TIME VALUE: PROBLEMS & SOLUTIONS SET C *(copyright © 2022 Joseph W. Trefzger)*

This problem set covers all of our basic time value of money applications. It is designed to provide yet additional practice for FIL 240 students beyond the main problem set and Set B. But whereas problems in the earlier sets are organized in the order of our coverage, generally progress in degree of difficulty, and are accompanied by complete explanations, here problems are in a random order, and long narrative explanations generally are not offered with the numerical answers. Set C therefore offers the chance to practice under something more like exam conditions.

1-C. Illinois State graduate Brad sets the long-term goal of buying a campus area condominium for his newborn daughter to live in while she attends ISU. He plans to save equal amounts of money every year until she graduates from high school in 18 years to amass the $218,000 that he believes a condo unit will cost at that time. If he expects to earn a 4.3% average annual rate of return on any money in his savings plan, how much should he deposit if he saves at the end of each year? At the start of each year?

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left(\frac{(1.043)^{18}-1}{.043}\right) = \$218{,}000$$

$$\text{PMT} \times 26.363310 = \$218{,}000$$

$218,000 ÷ 26.363310 = PMT = $<u>8,269.07</u> if deposits are at the end of each year or a smaller

$$\text{PMT}\left[\left(\frac{(1.043)^{18}-1}{.043}\right)(1.043)\right] = \$218{,}000$$

$$\text{PMT} \times 27.496933 = \$218{,}000$$

$218,000 ÷ 27.496933 = PMT = $<u>7,928.16</u> if deposits are at the beginning of each year.

(This is a future value of an annuity problem with an unknown regular periodic payment.)

2-C. Amy wants to save $26,000 to buy a bass fishing boat. If she can save $1,675 at the end of each year, and if she can earn a 3.92% average annual rate of return on her account's balance, how many years will it take for her to amass the $26,000? What if instead she makes the $1,675 deposit at the beginning of each year?

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$1{,}675\left(\frac{(1.0392)^{n}-1}{.0392}\right) = \$26{,}000$$

$$\frac{(1.0392)^{n}-1}{.0392} = 15.522388$$

$$(1.0392)^n - 1 = .608478 \quad \text{so} \quad (1.0392)^n = 1.608478$$

$$\ln[(1.0392)^n] = \ln 1.608478$$

$$n \times \ln 1.0392 = \ln 1.608478$$

$$n\,(.038451) = .475288$$

n = <u>12.360819</u>, or a little more than 12 years

with end-of-year savings deposits and a little bit shorter:

$$\$1{,}675\left[\left(\frac{(1.0392)^{n}-1}{.0392}\right)(1.0392)\right] = \$26{,}000$$

$$\left[\left(\frac{(1.0392)^{n}-1}{.0392}\right)(1.0392)\right] = 15.522388$$

$$\frac{(1.0392)^{n}-1}{.0392} = 14.936863$$

$$(1.0392)^n - 1 = .585525 \quad \text{so} \quad (1.0392)^n = 1.585525$$

$$\ln[(1.0392)^n] = \ln 1.585525$$

$$n \times \ln 1.0392 = \ln 1.585525$$

$$n\,(.038451) = .460916$$

n = <u>11.987032</u>, just a little under 12 years with beginning-of-year deposits. (This is a future value of an annuity problem with an unknown number of time periods.)

3-C. Wealthy graduate Aaron contributes $1,350,000 to the Illinois State University Foundation to establish a scholarship fund. He directs that the fund should provide scholarships for the next 14 years, in honor of legendary ISU quarterback Red G. Redbird, who wore Number 14. If the foundation's investment director can earn a 5.31% average annual rate of return on any money that she manages, how much can be paid out annually in scholarships if the awards are made at the end of each year? What if the money is paid out at the start of each of the 14 years?

PMT x FAC = TOT

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.0531}\right)^{14}}{.0531}\right) = \$1{,}350{,}000$$

PMT x 9.705332 = $1,350,000

$1,350,000 ÷ 9.705332 = PMT = $<u>139,098.80</u>

if scholarship recipients get the money at the end of each year (level ordinary annuity) or a smaller

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1}{1.0531}\right)^{14}}{.0531}\right)(1.0531)\right] = \$1{,}350{,}000$$

PMT x 14.404316 = $1,350,000

$1,350,000 ÷ 10.220685 = PMT = $<u>132,085.08</u>

if scholarship winners are paid at the beginning of each year (level annuity due). (This is a present value of an annuity problem with an unknown regular periodic payment.)

4-C. Tom just netted $33,625 as a winner on the TV game show *Wheel of Jeopardy* (he actually won $50,000, but after paying applicable income taxes has $33,625 left). He wants to use this single amount of money to help build a retirement nest egg. What average annual rate of return will he need to earn for the $33,625 to grow to $100,000 by the time he retires in 21 years? What average annual rate of return would grow his balance to $150,000?

$$\text{BAMT}\,(1 + r)^n = \text{EAMT}$$

$$\$33{,}625\,(1 + r)^{21} = \$100{,}000$$

$$(1 + r)^{21} = \$100{,}000 \div \$33{,}625 = 2.973978$$

$$\sqrt[21]{(1+r)^{21}} = \sqrt[21]{2.973978}$$

$$1 + r = 2.973978^{1/21} = 2.973978^{.047619} = 1.053270$$

r = .053270 = <u>5.3270</u>% for the single $33,625 figure to grow to $100,000 or

$$\$33{,}625\,(1 + r)^{21} = \$150{,}000$$

$$(1 + r)^{21} = \$150{,}000 \div \$33{,}625 = 4.460967$$

$$\sqrt[21]{(1+r)^{21}} = \sqrt[21]{4.460967}$$

$$1 + r = 4.460967^{1/21} = 4.460967^{.047619} = 1.073804$$

r = .073804 = <u>7.3804</u>% for the $33,625 to grow to $150,000.

(This is a non-annuity problem with the rate of return unknown; no series of regular payments.)

5-C. Betty just opened an account at BloNoBank. The bank manager expects to collect savings deposits from Betty of $530 in each of years 1 through 11 and $660 in each of years 12 through 17, and feels the bank will pay savers a 3.22% average annual interest rate in the coming years. How much does the banker expect to owe Betty at the end of year 17 if she makes the deposits at the end of each year? What if instead she saves at the start of each year?

