ANALYZING BONDS: PROBS. & SOLUTIONS ***(****copyright © 2021 Joseph W. Trefzger****)***

This problem set covers all of our bond valuation situations, with a general increase in degree of difficulty as we progress (some of the later problems are for FIL 404 only). Be sure that you have mastered the easier problems before moving ahead, because the more difficult examples tend to expand on the ideas presented in the easier ones.

1. You are analyzing a bond that will mature in four years. Actually, ABC Company issued several hundred million dollars' worth of these bonds, but we conduct our analysis in terms of the individual units, described as follows. These bonds had 25-year original maturities, but they were issued twenty-one years ago; thus they will mature four years from today. Each individual bond has a $1,000 par value ($1,000 was the amount originally lent to the company and the amount the company will pay back at maturity; par value for bonds issued by U.S. corporations is typically $1,000) and a 7% annual coupon interest rate, with interest paid *annually*. Today, rational buyers of bonds with similar risk, 4-year remaining lives, and annual interest payments require a 10% effective annual rate of return (also called the bond's yield to maturity). What should a rational investor pay for one of these ABC bonds?

Type: Bond Valuation; Annual Coupon Payments. We compute the value of any financial asset as the sum of the present values of all expected future periodic cash flows, discounted for the appropriate number of periods and by the required periodic rate of return. The most general way to state this relationship is

$$V_{Asset}\ (= CF_0) = CF_1\left(\frac{1}{1+r}\right)^1 + CF_2\left(\frac{1}{1+r}\right)^2 + CF_3\left(\frac{1}{1+r}\right)^3 + CF_4\left(\frac{1}{1+r}\right)^4 + \cdots + CF_n\left(\frac{1}{1+r}\right)^n$$

(Note that we are really just working with our Net Present Value equation:

$$NPV = CF_0\left(\frac{1}{1+r}\right)^0 + CF_1\left(\frac{1}{1+r}\right)^1 + CF_2\left(\frac{1}{1+r}\right)^2 + CF_3\left(\frac{1}{1+r}\right)^3 + \cdots + CF_n\left(\frac{1}{1+r}\right)^n$$

solving for V_{Asset}, which is CF_0: the price we would expect to pay today – and with NPV set equal to $0, which defines the case of a very competitive transaction in which the invested money earns a fair rate of return but nothing extra.)

When we compute bond values, the cash flows are the once-per-period coupon payments, along with the one-time-only expected receipt of par (the $1,000 originally lent) by whoever holds the bond at the maturity date. One way to represent the above equation when our financial asset is a bond is

$$V_B = \begin{matrix}\text{Coupon}\\ \text{Payment}\end{matrix}\left(\frac{1}{1+r}\right)^1 + \begin{matrix}\text{Coupon}\\ \text{Payment}\end{matrix}\left(\frac{1}{1+r}\right)^2 + \cdots + \begin{matrix}\text{Coupon}\\ \text{Payment}\end{matrix}\left(\frac{1}{1+r}\right)^{n-1} + \left[\begin{matrix}\text{Coupon}\\ \text{Payment}\end{matrix} + \begin{matrix}\text{Ending}\\ \text{Amount}\end{matrix}\right]\left(\frac{1}{1+r}\right)^n$$

Coupon Payment is the periodic interest payment, n is the number of periods that remain until the bond matures, r is the required periodic rate of return, and Ending Amount is a lump-sum payment to be received when the bond holder's investment period ends (in valuation situations, but not all of our bond analysis situations, this Ending Amount is the $1,000 par value to be received when the bond matures). If the coupon payments are to be received annually, then n is the number of full *years* that will pass before the final payment of interest (along with the return of the $1,000 principal lent) is to be received, and r is the required *annual* rate of return. In this case, the coupon payment is .07 x $1,000 = $70 per year; the Coupon Payment + Ending Amount to be received in the final period is $1,070 (the last $70 interest payment + the $1,000 of principal to be repaid); n is 4 years; and r is the 10% required effective annual rate of return. Thus we compute:

$$V_B = \$70\left(\frac{1}{1.10}\right)^1 + \$70\left(\frac{1}{1.10}\right)^2 + \$70\left(\frac{1}{1.10}\right)^3 + \$1{,}070\left(\frac{1}{1.10}\right)^4$$
$$= \$70\ (.909091) + \$70\ (.826446) + \$70\ (.751315) + \$1{,}070\ (.683013)$$
$$= \$63.64 + \$57.85 + \$52.59 + \$730.82 = \$\underline{\underline{904.90}}$$

