g}{1+r}\right)^n}{r-g}\right)$$

$$\therefore \quad (1+g)^0\left(\frac{1}{1+r}\right)^1 + (1+g)^1\left(\frac{1}{1+r}\right)^2 + \cdots + (1+g)^{n-2}\left(\frac{1}{1+r}\right)^{n-1} + (1+g)^{n-1}\left(\frac{1}{1+r}\right)^n = \left(\frac{1-\left(\frac{1+g}{1+r}\right)^n}{r-g}\right).$$

The PV of a changing annuity due factor, as with all annuities involving beginning-of-period payments, is merely the changing ordinary annuity factor with another $(1 + r)$ distributed over all individual terms. If payments occur at the start of each period the exponents on the related PV of \$1 factors are 0 to $n - 1$ rather than 1 to n, and those factors' sum is

$(1+g)^0\left(\frac{1}{1+r}\right)^0 + (1+g)^1\left(\frac{1}{1+r}\right)^1 + \cdots + (1+g)^{n-2}\left(\frac{1}{1+r}\right)^{n-2} + (1+g)^{n-1}\left(\frac{1}{1+r}\right)^{n-1}$, which can be factored as

$$\left[(1+g)^0\left(\frac{1}{1+r}\right)^1 + (1+g)^1\left(\frac{1}{1+r}\right)^2 + \cdots + (1+g)^{n-2}\left(\frac{1}{1+r}\right)^{n-1} + (1+g)^{n-1}\left(\frac{1}{1+r}\right)^n\right](1+r)$$

$$= \left[\left(\frac{1-\left(\frac{1+g}{1+r}\right)^n}{r-g}\right)(1+r)\right].$$

As with the future value case, the expected rate of change in periodic CFs in a PV of a changing annuity situation can bring interesting computational results. If $g = 0$ the PV of a changing ordinary annuity factor becomes the PV of a level ordinary annuity factor; the latter is a special case of the former. A PV of a changing annuity's CFs also can change by a negative periodic percentage, as with an endowed plan that provides lower payouts as time passes. As in the FV case we must use caution in working with the negative signs, but also must watch for CFs becoming too small to work with meaningfully. Negative signs also arise in a PV of a changing annuity example when expected periodic change rate g is greater than expected periodic return rate r. Withdrawals taken from a retirement nest egg could be planned to increase each period by a percentage higher than the periodic return earned on the account's remaining balance, at least over a finite interval not too long in duration.

And again the $g = r$ case might seem troublesome because of the resulting 0 denominator in the PV of a changing annuity factor, but in a manner similar to that used with the FV of a changing annuity we can discount individual expected changing CFs to present values. If an initial withdrawal in amount PMT is followed by two additional CFs that each grow by 6% and are taken from an account whose declining balance earns 6% per period, the amount needed today to fund the plan is

$$[\text{PMT}\,(1.06)^0]\left(\frac{1}{1.06}\right)^1 + [\text{PMT}\,(1.06)^1]\left(\frac{1}{1.06}\right)^2 + [\text{PMT}\,(1.06)^2]\left(\frac{1}{1.06}\right)^3$$

$$= \text{PMT}\left[\left(\frac{1}{1.06}\right)^1 + \left(\frac{1}{1.06}\right)^1 + \left(\frac{1}{1.06}\right)^1\right] = \text{PMT}\left[(3)\left(\frac{1}{1.06}\right)\right] = \text{PMT}\ (2.83018868).$$

More generally, the PV of a changing ordinary finite annuity factor when $g = r$ is $n\left(\frac{1}{1+r}\right)$. As in the earlier FV of a changing annuity case with $g = r$ we could directly compute an almost-correct solution by setting g's magnitude a tiny bit below r's, replacing 0 with a very small nonzero value. If told that $g = r = 6\%$, we can treat g instead as 5.99999%, and end up quite close to the theoretically correct value:

$$\text{PMT}\left(\frac{1-\left(\frac{1.0599999}{1.06}\right)^3}{.06-.0599999}\right) = \text{PMT}\left(\frac{.00000028}{.00000010}\right) = \text{PMT}\ (2.83018800).$$

And of course the PV of a changing annuity due factor with $g = r$ is just $n\left(\frac{1}{1+r}\right)$ multiplied by $(1 + r)$, which is, strange though it might seem, exactly n. After an immediate withdrawal of initial value PMT, interest applied to the declining balance exactly covers the periodic growth in the remaining payments.

Changing Annuity: Infinite Time Periods

The case of a perpetual series of payments that grows by g% per period is conceptually more complicated than the level perpetuity, in that each payment does not equal the interest generated during the specified period on the remaining principal. But as with its level payment cousin, the PV of a changing perpetuity factor is a handy streamlining of the factor for the PV of a changing finite annuity. If $r > 0$ (a periodic rate of return almost always would be positive), and if $g < r$ (which must be true in the long run, since expected growth is a component of an overall expected return), then as the exponent on the fraction that is the upper-right term approaches infinity that term's magnitude approaches 0, and the PV of a changing perpetuity factor simplifies to $1/(r - g)$;

$$\left(\frac{1-\left(\frac{1+g}{1+r}\right)^\infty}{r-g}\right) = \left(\frac{1-0}{r-g}\right) = \left(\frac{1}{r-g}\right).$$