In the two-stage savings plan described the depositor contributes $530 each year for 11 years and then earns interest on that stage's accumulated balance for 6 years while making the second deposit series; the grand total (it could be in two separate accounts or all in one account) will be

$$\$530\left(\frac{(1.0322)^{11}-1}{.0322}\right)(1.0322)^{6} + \$660\left(\frac{(1.0322)^{6}-1}{.0322}\right)(1.0322)^{0}$$

$$= \$530\ (12.953609)\ (1.209437) + \$660\ (6.504244)\ (1.000)$$

$$= \$530\ (15.666570) + \$660\ (6.504244) = \$8{,}303.28 + \$4{,}292.80 = \$\underline{\underline{12{,}596.08}}$$

if year-end deposits are made or, if contributions occur at the start of each year:

$$\$530\left(\frac{(1.0322)^{11}-1}{.0322}\right)(1.0322)(1.0322)^{6} + \$660\left(\frac{(1.0322)^{6}-1}{.0322}\right)(1.0322)(1.0322)^{0}$$

$$= \$530\ (12.953609)\ (1.0322)\ (1.209437) + \$660\ (6.504244)\ (1.0322)\ (1.000)$$

$$= \$530\ (16.171033) + \$660\ (6.713681) = \$8{,}570.65 + \$4{,}431.03 = \$\underline{\underline{13{,}001.68}}$$

(This problem combines the FV of a "truncated" annuity with the more typical FV of annuity case.)

6-C. Billie is retiring today with $1,265,000. She expects to survive for 31 years, and to earn a 4.26% average annual return on any money in her accounts. If her hope is to withdraw $53,000 to live on at the end of each of the 31 years and then leave $1,250,000 to Redbird Charities in her will, does she have enough money saved? What if instead she more realistically expects to withdraw the $53,000 to pay living expenses at the beginning of each year?

The easiest way to solve in this situation, in which the retiree has two different financial goals within the same plan, is to work two separate problems and then add the results. The amount needed today to fund a single withdrawal of $1,500,000 in 31 years if a 4.26% average annual rate of return can be earned is

$$\text{BAMT}\ (1 + r)^{n} = \text{EAMT}$$

$$\text{BAMT}\ (1.0426)^{31} = \$1{,}250{,}000$$

$$\$1{,}250{,}000 \div (1.0426)^{31} = \text{BAMT} \quad \text{or}$$

$$\$1{,}250{,}000\left(\frac{1}{1.0426}\right)^{31} = \$1{,}250{,}000\ (.274379) = \text{BAMT} = \$\underline{342{,}973.59}$$

The amount needed today to fund the 31-year series of $53,000 annual withdrawals is

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$53{,}000\left(\frac{1-\left(\frac{1}{1.0426}\right)^{31}}{.0426}\right) = \$53{,}000\ (17.033360) = \$\underline{902{,}768.07}$$

if the $53,000 is to be taken at the end of each year and a larger

$$\$53{,}000\left[\left(\frac{1-\left(\frac{1}{1.0426}\right)^{31}}{.0426}\right)(1.0426)\right] = \$53{,}000\ (17.758981) = \$\underline{941{,}225.99}$$

if the \$53,000 is to come out at the start of each year. So the two amounts she needs to have on hand today – which could be in two separate accounts or all in one account – must total \$342,973.59 + \$902,768.07 = \$1,245,741.66 if she plans to take out \$53,000 at the end of each year or a higher \$342,973.59 + \$941,225.99 = \$1,284,199.58 if the \$53,000 is to be taken out at the beginning of each year. Thus she has enough money on hand today to fund the plan if the \$53,000 annual withdrawals are to be end-of-year but not if they are to be taken at the beginning of each year. (This problem combines non-annuity and present value of an annuity applications.)

7-C. Tim has some very old carpentry tools that he got from his late grandfather. An antiques appraiser feels that they are worth \$9,400 today. If collectible items of this type have been increasing in value by an average of 4.65% per year, what would we estimate the tools to have been worth when Tim received them seven years ago?

$$\text{BAMT}\,(1 + r)^n = \text{EAMT}$$
$$\text{BAMT}\,(1.0465)^7 = \$9{,}400$$
$$\$9{,}400 \div (1.0465)^7 = \$9{,}400 \div (1.374595) = \text{BAMT} \quad \text{or}$$
$$\$9{,}400 \left(\frac{1}{1.0465}\right)^7 = \$9{,}400\,(.727487) = \text{BAMT} = \$\underline{\underline{6{,}838.38}}$$

(This is a non-annuity problem with the beginning amount unknown; no series of regular payments.)

8-C. Marilyn is retiring today with \$804,000 saved. What average annual rate of return will she have to earn on any balance remaining in her account from year to year if she wants to take out \$73,000 at the end of each year in the 32 years she expects to remain living? What if instead she wants to take the \$73,000 out at the start of each year?

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$73{,}000 \left(\frac{1-\left(\frac{1}{1+r}\right)^{32}}{r}\right) = \$804{,}000$$

With both r (which is r^1) and r^{32} in the same equation we can not solve for r directly; solving requires trial and error. (You would not be asked to solve with trial and error on an exam. But being able to set up the equation to solve for r in an annuity problem as part of your homework practice helps to assure that there are no gaps in your understanding.) If we try a series of different rates, the correct average annual rate for end-of-year \$73,000 withdrawals turns out to be 8.390372%; double-check:

$$\$73{,}000 \left(\frac{1-\left(\frac{1}{1.08390372}\right)^{27}}{.08390372}\right) = \$73{,}000\,(11.013699) = \$804{,}000 \;\checkmark$$

We also need trial and error to find r if there are start-of-year \$73,000 withdrawals, of course:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$73{,}000 \left[\left(\frac{1-\left(\frac{1}{1+r}\right)^{32}}{r}\right)(1+r)\right] = \$804{,}000$$

A higher average 9.363503% return is needed than for year-end withdrawals; double-check:

$$\$73{,}000\left[\left(\frac{1-\left(\frac{1}{1.09363503}\right)^{32}}{.09363503}\right)(1.09363503)\right] = \$73{,}000\ (11.013699) = \$804{,}000\ ✓$$

(This is a present value of an annuity problem with an unknown average periodic rate of return.)

9-C. All-Season Sports is planning to borrow $214,000 to replace sold inventory. The firm's managers want to repay the loan over 15 years, and the interest rate the bank charges can be represented as a 9.36% annual percentage rate (APR). What must each equal regular payment be if payments are to me made at the end of each year? At the beginning of each year? What if instead payments are to be made semi-annually, quarterly, or monthly, and interest accrues on the loan with the same frequency as the payments occur? With the 9.36% APR, what effective annual rate (EAR) of cost would the borrower incur under each of the four payment and compounding frequencies?

If one payment is made per year (annual) then the number of periods is 15 x 1 = 15 and the periodic rate is the .0936 APR divided by 1, which is .0936 ÷ 1 = .0936, and the needed annual payment is:

PMT x FAC = TOT

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.0936}\right)^{15}}{.0936}\right) = \$214{,}000$$

PMT x 7.892203 = $214,000

$214,000 ÷ 7.892203 = PMT = $<u>27,115.37</u> if payments occur at the end of each year and a lower

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1}{1.0936}\right)^{15}}{.0936}\right)(1.0936)\right] = \$214{,}000$$

PMT x 8.630913 = $214,000

$214,000 ÷ 8.630913 = PMT = $<u>24,794.60</u> if beginning-of-year payments are made.