Here we have accounted for the present value of each expected cash flow (interest payment and/or return of principal lent). There are other ways we could group the cash flows to account for all of the present values; for example, we might separate the last interest payment from the return of the $1,000 in principal to be received, for two separate end-of-year-4 cash flows:

$$V_B = \$70\left(\frac{1}{1.10}\right)^1 + \$70\left(\frac{1}{1.10}\right)^2 + \$70\left(\frac{1}{1.10}\right)^3 + \$70\left(\frac{1}{1.10}\right)^4 + \$1{,}000\left(\frac{1}{1.10}\right)^4$$

$$= \$70\ (.909091) + \$70\ (.826446) + \$70\ (.751315) + \$70\ (.683013) + \$1{,}000\ (.683013)$$

$$= \$70\ (.909091 + .826446 + .751315 + .683013) + \$1{,}000\ (.683013)$$

$$= \$70\ (3.169865) + \$1{,}000\ (.683013)$$

$$= \$221.89 + \$683.01 = \$\underline{\underline{904.90}}$$

The two methods are computationally equivalent, but the latter approach helps us to see how the stream of interest payments constitutes one major component of a bond's value, while the return of principal constitutes the other. In fact, investment firms sometimes *strip* bonds into the interest and principal components, selling the interest stream (in this case, for $221.89) to an investor who wants steady payments and selling the principal claim (in this case, for $683.01) to an investor who wants a zero-coupon instrument, which gives one large cash infusion at some point in the future.

Then note that, because the coupon payments are all scheduled to be the same, we could lump them together for computational ease and find their combined value as the present value of an annuity, which we add to the present value of the repayment of principal to get the bond's value. Let's restate our valuation equation now as:

$$V_B = \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right) + \begin{matrix}\text{Ending}\\\text{Amount}\end{matrix}\left(\frac{1}{1+r}\right)^n$$

In the problem at hand, we would see

$$V_B = \$70\left[\left(\frac{1}{1.10}\right)^1 + \left(\frac{1}{1.10}\right)^2 + \left(\frac{1}{1.10}\right)^3 + \left(\frac{1}{1.10}\right)^4\right] + \$1{,}000\left(\frac{1}{1.10}\right)^4$$

$$V_B = \$70\left(\frac{1-\left(\frac{1}{1.10}\right)^4}{.10}\right) + \$1{,}000\left(\frac{1}{1.10}\right)^4$$

$$= \$70\ (3.169865) + \$1{,}000\ (.683013)$$

$$= \$221.89 + \$683.01 = \$\underline{\underline{904.90}}$$

or 90.49% of the par value, as bond prices often are quoted. [On the Texas Instruments BA II Plus financial calculator we would solve as follows: enter $70 PMT; $1,000 FV; 10 I/Y; 4 N; CPT PV; we get –$904.90 (shown as negative because you would pay that amount today to buy the bond).]

Way back when this bond was issued 21 years ago, lenders required a 7% annual rate of return for lending to ABC for 25 years. ABC wanted to know its annual cost of using $1,000 for the long term, so it signed a contract to pay .07 x $1,000 = $70 per year until each bond matured; that amount does not change as time passes. But today lenders require a 10% effective annual return for lending to ABC for the remaining 4 years of the bond's life (perhaps interest rates across the economy are higher now than they were two decades ago, or perhaps ABC is seen as a riskier firm to lend to than it was when these bonds were issued). If ABC were to issue new 4-year bonds today they would pay a 10% annual coupon rate, but here we want to find the value of a bond with a 7% cash flow stream in a 10% world. Bond buyers get a 10% annual return on this 7% bond by paying a price less than par, thereby forcing the unchanging payments to represent a higher rate of return on the smaller amount invested.

