Annuities Due and Time Value Tables

1. The future value (FV) of a level *ordinary* annuity (with end-of-year payments) of $4,000 per year for ten years, if the account's growing balance earns a 6% annual rate of return, is computed as

$$[\$4{,}000\ (1.06)^9] + [\$4{,}000\ (1.06)^8] + \cdots + [\$4{,}000\ (1.06)^1] + [\$4{,}000\ (1.06)^0]$$

which, by the distributive property (the mathematical basis for all annuity computations), can be restated as

$$\$4{,}000\ [(1.06)^9 + (1.06)^8 + \cdots + (1.06)^1 + (1.06)^0]$$

$$= \$4{,}000 \left(\frac{(1.06)^{10} - 1}{.06}\right) = \$4{,}000\ (13.180795) = \$\underline{\underline{52{,}723.18}}$$

A year-by-year breakdown appears as follows:

Year	Beginning Balance	Plus Interest	Total Available	Plus Deposit	Ending Balance
1	$0.00	$0.00	$0.00	$4,000.00	$4,000.00
2	$4,000.00	$240.00	$4,240.00	$4,000.00	$8,240.00
3	$8,240.00	$494.40	$8,734.40	$4,000.00	$12,734.40
4	$12,734.40	$764.06	$13,498.46	$4,000.00	$17,498.46
5	$17,498.46	$1,049.91	$18,548.37	$4,000.00	$22,548.37
6	$22,548.37	$1,352.90	$23,901.27	$4,000.00	$27,901.27
7	$27,901.27	$1,674.08	$29,575.35	$4,000.00	$33,575.35
8	$33,575.35	$2,014.52	$35,589.87	$4,000.00	$39,589.87
9	$39,589.87	$2,375.39	$41,965.26	$4,000.00	$45,965.26
10	$45,965.26	$2,757.92	$48,723.18	$4,000.00	$52,723.18

Notice that *the 10-year FV of a level ordinary annuity factor is consistent with ten cash flows but only nine applications of interest* (no interest is earned or paid in year 1). If payments instead are to occur at the start of each year (annuity *due*), the only difference is that interest occurs one *additional* time over the plan's life; there are *ten cash flows and ten applications of interest*, as shown in the breakdown below:

Year	Beginning Balance	Plus Deposit	Total Available	Plus Interest	Ending Balance
1	$0.00	$4,000.00	$4,000.00	$240.00	$4,240.00
2	$4,240.00	$4,000.00	$8,240.00	$494.40	$8,734.40
3	$8,734.40	$4,000.00	$12,734.40	$764.06	$13,498.46
4	$13,498.46	$4,000.00	$17,498.46	$1,049.91	$18,548.37
5	$18,548.37	$4,000.00	$22,548.37	$1,352.90	$23,901.27
6	$23,901.27	$4,000.00	$27,901.27	$1,674.08	$29,575.35
7	$29,575.35	$4,000.00	$33,575.35	$2,014.52	$35,589.87
8	$35,589.87	$4,000.00	$39,589.87	$2,375.39	$41,965.26
9	$41,965.26	$4,000.00	$45,965.26	$2,757.92	$48,723.18
10	$48,723.18	$4,000.00	$52,723.18	$3,163.39	$55,886.57

The most direct and easily understood way to compute the annuity due's value is to multiply the ordinary annuity factor by (1 + *r*) and then let algebra guide the steps that follow:

$$\$4{,}000\ [(1.06)^{10} + (1.06)^{9} + \cdots + (1.06)^{2} + (1.06)^{1}]$$

$$= \$4{,}000\ [(1.06)^{9} + (1.06)^{8} + \cdots + (1.06)^{1} + (1.06)^{0}]\ (1.06)$$

$$= \$4{,}000 \left[\left(\frac{(1.06)^{10}-1}{.06}\right)(1.06)\right] = \$4{,}000\ (13.971643) = \$\underline{\underline{55{,}886.57}}$$

But instead of multiplying the level ordinary factor by (1 + *r*), students using a standard FV of a level ordinary annuity table are told to subtract 1 from the factor for 6% and *eleven* periods. Recall that the 10-year FV of a level ordinary annuity factor is consistent with ten cash flows and nine applications of interest, thus the 11-period level ordinary annuity factor relates to ten applications of interest but also to eleven cash flows, the last of which would occur at the end of year 11. Subtracting 1.0, representing that final phantom CF that would accrue no interest, leaves ten CFs and ten applications of interest.

Future Value of a Level Ordinary Annuity Factor @ 6%

7 Years	8 Years	9 Years	10 Years	11 Years	12 Years
8.393838	9.897468	11.491316	13.180795	14.971643	16.869941

So here we get 14.971643 – 1 = 13.971643, as computed above as the level ordinary factor multiplied by 1.06. This convoluted process of subtracting 1.0 from the ordinary annuity factor for the next higher number of periods is far more difficult and confusing than merely multiplying the ordinary annuity factor by (1 + *r*); note that 13.180795 (1.06) = 13.971643 as well.