With two payments yearly (semi-annual) the number of periods is 15 x 2 = 30 and the periodic rate is the .0936 APR divided by 2, which is .0936 ÷ 2 = .0468, and the needed semi-annual payment is:

PMT x FAC = TOT

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.0468}\right)^{30}}{.0468}\right) = \$214{,}000$$

PMT x 15.949475 = $214,000

$214,000 ÷ 15.949475 = PMT = $<u>13,417.37</u> if payments are at the end of each half-year and a lower

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1}{1.0468}\right)^{30}}{.0468}\right)(1.0468)\right] = \$214{,}000$$

PMT x 16.695910 = $214,000

$214,000 ÷ 16.695910 = PMT = $<u>12,817.51</u> if beginning-of-half-year payments are made.

With four payments each year (quarterly) the number of periods is 15 x 4 = 60 and the periodic rate is the .0936 APR divided by 4, which is .0936 ÷ 4 = .0234, and the quarterly payment should be:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.0234}\right)^{60}}{.0234}\right) = \$214{,}000$$

$$\text{PMT} \times 32.067623 = \$214{,}000$$

\$214,000 ÷ 32.067623 = PMT = \$<u>6,673.40</u> if payments are at the end of each quarter and a lower

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1}{1.0234}\right)^{60}}{.0234}\right)(1.0234)\right] = \$214{,}000$$

$$\text{PMT} \times 32.818005 = \$214{,}000$$

\$214,000 ÷ 32.818005 = PMT = \$<u>6,520.81</u> if beginning-of-quarter payments are scheduled.

And with twelve yearly payments (monthly) the number of periods is 15 x 12 = 180 and the periodic rate is the .0936 APR divided by 12 or .0936 ÷ 12 = .0078, with a required monthly payment of:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.0078}\right)^{180}}{.0078}\right) = \$214{,}000$$

$$\text{PMT} \times 96.544344 = \$214{,}000$$

\$214,000 ÷ 96.544344 = PMT = \$<u>2,216.60</u> if payments happen at the end of each month and a lower

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1}{1.0078}\right)^{180}}{.0078}\right)(1.0078)\right] = \$214{,}000$$

$$\text{PMT} \times 97.297389 = \$214{,}000$$

\$214,000 ÷ 97.297389 = PMT = \$<u>2,199.44</u> if they are made at the beginning of each month.

A borrower paying a .0936 annual periodic r is paying an APR of .0936 x 1 = .0936 and an EAR of $(1.0936)^1$ – 1 = .0936; when cash flows and compounding occur annually the APR and EAR are the same. A borrower paying a .0468 semi-annual periodic r pays an APR of .0468 x 2 = .0936 and an EAR of $(1.0468)^2$ – 1 = .09579. Someone paying a .0234 quarterly periodic r is paying an APR of .0234 x 4 = .0936 and an EAR of $(1.0234)^4$ – 1 = .096937. And paying a .0078 monthly periodic r corresponds to an APR of .0078 x 12 = .0936 and an EAR of $(1.0078)^{12}$ – 1 = .097722. Here the APR is the same 9.36% regardless of the compounding or discounting frequency, but when cash flows and compounding occur more often than annually the compounded EAR exceeds the simple APR, and the EAR exceeds the APR by a wider margin if there are more compounding periods within the year.

Note also that in this story the annual rate discussed could have been the EAR instead of the APR. If told that the EAR is 9.579% and interest is compounded semi-annually we would have computed the semiannual periodic r to compute with as $\sqrt[2]{1.09579}$ – 1 = .0468. If told that the EAR is 9.6937% and interest is compounded quarterly we would have computed the quarterly periodic r as $\sqrt[4]{1.096937}$ – 1 = .0234. And if the annual rate were discussed as a 9.7722% EAR with compounding occurring monthly we would have computed the monthly periodic r to compute with as $\sqrt[12]{1.097722}$ – 1 = .0078.

(This is a PV of annuity problem with the payment unknown, and with both annual and non-annual cash flows & discounting.)

10-C. Nancy just graduated from ISU at age 23 with a BS/MPA degree in accounting and is starting work as an auditor with Jesse, Fell & Associates. She expects to work until age 66, and plans to save $13,800 every year in the accounting firm's 401-K plan. What average annual rate of return will she have to earn on her account's balance if she is to retire in 43 years with $1,850,000 saved? What if instead she deposits the $13,800 at the beginning of each year?

PMT x FAC = TOT

$$\$13{,}800\left(\frac{(1+r)^{43}-1}{r}\right) = \$1{,}850{,}000$$

With both r (which is r^1) and r^{43} in just one equation we are unable to solve directly. Using trial and error, we ultimately find the answer to be 4.757167%; double-check to make sure it is correct:

$$\$13{,}800\left(\frac{(1.04757167)^{43}-1}{.04757167}\right) = \$13{,}800\ (134.057971) = \$1{,}850{,}000\ \checkmark$$

(You would not be asked to solve with trial and error on an exam. But being able to set up the equation to solve for r in an annuity problem as part of your homework practice helps to assure that there are no gaps in your understanding.) For beginning-of-year deposits:

PMT x FAC = TOT

$$\$13{,}800\left[\left(\frac{(1+r)^{43}-1}{r}\right)(1+r)\right] = \$1{,}850{,}000$$

Using trial and error, we would find r to be a slightly lower 4.587181% (with earlier deposits she could earn a lower average annual return and still reach the desired total); a double-check shows:

$$\$13{,}800\left[\left(\frac{(1.04587181)^{19}-1}{.04587181}\right)(1.04587181)\right] = \$13{,}800\ (134.057971) = \$1{,}850{,}000\ \checkmark$$

(This is a future value of an annuity problem with an unknown average periodic rate of return.)

11-C. Clearing Publishers House, headquartered near Midway Airport, sponsors the $10 Million Sweepstakes, whose winner will receive $500,000 per year for 20 years. How much money must Clearing's managers have on deposit today to fund the promised stream of payments if they expect to earn a 4.17% average annual rate of return on any money they have invested and the winner is to get the $500,000 at the end of each year? What if instead the winner is to collect at the beginning of each year?

PMT x FAC = TOT

$$\$500{,}000\left(\frac{1-\left(\frac{1}{1.0417}\right)^{20}}{.0417}\right) = \text{TOT}$$

$500,000 x 13.388018 = $<u>6,694,009.03</u> if the winner collects at the end of each year and a bigger

$$\$500{,}000\left[\left(\frac{1-\left(\frac{1}{1.0417}\right)^{20}}{.0417}\right)(1.0417)\right] = \text{TOT}$$

$500,000 x 13.946298 = $<u>6,973,149.20</u> if the winner is to be paid at the beginning of each year.

(This is a present value of an annuity problem with an unknown large total.)