Specifically, if they pay $904.90, then the receipt of four $70 annual interest payments and the receipt of the $1,000 principal value at the end of year 4 represents a 10% annual return on the $904.90 investment. Indeed, we got the $904.90 value by discounting the periodic (here, annual) cash flows at a 10% periodic rate – thus building a 10% periodic (annual) rate of return into the value estimate. Paying $904.90 and then getting back a total of $1,280 (4 x $70 in interest plus $1,000 in principal) represents a recouping of the $904.90 plus a 10% annual return on the portion of the $904.90 that remains invested from year to year over the bond's 4-year remaining life.

Finally, we have discussed an "effective annual rate" of return without mentioning the corresponding "annual percentage rate" of return. The distinction between the two is meaningful only when we deal with non-annual payments; if coupon interest payments are made annually the APR and EAR are the same. So it is only with semiannual coupon payments that we must contend with the APR vs. EAR issue in bond problems.

2. BCD Company issued several hundred million dollars' worth of bonds twenty-one years ago. These bonds had 25-year original maturities; therefore they will mature in four years. Each individual bond has a $1,000 par value and a 7% annual coupon interest rate, with interest paid *semiannually* (unlike what question 1 above might seem to suggest, bonds issued by U.S.-based corporations typically provide interest payments semiannually, not annually). Today, rational buyers of bonds with similar risk, 4-year remaining lives, and semiannual interest payments require a 10% *annual percentage rate* (APR) of return. What should a rational investor pay for one of these BCD bonds? If BCD were to issue 4-year bonds today, what annual coupon interest rate would we expect them to pay?

Type: Bond Valuation; Semiannual Coupon Payments. Again, we compute any financial asset's value as the sum of the present values of the expected periodic cash flows, discounted for the appropriate number of periods and by the required periodic rate of return. As noted, one way to present our bond valuation equation is

$$V_B = \begin{matrix}\text{Coupon}\\ \text{Payment}\end{matrix}\left(\frac{1}{1+r}\right)^1 + \begin{matrix}\text{Coupon}\\ \text{Payment}\end{matrix}\left(\frac{1}{1+r}\right)^2 + \cdots + \begin{matrix}\text{Coupon}\\ \text{Payment}\end{matrix}\left(\frac{1}{1+r}\right)^{n-1} + \left[\begin{matrix}\text{Coupon}\\ \text{Payment}\end{matrix} + \begin{matrix}\text{Ending}\\ \text{Amount}\end{matrix}\right]\left(\frac{1}{1+r}\right)^n$$

Coupon Payment is the periodic interest payment, which here is to be received *semiannually*. And here n is the number of *half-year* periods that will pass before the final payment of interest (along with the return of the $1,000 principal lent) is to be received. So in this case n is 8 half-years, the Coupon Payment is (.07 ÷ 2) x $1,000 = $35 every six months; and the amount in brackets to be received at the end of the final six-month period is $1,035 (the last $35 interest payment + the $1,000 of principal to be repaid). Finally, r has to be the *semiannual* required rate of return. Because the annual return measure talked about in the discussion phase is the annual percentage rate (APR), the simple or convenient annual rate measure that is not adjusted for the impact of intra-year compounding, we find the semiannual required rate of return simply by dividing the APR by the number of periods in a year, which here is 10% ÷ 2 = 5%, and we compute:

$$V_B = \$35\left(\frac{1}{1.05}\right)^1 + \$35\left(\frac{1}{1.05}\right)^2 + \$35\left(\frac{1}{1.05}\right)^3 + \$35\left(\frac{1}{1.05}\right)^4 + \$35\left(\frac{1}{1.05}\right)^5$$

$$+ \$35\left(\frac{1}{1.05}\right)^6 + \$35\left(\frac{1}{1.05}\right)^7 + \$1{,}035\left(\frac{1}{1.05}\right)^8$$

= \$35 (.952381) + \$35 (.907029) + \$35 (.863838) + \$35 (.822702) + \$35 (.783526) + \$35 (.746215) + \$35 (.710681) + \$1,035 (.676839)
= \$33.33 + \$31.75 + \$30.23 + \$28.79 + \$27.42 + \$26.12 + \$24.87 + \$700.53 = \$903.04