If a generous alumnus wants to fund a scholarship plan that will continue indefinitely, starting with a \$12,000 payout at the end of period 1 and then increasing by 3% per period, and the university expects to earn a 7% average periodic return on money in its foundation, then the donor should contribute \$12,000 $\left(\frac{1}{.07-.03}\right)$ = \$300,000; the \$12,000 initial payment certainly is not the .07 (\$300,000) = \$21,000 in expected period 1 interest earnings. And as in all previous annuity cases, if the stream of growing scholarships is to be awarded at the beginning of each period a withdrawal will be taken from the fund immediately, before any interest builds up, so the donor will have to give a somewhat higher initial endowment. We compute that figure, of course, by multiplying the initial payment by the factor for corresponding end-of-period payments, increased by (1 + r):

$$\$12{,}000\left[\left(\frac{1}{.07-.03}\right)(1.07)\right] = \$12{,}000\ (26.7500) = \$321{,}000.$$

After the immediate $12,000 payout the balance would be the sum needed to fund end-of-period flows starting at $12,000 (1.03) and rising by 3% each period thereafter: $12,360 [1/(.07 – .03)] = $309,000.

A changing perpetuity also could, in theory, involve a negative average constant rate of change (expected decline in CFs over time); just treat the negative rate of change correctly in the formula. But while CFs in a declining percentage perpetuity never actually reach zero, eventually they reduce to such tiny fractions of a cent that paying anything to a beneficiary would be unworkable. Unlike with finite annuities, a perpetuity's periodic rate of change g cannot exceed (or even equal) the expected periodic return r. If the university expects to earn a 7% average periodic return on scholarship endowments, but the contributor wants the award to be $12,000 in period 1 and then increase by 10% per period, the scholarship program cannot continue indefinitely (the endowment's balance will eventually reach zero). We get a nonsensical answer of

$$\$12{,}000\left(\frac{1}{.07-.10}\right) = \$12{,}000\left(\frac{1}{-.03}\right) = \frac{\$13{,}000}{-.03} = \$12{,}000\ (-\,33.333333) = -\ \$400{,}000$$

(or [$12,000/– .03] [1.07] = – $428,000 if the first of the changing awards were to be paid immediately); a negative initial balance, with the school owing the donor, surely cannot fund a series of increasing awards over an unending time span. Actually, an infinite initial TOT amount would be needed to fund an infinite series of withdrawals that grow by a periodic rate equal to or greater than the periodic rate of return (the factor [1/(r – g)] for PV of a changing perpetuity would approach ∞ even as g approaches r from below). As noted, over the long term, periodic growth rate g should be a portion of average periodic return rate r.

Checking Student Understanding

A closed-book pre-test with two multiple choice conceptual questions and five computational questions was administered to 36 undergraduate finance majors who had worked with TVM applications in some prior courses. Questions involved a mix of FV/PV of ordinary/annuity due situations, with level/changing payments over finite/infinite periods. Subjects were asked to do their best, assured that performance would not affect course grades. They were encouraged to show the algebra steps needed for correct answers, with computations not required, though it also was acceptable to compute actual answers with financial or even graphing calculators (internet access was not permitted). In the remainder of that class period and part of the subsequent class the instructor talked through the proofs seen in equations 1 through 10 discussed earlier; a handout was provided. A closed-book post-test then was administered, with the same structure as the pre-test but different conceptual questions and numerical examples, and comments were invited. The scoring rubric for both tests gave full credit for correct answers, partial credit based on the severity of errors, and zero credit for incorrect problem identification or no answers given. Pre-test percentage outcomes ranged from 0 to .714, with mean .265 (median .22) and standard deviation .21; scores clearly were low, with many questions left blank, despite thirty minutes' time allowed. Post-test scores were markedly higher, ranging from 0 to 1.00, with mean .45 (median .5) and standard deviation .27. A two-tailed T-test shows improvement in scores to be significant at a 99% level.

We must be cautious not to oversell the post-test results. First, the sample size was small. Second, we cannot separate the impact of showing the distributive property's role and other algebra details from that attributable to the general TVM overview embedded in the algebra discussion. But even more encouraging than the higher average scores were the comments; students generally reported being very pleased with the discussion and having a much better grasp of TVM mechanics, with some asking why they had not seen these straightforward proofs before.

Conclusions

Many students are content to let a phone app or web site do financial computations for them. But some are willing to give TVM fundamentals a serious effort, and while $(1 + r)^n$ and its reciprocal make perfect sense, they can become frustrated trying to explain annuity factors to themselves, asking questions such as "why do we subtract the 1?" The preceding discussion provides the answers, for level annuity and even changing annuity factors, which algebraic proofs show to be only minor extensions of the level annuity factors. Serious learners appreciate being shown these breakdowns; in fact, working proofs is seen by experts in math education as "critically important to knowing and doing mathematics" (Nardi and Knuth, 2017, p. 268). Examining the algebra also shows that the structure of the money flows is what matters in identifying any TVM problem. Indeed, that structural understanding makes clear the folly of the frequent oversimplification that "in FV of annuity you put money in and in PV of annuity you take money out." After all, the cash flow structure looks the same from either transactor's perspective; the question is whether equal or related payments are connected to a large future or present total. The algebra that explains annuity fundamentals is based largely on the distributive property. That mathematical tool demonstrates both how the FV and PV of $1 factors sum to the annuity factors, and how the FV and PV of both level and changing ordinary annuity factors are adjusted with $(1 + r)$ in producing the corresponding annuity due factors. The steps presented here also show how perpetuities are uncomplicated offshoots of PV of finite annuity situations. Truly understanding time value's foundations enhances a student's ability to pursue analytical jobs; exploring the algebra behind the annuity factors helps elevate learning TVM from memorization to a substantive critical thinking exercise.

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