12-C. French wine dealer John just paid €2,322 (2,322 euros) for a bottle of a rare Chenin blanc at an auction in Angers. If it increases in value, as expected, by an average rate of 11.4% annually, how many years will it take for the bottle to be worth €10,000? To be worth €20,000?

$$\text{BAMT}\,(1 + r)^n = \text{EAMT}$$
$$€2{,}322\,(1.114)^n = €10{,}000$$
$$(1.114)^n = 4.306632$$
$$\ln[(1.114)^n] = \ln 4.306632$$
$$n \times \ln 1.114 = \ln 4.306632$$
$$n\,(.107957) = 1.460156$$

$n = \underline{\underline{13.525332}}$, or about 13 ½ years to reach a €10,000 value   or

$$€2{,}322\,(1.114)^n = €20{,}000$$
$$(1.114)^n = 8.613264$$
$$\ln[(1.114)^n] = \ln 8.613264$$
$$n \times \ln 1.114 = \ln 8.613264$$
$$n\,(.107957) = 2.153303$$

$n = \underline{\underline{19.945910}}$, or just under 20 years to reach a €20,000 value

(This is a non-annuity problem with the number of periods unknown; no series of regular payments.)

13-C. a. Richard wants Mid-Illinois Food Bank to have $130,000 available at the end of each year so it can continue operations indefinitely into the future. If the food bank's trustees can earn a 4.34% average annual rate of return on money under their care, how much should Richard give to the organization today? What if instead the $130,000 yearly budget should be provided at the beginning of each year forever? If he is able to give only $2,000,000 how much can the food bank budget to spend at the end *vs*. beginning of each year forever?

The endowment needs to be

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$130{,}000\left(\frac{1-\left(\frac{1}{1.0434}\right)^{\infty}}{.0434}\right) = \$130{,}000\left(\frac{1-0}{.0434}\right) = \$130\text{ K}\left(\frac{1}{.0434}\right) \text{ or } \frac{\$130{,}000}{.0434} = \text{TOT} = \$\underline{\underline{2{,}995{,}391.71}}$$

if the $130,000 is to be accessed at the end of each year (a level ordinary perpetuity) and

$$\$130{,}000\left[\left(\frac{1}{.0434}\right)(1.0434)\right] = \text{TOT} = \$\underline{\underline{3{,}125{,}391.71}}$$

if the money is to be available at the beginning of each year (a level perpetuity due). The amount that can be spent helping the food bank every year forever with a smaller $2 million endowment is

$$\text{PMT}\left(\frac{1}{.0434}\right) \quad \text{or} \quad \frac{\text{PMT}}{.0434} = \$2{,}000{,}000$$

$$\$2{,}000{,}000 \times .0434 = \text{PMT} = \$\underline{\underline{86{,}800}}$$

if the money is to be made available at the end of each year forever or an even smaller

$$\text{PMT}\left[\left(\frac{1}{.0434}\right)(1.0434)\right] = \$2{,}000{,}000$$
$$(\$2{,}000{,}000 \div 1.0434) \times .0434 = \$1{,}916{,}810.43 \times .0434 = \text{PMT} = \$\underline{\underline{83{,}189.57}}$$

if the money is to be available at the beginning of each year forever. (This is a present value of a level perpetuity problem.)

b. Now assume that the food bank trustees can invest donations in a plan that provides investment returns more frequently than once per year. How much must Richard contribute today to generate $65,000 in receipts every six months ($130,000 annual total) if earnings compound semiannually and the rate of return is represented as a 5.4% annual percentage rate (APR)? What if the plan is to provide $32,500 every three months (again $130,000 annually) if earnings compound quarterly and the rate of return is represented as a 4.88% APR?

With semiannual payments and compounding we must convert the annual rate we talk about in the talking phase to a semiannual r we work with in the computing phase. Recall that APR is the simple or "convenient" presentation of an annual rate; it is just the related semiannual r multiplied by 2, such that if we know the APR we find the corresponding semiannual r by dividing it by 2: .054 ÷ 2 = .027. (We could have been told instead in the talking phase that the more complicated, compounded effective annual rate of return, or EAR, is 5.4729%, and then would have found the semiannual r as $\sqrt[2]{1.054729}$ – 1 = .027.) So the endowment amount to produce $65,000 semiannually should be

$$\$65{,}000\left(\frac{1-\left(\frac{1}{1.027}\right)^{\infty}}{.027}\right) = \$65{,}000\left(\frac{1-0}{.027}\right) = \$65{,}000\left(\frac{1}{.027}\right) \text{ or } \frac{\$65{,}000}{.027} = \text{TOT} = \$\underline{\underline{2{,}407{,}407.41}}$$

if $65,000 is to be available at the end of each half-year (level ordinary perpetuity) and a larger

$$\$65{,}000\left[\left(\frac{1}{.027}\right)(1.027)\right] = \text{TOT} = \$\underline{\underline{2{,}472{,}407.41}}$$

if the money is to be received at the beginning of each six-month period (level perpetuity due). If the APR is 4.88% and payments and compounding occur quarterly, the quarterly r is .0488 ÷ 4 = .0122 (or if instead told the EAR is 4.97% we would find the quarterly r as $\sqrt[4]{1.0497}$ – 1 = .0122). Under these conditions the donor should give

$$\$32{,}500\left(\frac{1-\left(\frac{1}{1.0122}\right)^{\infty}}{.0122}\right) = \$32{,}500\left(\frac{1-0}{.0122}\right) = \$32{,}500\left(\frac{1}{.0122}\right) \text{ or } \frac{\$32{,}500}{.0122} = \text{TOT} = \$\underline{\underline{2{,}663{,}934.43}}$$

if the $32,500 is to be collected at the end of each quarter (level ordinary perpetuity) and a larger

$$\$32{,}500\left[\left(\frac{1}{.0122}\right)(1.0122)\right] = \text{TOT} = \$\underline{\underline{2{,}696{,}434.43}}$$

if the money is to be available at the beginning of each quarter (level perpetuity due).

14-C. James wants to save $172,000 over the next 11 years to buy a lakeside cabin. He expects to earn a rate of return on his savings plan that can be represented as a 4.86% annual percentage rate (APR). How big must his regular equal deposits be if he wants to reach the $172,000 goal by putting money in at the end of each year? At the beginning of each year? What if instead he makes deposits semi-annually, quarterly, or monthly, and interest compounds on the account with the same frequency as the deposits? With the 4.86% APR, what effective annual rate (EAR) of return would he earn under each of the four deposit and compounding frequencies?

If one deposit is made per year (annual) then the number of periods is 11 x 1 = 11 and the periodic rate is the .0486 APR divided by 1, which is .0486 ÷ 1 = .0486, and the needed annual deposit is:

PMT x FAC = TOT

$$\text{PMT}\left(\frac{(1.0486)^{11}-1}{.0486}\right) = \$172{,}000$$

PMT x 14.103312 = $172,000

$172,000 ÷ 14.103312 = PMT = $<u>12,195.72</u> if deposits are made at the end of each year and a lower

$$\text{PMT}\left[\left(\frac{(1.0486)^{11}-1}{.0486}\right)(1.0486)\right] = \$172{,}000$$

PMT x 14.788733 = $172,000

$172,000 ÷ 14.788733 = PMT = $<u>11,630.48</u> if beginning-of-year deposits are made.