Here we have accounted for the present value of each expected cash flow (interest payment and/or return of principal lent). There are other ways we could group the cash flows to account for all of the present values; for example, we might separate the last interest payment from the return of the \$1,000 in principal to be received at the end of year 4 (the end of half-year 8):

$$V_B = \$35\left(\frac{1}{1.05}\right)^1 + \$35\left(\frac{1}{1.05}\right)^2 + \$35\left(\frac{1}{1.05}\right)^3 + \$35\left(\frac{1}{1.05}\right)^4 + \$35\left(\frac{1}{1.05}\right)^5$$

$$+ \$35\left(\frac{1}{1.05}\right)^6 + \$35\left(\frac{1}{1.05}\right)^7 + \$35\left(\frac{1}{1.05}\right)^8 + \$1{,}000\left(\frac{1}{1.05}\right)^8$$

= \$35 (.952381) + \$35 (.907029) + \$35 (.863838) + \$35 (.822702) + \$35 (.783526) + \$35 (.746215) + \$35 (.710681) + \$35 (.676839) + \$1,000 (.676839)

= \$35 (.952381 + .907029 + .863838 + .822702 + .783526 + .746215 + .710681 + .676839) + \$1,000 (.676839)
= \$35 (6.463211) + \$ 1,000 (.676839)
= \$226.21 + \$676.84 = \$903.05

(a slight rounding difference). As we saw with annual interest payments, these two methods are equivalent computationally, but the latter approach helps us see how the stream of coupon payments provides one major component of a bond's value (here the stripped value of the right to collect the interest payments is \$226.21), while the return of principal at maturity constitutes the other (here the stripped value of the right to collect \$1,000 in 4 years is \$676.84). And because the coupon payments are all the same, we could lump them together for computational ease and find their combined value as the PV of an annuity, which we add to the PV of the principal repayment to get the bond's value (the price a rational buyer should be willing to pay):

$$V_B = \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right) + \begin{matrix}\text{Ending}\\\text{Amount}\end{matrix}\left(\frac{1}{1+r}\right)^n$$

In the problem at hand, with semiannual interest payments, it would become

$$V_B = \$35\left[\left(\frac{1}{1.05}\right)^1 + \left(\frac{1}{1.05}\right)^2 + \left(\frac{1}{1.05}\right)^3 + \left(\frac{1}{1.05}\right)^4 + \left(\frac{1}{1.05}\right)^5 + \left(\frac{1}{1.05}\right)^6 + \left(\frac{1}{1.05}\right)^7 + \left(\frac{1}{1.05}\right)^8\right] + \$1{,}000\left(\frac{1}{1.05}\right)^8$$

$$V_B = \$35\left(\frac{1-\left(\frac{1}{1.05}\right)^8}{.05}\right) + \$1{,}000\left(\frac{1}{1.05}\right)^8$$

= \$35 (6.463213) + \$1,000 (.676839)
= \$226.21 + \$676.84 = \$903.05

or 90.305% of the par value, as bond prices often are quoted. [On the TI BA II Plus calculator we would solve as follows: enter \$70 ÷ 2 = PMT; \$1,000 FV; 10 ÷ 2 = I/Y; 4 x 2 = N; CPT PV; we get –\$903.05 (shown as negative because you would pay that amount today to buy the bond).]

Twenty-one years ago when this bond was issued, lenders required a 7% annual percentage rate of return (a coupon rate is a non-compounded APR), which equates to a 7% ÷ 2 = 3.5% semiannual periodic rate of return, for lending to BCD for 25 years. BCD wanted to know what its annual cost of using $1,000 would be, so it signed a contract to pay (.07 ÷ 2) x $1,000 = $35 every six months until the bond matured; that amount does not change as time passes. But today lenders require a 10% APR, which equates to a 10% ÷ 2 = 5% semiannual rate of return, for lending to BCD for the remaining four years = eight half-years of the bond's life. If BCD were to issue new 4-year bonds today they would pay a 10% annual (or 5% semiannual) coupon interest rate, so here we want to find the value of a bond with a 3.5% cash flow stream in a 5% world. Bond buyers get a 5% semiannual return on this 3.5% semiannual coupon bond by paying a price less than par.