2. The present value (PV) of a level ordinary annuity (with end-of-year payments) of $3,000 per year for ten years, if the account's declining balance earns a 5% annual rate of return, is computed as

$$\left[\$3{,}000\left(\frac{1}{1.05}\right)^{1}\right] + \left[\$3{,}000\left(\frac{1}{1.05}\right)^{2}\right] + \cdots + \left[\$3{,}000\left(\frac{1}{1.05}\right)^{9}\right] + \left[\$3{,}000\left(\frac{1}{1.05}\right)^{10}\right]$$

which, by the distributive property (which works with multiplication but not division), can be restated as

$$\$3{,}000\left[\left(\frac{1}{1.05}\right)^{1} + \left(\frac{1}{1.05}\right)^{2} + \cdots + \left(\frac{1}{1.05}\right)^{9} + \left(\frac{1}{1.05}\right)^{10}\right]$$

$$= \$3{,}000\left(\frac{1-\left(\frac{1}{1.05}\right)^{10}}{.05}\right) = \$3{,}000\ (7.721735) = \$\underline{\underline{23{,}165.20}}$$

A year-by-year breakdown appears as follows:

Year	Beginning Balance	Plus Interest	Total Available	Minus Withdrawal	Ending Balance
1	$23,165.20	$1,158.26	$24,323.47	$3,000.00	$21,323.47
2	$21,323.47	$1,066.17	$22,389.64	$3,000.00	$19,389.64
3	$19,389.64	$969.48	$20,359.12	$3,000.00	$17,359.12
4	$17,359.12	$867.96	$18,227.08	$3,000.00	$15,227.08
5	$15,227.08	$761.35	$15,988.43	$3,000.00	$12,988.43
6	$12,988.43	$649.42	$13,637.85	$3,000.00	$10,637.85
7	$10,637.85	$531.89	$11,169.74	$3,000.00	$8,169.74
8	$8,169.74	$408.49	$8,578.23	$3,000.00	$5,578.23
9	$5,578.23	$278.91	$5,857.14	$3,000.00	$2,857.14
10	$2,857.14	$142.86	$3,000.00	$3,000.00	$0.00

Notice that the 10-year *PV of a level ordinary annuity factor is consistent with ten cash flows and ten applications of interest*. If payments instead are to occur at the start of each year (annuity due), the only difference is that interest occurs one *less* time over the plan's life; there are *ten cash flows and nine applications of interest*, as shown in the breakdown below:

Year	Beginning Balance	Minus Withdrawal	Total Available	Plus Interest	Ending Balance
1	$24,323.47	$3,000.00	$21,323.47	$1,066.17	$22,389.64
2	$22,389.64	$3,000.00	$19,389.64	$969.48	$20,359.12
3	$20,359.12	$3,000.00	$17,359.12	$867.96	$18,227.08
4	$18,227.08	$3,000.00	$15,227.08	$761.35	$15,988.43
5	$15,988.43	$3,000.00	$12,988.43	$649.42	$13,637.85
6	$13,637.85	$3,000.00	$10,637.85	$531.89	$11,169.74
7	$11,169.74	$3,000.00	$8,169.74	$408.49	$8,578.23
8	$8,578.23	$3,000.00	$5,578.23	$278.91	$5,857.14
9	$5,857.14	$3,000.00	$2,857.14	$142.86	$3,000.00
10	$3,000.00	$3,000.00	$0.00	$0.00	$0.00

The most direct and easily understood way to compute the annuity due's value is to multiply the ordinary annuity factor by (1 + *r*) and then let algebra guide the steps that follow:

$$\$3{,}000\left[\left(\frac{1}{1.05}\right)^0 + \left(\frac{1}{1.05}\right)^1 + \cdots + \left(\frac{1}{1.05}\right)^8 + \left(\frac{1}{1.05}\right)^9\right]$$

$$= \$3{,}000\left[\left(\frac{1}{1.05}\right)^1 + \left(\frac{1}{1.05}\right)^2 + \cdots + \left(\frac{1}{1.05}\right)^9 + \left(\frac{1}{1.05}\right)^{10}\right](1.05)$$

$$= \$3{,}000\left[\left(\frac{1-\left(\frac{1}{1.05}\right)^{10}}{.05}\right)(1.05)\right] = \$3{,}000\ (8.107822) = \$\underline{\underline{24{,}323.47}}$$

But instead of multiplying the level ordinary factor by (1 + *r*), students using a standard PV of a level ordinary annuity table are told to add 1 to the factor for 5% and *nine* periods. Recall that the 10-year PV of a level ordinary annuity factor is consistent with ten cash flows and ten interest applications, while the 10-year annuity due should have nine applications of interest and ten cash flows, the first occurring at the start of year 1. Adding 1.0, representing the new CF that would occur before any interest could accrue, to the 9-year factor leaves ten CFs and nine applications of interest.

Present Value of a Level Ordinary Annuity Factor @ 5%

7 Years	8 Years	9 Years	10 Years	11 Years	12 Years
5.786373	6.463213	7.107822	7.721735	8.306414	8.863252

So here we get 7.107822 + 1 = 8.107822, as also computed above as the level ordinary factor multiplied by 1.05. Again this awkward adjusting of the ordinary annuity factor for an adjacent number of periods is more difficult to understand than merely multiplying the ordinary annuity factor by (1 + *r*); note that 7.721735 (1.05) = 8.107822 as well.