With two deposits per year (semi-annual) the number of periods is 11 x 2 = 22 and the periodic rate is the .0486 APR divided by 2, which is .0486 ÷ 2 = .0243, and the needed semi-annual deposit is:

PMT x FAC = TOT

$$\text{PMT}\left(\frac{(1.0243)^{22}-1}{.0243}\right) = \$172{,}000$$

PMT x 28.637468 = $172,000

$172,000 ÷ 28.637468 = PMT = $<u>6,006.12</u> if deposits are at the end of each half-year and a lower

$$\text{PMT}\left[\left(\frac{(1.0243)^{22}-1}{.0243}\right)(1.0243)\right] = \$172{,}000$$

PMT x 29.33358 = $172,000

$172,000 ÷ 29.333358 = PMT = $<u>5,863.63</u> if beginning-of-half-year deposits are made.

With four deposits made per year (quarterly) the number of periods is 11 x 4 = 44 and the periodic rate is the .0486 APR divided by 4, which is .0486 ÷ 4 = .01215, and the needed quarterly deposit is:

PMT x FAC = TOT

$$\text{PMT}\left(\frac{(1.01215)^{44}-1}{.01215}\right) = \$172{,}000$$

PMT x 57.718164 = $172,000

$172,000 ÷ 57.718164 = PMT = $<u>2,980.00</u> if deposits are at the end of each quarter and a lower

$$\text{PMT}\left[\left(\frac{(1.01215)^{44}-1}{.01215}\right)(1.01215)\right] = \$172{,}000$$

PMT x 58.419439 = $172,000

$172,000 ÷ 58.419439 = PMT = $<u>2,944.23</u> if beginning-of-quarter deposits are made.

And with twelve deposits per year (monthly) the number of periods is 11 x 12 = 132 and the periodic rate is the .0486 APR divided by 12 or .0486 ÷ 12 = .00405, with a needed monthly deposit of:

PMT x FAC = TOT

$$\text{PMT}\left(\frac{(1.00405)^{132}-1}{.00405}\right) = \$172{,}000$$

PMT x 174.055230 = $172,000

$172,000 ÷ 174.055230 = PMT = $<u>988.19</u> if deposits are made at the end of each month and a lower

$$\text{PMT}\left[\left(\frac{(1.00405)^{132}-1}{.00405}\right)(1.00405)\right] = \$172{,}000$$

$$\text{PMT} \times 174.760154 = \$172{,}000$$

$172,000 ÷ 174.760154 = PMT = $984.21 if deposits take place at the beginning of each month.

Note that a saver earning a .0486 annual periodic r is earning an APR of .0486 x 1 = .0486 and an EAR of $(1.0486)^1 - 1 = .0486$; when cash flows and compounding occur annually the simple APR and compounded EAR are the same. A saver earning a .0243 semi-annual periodic r is earning an APR of .0243 x 2 = .0486 and an EAR of $(1.0243)^2 - 1 = .04919$. Someone earning a .01215 quarterly periodic r is earning an APR of .01215 x 4 = .0486 and an EAR of $(1.01215)^4 - 1 = .049493$. And earning a .00405 monthly periodic r corresponds to an APR of .00405 x 12 = .0486 and an EAR of $(1.00405)^{12} - 1 = .049697$. Here the saver's measured APR is 4.86% regardless of the compounding frequency, but when cash flows and compounding occur more often than annually the EAR exceeds the APR, and the EAR exceeds the APR by a wider margin as the number of compounding periods within the year increases.

Note also that in this story the annual rate discussed could have been the EAR instead of the APR. If told that the EAR is 4.919% and interest is compounded semi-annually we would have computed the semiannual periodic r to compute with as $\sqrt[2]{1.04919} - 1 = .0243$. If told that the EAR is 4.9493% and interest is compounded quarterly we would have computed the quarterly periodic r as $\sqrt[4]{1.049493} - 1 = .01215$. And if the annual rate were discussed as a 4.9697% EAR with compounding occurring monthly we would have found the monthly periodic r to compute with as $\sqrt[12]{1.049697} - 1 = .00405$.

(This is an FV of annuity problem with payments unknown, and with both annual and non-annual cash flows and compounding.)

15-C. Dorothy hopes to buy her neighbor's motor home when the neighbor retires to Florida in nine years. If Dorothy can save $2,700 every year in a savings plan that earns a 3.6% average compounded annual rate of return, how much will she be able to offer for the vehicle? What if instead she makes the $2,700 deposits at the beginning of each year?

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$2{,}700 \left(\frac{(1.036)^9 - 1}{.036}\right) = \text{TOT}$$

$2,700 x 10.410959 = $28,109.59 with end-of-year deposits (level ordinary annuity) and a larger

$$\$2{,}700 \left[\left(\frac{(1.036)^9 - 1}{.036}\right)(1.036)\right] = \text{TOT}$$

$2,700 x 10.785754 = $29,121.54 with beginning-of-year deposits (level annuity due).

(This is a future value of an annuity problem with an unknown large total.)

16-C. Robert just won $54,000 (net of income taxes) playing the Illinois State Lottery. With those winnings he creates an account to pay for annual year-end trips to Monte Carlo that are expected to cost $6,250 each. If he can earn a 3.14% average annual rate of return on any money in his account, for how many years will he be able to fund his trips? What if instead he withdraws the $6,250 to pay for his Monte Carlo visits at the beginning of each year?

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$6{,}250 \left(\frac{1 - \left(\frac{1}{1.0314}\right)^n}{.0314}\right) = \$54{,}000$$

$$\frac{1 - \left(\frac{1}{1.0314}\right)^n}{.0314} = 8.64000$$

$$1-\left(\frac{1}{1.0314}\right)^{n} = .271296$$

$$\left(\frac{1}{1.0314}\right)^{n} = .728704 \quad \text{so} \quad (.969556)^{n} = .728704$$

$$\ln[(.969556)^{n}] = \ln .728704$$

$$n \times \ln .969556 = \ln .728704$$

$$n\,(-.030917) = -.316488$$

n = <u>10.236653</u> years of trips paid for with end-of-year \$6,250 cash flows and

$$\$6{,}250\left[\left(\frac{1-\left(\frac{1}{1.0314}\right)^{n}}{.0314}\right)(1.0314)\right] = \$54{,}000$$

$$\left(\frac{1-\left(\frac{1}{1.0314}\right)^{n}}{.0314}\right)(1.0314) = 8.64000$$

$$\frac{1-\left(\frac{1}{1.0314}\right)^{n}}{.0314} = 8.376963$$

$$1-\left(\frac{1}{1.0314}\right)^{n} = .263037$$

$$\left(\frac{1}{1.0314}\right)^{n} = .736963 \quad \text{so} \quad (.969556)^{n} = .736963$$

$$\ln[(.969556)^{n}] = \ln .736963$$

$$n \times \ln .969556 = \ln .736963$$

$$n\,(-.030917) = -.305217$$

n = <u>9.872113</u> years of \$6,250 trips paid for if taken at the beginning of each year

(This is a present value of an annuity problem with an unknown number of time periods.)