Specifically, if they pay $903.05, then getting back eight $35 semiannual interest payments and receiving the $1,000 principal value at the end of half-year 8 represents a 5% semiannual return, for a 5% x 2 = 10% APR. (Questions 1 and 2 involve similar time periods and expected annual rates of return, but the interest payments in problem 2 are to be received semiannually, leading to a slightly different computed value.) Indeed, we got the $903.05 value by discounting the semiannual cash flows at a 5% rate – thus building a 5% semiannual rate of return into the value estimate. Paying $903.05 and then getting back a total of $1,280 (8 x $35 in interest and $1,000 in principal) represents a recouping of the $903.05 plus a 5% semiannual return on the portion of the $903.05 that remains invested from year to year over the bond's 8-half-year remaining life.

Finally, we know that the bond buyer earns a 5% semiannual return, but we like to talk about rates of return in annual terms. Two types of annual rates can be given, one that accounts for intra-year compounding (EAR) and one that does not (APR). Here, as noted, the investor would be getting a .05 x 2 = 10% APR, and thus we would expect BCD to pay a 10% annual coupon interest rate if they were to issue new 4-year bonds today. In receiving 5% semiannually the investor would be earning a $(1.05)^2$ – 1 = 10.25% effective annual rate (EAR) of return. In our coverage, when dealing with bonds, we refer to the EAR as the yield to maturity, or YTM (textbooks sometimes treat the YTM as an APR, so be alert). If interest is paid annually, the APR and EAR are equal, but with semiannual interest payments the EAR slightly exceeds the APR. In problem 3 our focus is on the EAR.

3. CDE Company issued several hundred million dollars' worth of bonds twenty-one years ago. These bonds had 25-year original maturities; therefore they will mature in four years. Each individual bond has a $1,000 par value and a 7% annual coupon interest rate, with interest paid *semiannually*. Today, rational buyers of bonds with similar risk, 4-year remaining lives, and semiannual interest payments require a 10% *effective annual rate* (EAR) of return (also called the bond's yield to maturity). What should a rational investor be willing to pay for one of these CDE bonds? If CDE were to issue 4-year bonds today, what coupon interest rate would we expect them to pay?

Type: Bond Valuation; Semiannual Coupon Payments. This question is fairly similar to question 2 above, with semiannual coupon payments, n representing the 8 half-years that will pass before the final payment of interest (along with the return of the $1,000 principal lent) is to be received, and the coupon payment again totaling (.07 ÷ 2) x $1,000 = $35 every six months, with the $1,000 of principal also to be repaid at the end of the 8th half-year. But now the annual return measure we are given in the discussion stage (recall that we talk about rates of return or cost in annual terms) is an EAR, the annual rate measure that is adjusted for the impact of intra-year compounding – so we must un-compound it to get the semiannual required rate of return r for use in our computations: $\sqrt[2]{1.10}$ – 1 = .048809, or 4.8809% (such that the semiannual rate compounds out to $[1.048809]^2$ – 1 = 10%). [In question 2 above 10% was an annual percentage rate, or APR, which we simply divided by 2 to find question #2's higher 5% required semiannual rate r.] Plugging these figures into the more compact form of our bond valuation equation, we find

$$V_B = \$35\left(\frac{1-\left(\frac{1}{1.048809}\right)^8}{.048809}\right) + \$1{,}000\left(\frac{1}{1.048809}\right)^8$$
$$= \$35\ (6.494448) + \$1{,}000\ (.683013)$$
$$= \$227.31 + \$683.01 = \$\underline{910.32}\ ,$$

or 91.032% of the par value, as bond prices are often quoted. [On BA II Plus calculator we enter \$70 ÷ 2 = PMT; \$1,000 FV; 1.1 $\sqrt[2]{x}$ – 1 = x 100 = I/Y; 4 x 2 = N; CPT PV; we get – \$910.32 (shown as negative because you would pay that amount today to buy the bond).]

Way back when this bond was issued, lenders required a 3.5% semiannual rate of return, which equated to an APR of .035 x 2 = 7% (so that was the coupon interest rate the borrowing firm paid) and an EAR of $(1.035)^2$ – 1 = 7.1225%, for lending to CDE for 25 years. But today lenders require a 10% EAR for lending to CDE for the remaining four years = eight half-years of the bond's life. If CDE were to issue new 4-year bonds today lenders would expect a 4.8809% semiannual rate of return, which would equate to a .048809 x 2 = 9.7618% APR (so that is the annual coupon interest rate the company would expect to pay if it issued new 4-year bonds today) and a $(1.048809)^2$ – 1 = 10% yield to maturity or EAR. So here we want to find the value of a bond with a 3.5% cash flow stream in a 4.8809% world. Bond buyers get a 4.8809% semiannual return on this 3.5% semiannual coupon bond by paying a price less than par.