17-C. Susan wants to have \$515,000 when she retires in 12 years. The average rate of return she expects to earn on any money she holds can be represented as a 5.36% annual percentage rate (APR), with compounding to take place quarterly. If she already has \$29,000, how much must she save at the end of each quarter for the next 12 years to reach her goal? What if instead she makes savings deposits at the beginning of each quarter? What effective annual rate (EAR) of return would she be earning?

The easiest way to solve in this situation, in which the saver has a two-part approach to reaching a financial goal, is to combine the results of two separate problems. First we recognize that while the world tends to talk about transactions in annual terms, our computational work has to reflect the timing of any payments and compounding. Here we are in a world of quarterly payments and compounding, so the number of periods is 12 x 4 = 48 quarters and the quarterly periodic rate is 0536 ÷ 4 = .0134. The amount to which the currently held \$29,000 (we can think of it as being in account 1) will grow by the end of year 12 (quarter 48) if a .0536 ÷ 4 = .0134 quarterly rate of return can be earned is

$$\text{BAMT}\,(1+r)^{n} = \text{EAMT}$$

$$\$29{,}000\,(1.0134)^{48} = \$29{,}000\,(1.894450) = \$\underline{54{,}939.06}$$

Since \$54,939.06 of quarter 48's desired \$515,000 total will be accounted for in account 1, the remaining \$515,000 – \$54,939.06 = \$<u>460,060.94</u> is the amount that the saver must accumulate in account 2 with deposits made over the next 48 quarters. The equal deposit she must make each quarter to reach that goal is

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left(\frac{(1.0134)^{48}-1}{.0134}\right) = \$460{,}060.94$$

$$\$460{,}060.94 \div 66.750016 = \text{PMT} = \$\underline{\underline{6{,}892.30}}$$

if she puts the money into account 2 at the end of each quarter and a smaller

$$\text{PMT}\left[\left(\frac{(1.0134)^{48}-1}{.0134}\right)(1.0134)\right] = \$460{,}060.94$$

$$\$460{,}060.94 \div 67.644467 = \text{PMT} = \$\underline{\underline{6{,}801.16}}$$

if the account 2 deposits are to go in at the beginning of each quarter. (It actually all could be handled within the same account; she could start by adding $6,801.16 at the beginning of quarter 1 to an existing account's $29,000 balance.) A .0134 quarterly rate of return compounds out to an EAR of $(1.0134)^4 - 1 = 5.4687\%$, which is higher of course than the simple .0134 x 4 = 5.36% APR. If the annual rate talked about instead were this 5.4687% EAR we would find the quarterly r to compute with as $\sqrt[4]{1.054687} - 1 = .0134$. (This problem combines non-annuity and future value of an annuity applications with non-annual payments and compounding periods.)

18-C. Redbird fan Allison plans to donate $243,000 to provide for a scholarship to be given yearly to Illinois State University's top ornithology student. She wants the first scholarship to be awarded in year 11, and for the program then to continue indefinitely. The ISU Foundation expects to earn a 4.12% average annual rate of return on any money in dedicated accounts of this type. How big an annual scholarship can be given if the money is paid at the end of each year? What if instead scholarships are to be awarded at the beginning of each year? How much would Allison have to donate today to fund an $18,000 annual scholarship starting in year 11 and continuing indefinitely?

The dollar amount of each year's scholarship can be

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1}{1.0412}\right)^{\infty}}{.0412}\right)\left(\frac{1}{1.0412}\right)^{10}\right] = \text{PMT}\left[\left(\frac{1}{.0412}\right)\left(\frac{1}{1.0412}\right)^{10}\right] = \$243{,}000$$

$$\text{PMT}\,(16.209185) = \$243{,}000$$

$$\$243{,}000 \div 16.209185 = \text{PMT} = \$\underline{\underline{14{,}991.50}}$$

if the money is to be awarded at the end of each of years 11 to infinity (the exponent is 10 because 10 years pass before the payment stream begins in year 11), or a little bit lower

$$\text{PMT}\left[\left(\frac{1}{.0412}\right)(1.0412)\left(\frac{1}{1.0412}\right)^{10}\right] = \$243{,}000$$

$$\text{PMT}\,(16.877004) = \$243{,}000$$

$$\$243{,}000 \div 16.877004 = \text{PMT} = \$\underline{\underline{14{,}398.29}}$$

in the perhaps more realistic case of scholarships being awarded at the beginning of each year. To fund somewhat larger $18,000 annual scholarships the endowment provided today would have to be

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$18{,}000\left[\left(\frac{1-\left(\frac{1}{1.0412}\right)^{\infty}}{.0412}\right)\left(\frac{1}{1.0412}\right)^{10}\right] = \$18{,}000\left[\left(\frac{1}{.0412}\right)\left(\frac{1}{1.0412}\right)^{10}\right] = \text{TOT}$$

$$\$18{,}000\,(16.209185) = \$\underline{\underline{291{,}765.34}}$$

if the money is to be paid out at the end of each of years 11 to infinity, or a higher

$$\$18{,}000\left[\left(\frac{1}{.0412}\right)(1.0412)\left(\frac{1}{1.0412}\right)^{10}\right] = \text{TOT}$$

$$\$18{,}000\ (16.877004) = \$\underline{\underline{303{,}786.07}}$$

if students get their scholarships at the beginning of each year. (This is a present value of a deferred perpetuity problem, with the regular periodic payment as the unknown being solved for.)

19-C. How much money can a bank justify lending Alex today if it charges a 7.3% annual interest rate and Alex agrees to make payments of $3,100 at the end of year 1; $4,200 at the end of year 2; $5,100 at the end of year 3; and $4,800 at the end of year 4? What if instead he agrees to pay the bank $4,600 at the end of each year? What if instead he is willing to make payments of $4,600 at the start of each of the four years?

The amount of loan the bank can extend is

$$\$3{,}100\left(\frac{1}{1.073}\right)^1 + \$4{,}200\left(\frac{1}{1.073}\right)^2 + \$5{,}100\left(\frac{1}{1.073}\right)^3 + \$4{,}800\left(\frac{1}{1.073}\right)^4$$

$$= \$3{,}100\ (.931996) + \$4{,}200\ (.868561) + \$5{,}100\ (.809470) + \$4{,}800\ (.754399)$$

$$= \$2{,}889.10 + \$3{,}647.96 + \$4{,}128.30 + \$3{,}621.12 = \$\underline{\underline{14{,}286.47}}$$

if the described unequal end-of-year payments are received, while it can lend

$$\$4{,}600\left(\frac{1}{1.073}\right)^1 + \$4{,}600\left(\frac{1}{1.073}\right)^2 + \$4{,}600\left(\frac{1}{1.073}\right)^3 + \$4{,}600\left(\frac{1}{1.073}\right)^4$$

$$= \$4{,}600\ (.931996) + \$4{,}600\ (.868561) + \$4{,}600\ (.809470) + \$4{,}600\ (.754399)$$

$$= \$4{,}287.05 + \$3{,}995.38 + \$3{,}723.56 + \$3{,}470.24 = \$\underline{\underline{15{,}476.23}}$$

(or note that, by the distributive property)

$$\$4{,}600\left(\frac{1}{1.073}\right)^1 + \$4{,}600\left(\frac{1}{1.073}\right)^2 + \$4{,}600\left(\frac{1}{1.073}\right)^3 + \$4{,}600\left(\frac{1}{1.073}\right)^4$$

$$= \$4{,}600\left[\left(\frac{1}{1.073}\right)^1 + \left(\frac{1}{1.073}\right)^2 + \left(\frac{1}{1.073}\right)^3 + \left(\frac{1}{1.073}\right)^4\right]$$

$$= \$4{,}600\left(\frac{1-\left(\frac{1}{1.073}\right)^4}{.073}\right) = \$4{,}600\ (3.364397) = \$\underline{\underline{15{,}476.23}}$$

if four $4,600 equal end-of-year payments are to be received. It can lend

$$\$4{,}600\left(\frac{1}{1.073}\right)^0 + \$4{,}600\left(\frac{1}{1.073}\right)^1 + \$4{,}600\left(\frac{1}{1.073}\right)^2 + \$4{,}600\left(\frac{1}{1.073}\right)^3$$

$$= \$4{,}600\ (1.000) + \$4{,}600\ (.931996) + \$4{,}600\ (.868561) + \$4{,}600\ (.809470)$$

$$= \$4{,}600 + \$4{,}287.05 + \$3{,}995.38 + \$3{,}723.56 = \$\underline{\underline{16{,}605.99}}$$

(or, by the distributive property)

$$\$4{,}600\left(\frac{1}{1.073}\right)^0 + \$4{,}600\left(\frac{1}{1.073}\right)^1 + \$4{,}600\left(\frac{1}{1.073}\right)^2 + \$4{,}600\left(\frac{1}{1.073}\right)^3$$

$$= \$4{,}600\left[\left(\frac{1}{1.073}\right)^0 + \left(\frac{1}{1.073}\right)^1 + \left(\frac{1}{1.073}\right)^2 + \left(\frac{1}{1.073}\right)^3\right]$$

$$= \$4,600 \left(\frac{1-\left(\frac{1}{1.073}\right)^4}{.073}\right)(1.073) = \$4,600\ (3.609998) = \$\underline{\underline{16,605.99}}$$

if four $4,600 equal beginning-of-year payments are to be received. (This problem involves the present value of a series of payments.)

20-C. The Income Boost Annuity plan offered by Redbird Investments provides the investor with payments of $12,365 at the end of each of years 1 through 15 and then $14,365 at the end of each of years 16 through 24. If Emily believes that the risk of being in this plan calls for an 8.17% average annual rate of return, what should she be willing to pay for this contract? What if instead Redbird pays the plan holder at the beginning of each year?

A brute-force approach to solving for the case of year-end payments would be

$$\$12,365 \left[\left(\frac{1}{1.0817}\right)^1 + \left(\frac{1}{1.0817}\right)^2 + \left(\frac{1}{1.0817}\right)^3 + \cdots + \left(\frac{1}{1.0817}\right)^{14} + \left(\frac{1}{1.0817}\right)^{15}\right]$$

$$+ \$14,365 \left[\left(\frac{1}{1.0817}\right)^{16} + \left(\frac{1}{1.0817}\right)^{17} + \left(\frac{1}{1.0817}\right)^{18} + \cdots + \left(\frac{1}{1.0817}\right)^{23} + \left(\frac{1}{1.0817}\right)^{24}\right].$$

The more efficient "compute a factor" approach gives

$$\$12,365 \left(\frac{1-\left(\frac{1}{1.0817}\right)^{15}}{.0817}\right) + \$14,365 \left[\left(\frac{1-\left(\frac{1}{1.0817}\right)^{24}}{.0817}\right) - \left(\frac{1-\left(\frac{1}{1.0817}\right)^{15}}{.0817}\right)\right] = \text{TOT}$$

$$\$12,365\ (8.471342) + \$14,365\ (10.381182 - 8.471342)$$

$$= \$104,748.14 + \$27,434.86 = \$\underline{\underline{132,182.99}}$$

while the arguably even more efficient "discount the annuity" approach shows

$$\$12,365 \left(\frac{1-\left(\frac{1}{1.0817}\right)^{15}}{.0817}\right) + \$14,365 \left[\left(\frac{1-\left(\frac{1}{1.0817}\right)^{9}}{.0817}\right)\left(\frac{1}{1.0817}\right)^{15}\right] = \text{TOT}$$

$$\$12,365\ (8.471342) + \$14,365\ (1.909840)$$

$$= \$104,748.14 + \$27,434.86 = \$\underline{\underline{132,182.99}}$$

If payments are to be received at the start of each year the "compute a factor" approach shows

$$\$12,365 \left(\frac{1-\left(\frac{1}{1.0817}\right)^{15}}{.0817}\right)(1.0817) + \$14,365 \left[\left(\frac{1-\left(\frac{1}{1.0817}\right)^{24}}{.0817}\right) - \left(\frac{1-\left(\frac{1}{1.0817}\right)^{15}}{.0817}\right)\right](1.0817) = \text{TOT}$$

$$\$12,365\ (9.163450) + \$14,365\ (10.381182 - 8.471342)\ (1.0817)$$

$$= \$113,306.06 + \$29,676.28 = \$\underline{\underline{142,982.34}}$$

while the "discount the annuity" result is the same

$$\$12,365 \left(\frac{1-\left(\frac{1}{1.0817}\right)^{15}}{.0817}\right) + \$14,365 \left[\left(\frac{1-\left(\frac{1}{1.0817}\right)^{9}}{.0817}\right)(1.0817)\left(\frac{1}{1.0817}\right)^{15}\right] = \text{TOT}$$

$$\$12,365\ (9.163450) + \$14,365\ (2.065874)$$

$$= \$113,306.06 + \$29,676.28 = \$\underline{\underline{142,982.34}}$$

(This problem combines the PV of a deferred annuity with the more typical PV of annuity case.)