Specifically, if they pay \$910.32, then the receipt of eight \$35 semiannual interest payments and the receipt of the \$1,000 principal value at the end of half-year 8 represents a 4.8809% semiannual return on the \$910.32 initially given up. Indeed, we got the \$910.32 value by discounting the semiannual cash flows at a 4.8809% rate – thus building a 4.8809% semiannual rate of return into the value estimate. Paying \$910.32 and then getting back a total of \$1,280 (8 x \$35 in interest and \$1,000 in principal) represents a recouping of the \$910.32 plus a 4.8809% semiannual return on the portion of the \$910.32 that remains invested from year to year over the bond's 8-half-year remaining life. And with a 4.8809% semiannual return, the investor would be getting a .048809 x 2 = 9.7618% APR and a $(1.048809)^2$ – 1 = 10% EAR.

Again, in our coverage when we are dealing with bonds that have semiannual interest payments we treat the yield to maturity as an EAR measure (but not all textbooks or other sources do – some are careful to label a bond investment's internal rate of return in EAR terms as the "effective yield to maturity"). Note that in a case such as this one, with semiannual coupon payments, the rate of return in EAR terms (YTM) slightly exceeds the rate of return in APR terms. And note that, accordingly, the value computed here (\$910.32) is slightly higher than the value computed in question 2 above (\$903.05), because a 10% EAR corresponds to a lower 4.8809% semiannual discount rate, whereas a 10% APR corresponds to a higher 5% semiannual discount rate (when expected cash flows are discounted at a higher rate their PV is lower).

4. DEF Company issued several hundred million dollars worth of bonds twenty-one years ago. These bonds had 25-year original maturities; therefore they will mature in four years. Each individual bond has a \$1,000 par value. But there are no annual or semiannual coupon interest payments; these bonds are of the *zero-coupon* variety. Rational buyers of zero-coupon bonds with similar risk and 4-year remaining lives require a 10% effective annual rate (EAR) of return, or yield to maturity (YTM). What should they pay for each of these DEF bonds? If rational

lenders had required a 7% EAR 21 years ago, what should they have paid for these bonds when they were issued? If rational investors required a 7% EAR (instead of 10%) today, what would each of these bonds be worth now?

Type: Zero-Coupon Bond Valuation. Once again, we compute any financial asset's value by summing the present values of the expected periodic cash flows, discounted for the appropriate number of periods and by the required periodic rate of return. And again we can show our bond value equation as

$$V_B = \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1}{1+r}\right)^1 + \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1}{1+r}\right)^2 + \cdots + \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1}{1+r}\right)^{n-1} + \left[\begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix} + \begin{matrix}\text{Ending}\\\text{Amount}\end{matrix}\right]\left(\frac{1}{1+r}\right)^n$$

or, making use of the distributive property, can simplify our computations as

$$V_B = \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right) + \begin{matrix}\text{Ending}\\\text{Amount}\end{matrix}\left(\frac{1}{1+r}\right)^n$$

But in this example there are no coupon payments, so the Coupon Payment term simply becomes zero, and the bond's value is computed as

$$V_B = \$0\left(\frac{1-\left(\frac{1}{1.10}\right)^4}{.10}\right) + \$1{,}000\left(\frac{1}{1.10}\right)^4$$

$$= \$1{,}000\ (.683013) = \$\underline{\underline{683.01}}$$

[On BA II Plus calculator we would enter $0 PMT; $1,000 FV; 10 I/Y; 4 N; CPT PV; we get –$683.01 (shown as negative because you would pay that amount today to buy the bond). But remember that you are not expected to have a financial calculator for our class; we encourage you to work problems by hand in this basic coverage to assure understanding, but finance majors will want to become familiar with financial calculator steps at some point for use in later courses.]