21-C. Paul expects money in his savings account to earn interest at a 5.9% compounded average annual rate. If he has $10,250 today what does he expect the account balance to grow to be in 24 years?

$$\text{BAMT } (1 + r)^n = \text{EAMT}$$
$$\$10{,}250\ (1.059)^{24} = \text{EAMT}$$
$$\$10{,}250\ (3.958248) = \$\underline{\underline{40{,}572.05}}$$

(This is a non-annuity problem with the ending amount unknown; no series of regular payments.)

22-C. Sarah's four-year old son is so argumentative that she feels he should be a lawyer. She plans to save systematically over the coming 14 years to accumulate the money to pay for his schooling over the subsequent 7 years (BS and JD degrees). She expects to earn a 3.64% average annual rate of return on any money in her account.

a. If she wants to be able to give him $42,500 at the start of each of the 7 school years, how much must she save at the end of each of the 14 deposit years? What if instead she saves at the beginning of each of the 14 deposit years?

We do not consider the case of giving the son money at the end of each year since school costs must be paid in advance. The amount she will need to have on hand the day he starts college (it will be a present value on that day) is

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$42{,}500 \left[\left(\frac{1-\left(\frac{1}{1.0364}\right)^7}{.0364}\right)(1.0364)\right] = \text{TOT}$$
$$\$42{,}500 \times 6.304136 = \$\underline{267{,}925.76}$$

To reach that total she must save

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\text{PMT}\left(\frac{(1.0364)^{14}-1}{.0364}\right) = \$267{,}925.76$$
$$\$267{,}925.76 \div 17.846677 = \text{PMT} = \$\underline{\underline{15{,}012.64}}$$

at the end of each year, or if she makes deposits at the start of each year a somewhat smaller

$$\text{PMT}\left[\left(\frac{(1.0364)^{14}-1}{.0364}\right)(1.0364)\right] = \$267{,}925.76$$
$$\$267{,}925.76 \div 18.496296 = \text{PMT} = \$\underline{\underline{14{,}485.37}}$$

b. Sarah comes to realize that she may have been overly optimistic in projecting her ability to save. If she can put aside only $10,980 at the end of each of the 14 savings years how much will she be able to give her son at the beginning of each of his 7 years of school? What if instead she saves at the beginning of each year?

If she saves $10,980 at the end of each year the total she will have when her son starts college is

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$10{,}980\left(\frac{(1.0364)^{14}-1}{.0364}\right) = \text{TOT}$$
$$\$10{,}980 \times 17.846677 = \$\underline{195{,}956.51}$$

and that amount will allow for 7 subsequent beginning-of-year withdrawals of

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1}{1.0364}\right)^7}{.0364}\right)(1.0364)\right] = \$195{,}956.51$$

$$\text{PMT} \times 6.304136 = \$195{,}956.51$$

$$\$195{,}956.51 \div 6.304136 = \text{PMT} = \$\underline{\underline{31{,}083.80}}$$

If she saves $10,980 at the beginning of each year the college fund will grow to a slightly higher

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$10{,}980\left[\left(\frac{(1.0364)^{14}-1}{.0364}\right)(1.0364)\right] = \text{TOT}$$

$$\$10{,}980 \times 18.496296 = \$\underline{\underline{203{,}089.33}}$$

which will allow for beginning-of-year withdrawals in the following 7 years of a slightly higher

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1}{1.0364}\right)^7}{.0364}\right)(1.0364)\right] = \$203{,}089.33$$

$$\text{PMT} \times 6.304136 = \$203{,}089.33$$

$$\$203{,}089.33 \div 6.304136 = \text{PMT} = \$\underline{\underline{32{,}215.25}}$$

(This is a problem that combines PV of annuity and FV of annuity applications.)

23-C. The Virginia State Bank expects to receive a $6,200 deposit from saver Roger at the beginning of year 1; $7,300 at the beginning of year 2; $8,500 at the beginning of year 3; and $9,900 at the beginning of year 4. If the bank managers predict that they will pay interest on deposit accounts at a 2.8% average annual rate, how much should they expect to owe Roger at the end of year 4? What will they owe him if instead he deposits $7,900 at the start of each of the four years? What if instead he deposits the $7,900 at the end of each year?

After four years the bank will owe the saver

$$\$6{,}200\ (1.028)^4 + \$7{,}300\ (1.028)^3 + \$8{,}500\ (1.028)^2 + \$9{,}900\ (1.028)^1$$

$$= \$6{,}200\ (1.116792) + \$7{,}300\ (1.086374) + \$8{,}500\ (1.056784) + \$9{,}900\ (1.028)$$

$$= \$6{,}924.11 + \$7{,}930.53 + \$8{,}982.66 + \$10{,}177.20 = \$\underline{\underline{34{,}014.51}}$$

if the described unequal beginning-of-year deposits are received. It will owe him

$$\$7{,}900\ (1.028)^4 + \$7{,}900\ (1.028)^3 + \$7{,}900\ (1.028)^2 + \$7{,}900\ (1.028)^1$$

$$= \$7{,}900\ (1.116792) + \$7{,}900\ (1.086374) + \$7{,}900\ (1.056784) + \$7{,}900\ (1.028)$$

$$= \$8{,}822.60 + \$8{,}582.35 + \$8{,}348.59 + \$8{,}121.20 = \$\underline{\underline{33{,}874.81}}$$

(or note that, by the distributive property)

$$\$7{,}900\ (1.028)^4 + \$7{,}900\ (1.028)^3 + \$7{,}900\ (1.028)^2 + \$7{,}900\ (1.028)^1$$

$$= \$7{,}900\ [(1.028)^4 + (1.028)^3 + (1.028)^2 + (1.028)^1]$$

$$= \$7{,}900\left[\left(\frac{(1.028)^4-1}{.028}\right)(1.028)\right] = \$7{,}900\ (4.287950) = \$\underline{\underline{33{,}874.81}}$$

if four $7,900 equal beginning-of-year deposits are received, and

$$\$7,900\ (1.028)^3 + \$7,900\ (1.028)^2 + \$7,900\ (1.028)^1 + \$7,900\ (1.028)^0$$
$$= \$7,900\ (1.086374) + \$7,900\ (1.056784) + \$7,900\ (1.028) + \$7,900\ (1.00)$$
$$= \$8,582.35 + \$8,348.59 + \$8,121.20 + \$7,900 = \$\underline{\underline{32,952.15}}$$

(or, by the distributive property)

$$\$7,900\ (1.028)^3 + \$7,900\ (1.028)^2 + \$7,900\ (1.028)^1 + \$7,900\ (1.028)^0$$
$$= \$7,900\ [(1.028)^3 + (1.028)^2 + (1.028)^1 + (1.028)^0]$$
$$= \$7,900 \left(\frac{(1.028)^4 - 1}{.028}\right) = \$7,900\ (4.171158) = \$\underline{\underline{32,952.15}}$$

if four $7,900 equal end-of-year deposits are received. (This problem involves the future value of a series of payments.)