Note that zero-coupon bonds have no periodic payments. Thus we can keep things simple and think in terms of full-year periods as long as the rate r we use in discounting the Ending Amount to a present value is the required effective annual rate (EAR) of return. But what if a particular analyst always likes to think in terms of semiannual periods when analyzing bonds? Then simply find the semiannual periodic rate that compounds out to equal the specified EAR. If 10% represents today's required EAR (recall that in our coverage we treat the yield to maturity as an EAR), then the corresponding semiannual periodic rate is $\sqrt[2]{1.10}$ – 1 = .048809 or 4.8809%, and the bond's value today (with 8 half-years remaining until maturity) is computed as

$$V_B = \$0\left(\frac{1-\left(\frac{1}{1.048809}\right)^8}{.048809}\right) + \$1{,}000\left(\frac{1}{1.048809}\right)^8$$

$$= \$1{,}000\ (.683013) = \$\underline{\underline{683.01}}\ ,$$

just as we computed with a 10% EAR for 4 years above (note that this is also the value of the stripped principal payment computed in question #3). [On the BA II Plus calculator enter $0 PMT; $1,000 FV; 1.10 $\sqrt[2]{x}$ – 1= x 100 = I/Y; 4 x 2 = N; CPT PV; we get –$683.01.]

Now assume that when these bonds were issued 21 years ago, with a 25-year life, lenders required a 7% EAR, or yield to maturity, for lending to DEF on a zero-coupon basis for 25 years. If so, then the price at which each of these bonds originally sold should have been

$$V_B = \$0\left(\frac{1-\left(\frac{1}{1.07}\right)^{25}}{.07}\right) + \$1{,}000\left(\frac{1}{1.07}\right)^{25}$$

$$= \$1{,}000\ (.184249) = \$\underline{\underline{184.25}}$$

[BA II Plus: enter $0 PMT; $1,000 FV; 7 I/Y; 25 N; CPT PV; we get –$184.25.] If 7% represents the EAR that applied when the bond was issued, the corresponding semiannual periodic rate should have been $\sqrt[2]{1.07}$ – 1 = 3.4408%, and the bond's value when it was originated (with 50 half-years remaining until maturity) should have been (just as we found above with the annual rate for 25 years):

$$V_B = \$0\left(\frac{1-\left(\frac{1}{1.034408}\right)^{50}}{.034408}\right) + \$1{,}000\left(\frac{1}{1.034408}\right)^{50}$$
$$= \$1{,}000\ (.184250) = \$\underline{\underline{184.25}}$$

[BA II Plus financial calculator: enter $0 PMT; $1,000 FV; 1.07 $\sqrt[2]{x}$ – 1 = x 100 = I/Y; 25 x 2 = N; CPT PV; we get –$184.25.] Finally, if today investors expected a 7% EAR or YTM for lending to DEF Company on a zero-coupon basis for 4 years, today's value would be

$$V_B = \$0\left(\frac{1-\left(\frac{1}{1.07}\right)^{4}}{.07}\right) + \$1{,}000\left(\frac{1}{1.07}\right)^{4}$$
$$= \$1{,}000\ (.762895) = \$\underline{\underline{762.90}}$$

or, if the analyst wanted to think in terms of 8 semiannual periods remaining and a semiannual periodic rate of $\sqrt[2]{1.07}$ – 1 = .034408 or 3.4408%, we would compute

$$V_B = \$0\left(\frac{1-\left(\frac{1}{1.034408}\right)^{8}}{.034408}\right) + \$1{,}000\left(\frac{1}{1.034408}\right)^{8}$$
$$= \$1{,}000\ (.762895) = \$\underline{\underline{762.90}}$$

[BA II Plus calculator: enter $0 PMT; $1,000 FV; 7 I/Y; 4 N; CPT PV; we get –$762.90 or else $0 PMT; $1,000 FV; 1.07 $\sqrt[2]{x}$ – 1 = x 100 = I/Y; 4 x 2 = N; CPT PV; we get the same –$762.90.] These three computed values tell an interesting story. When the bond was issued an investor who required a 7% effective annual rate of return paid $184.25 for the right to collect $1,000 after waiting 25 years. An investor today who required a 7% EAR would pay a much higher $762.90 for the right to collect $1,000 after waiting just 4 years. (All you get with a zero-coupon bond is the right to collect the par value at maturity, so the value rises as we get closer to the maturity date.) But the current state of the economy, and of DEF, cause investors to require a 10% effective annual rate of return for lending to DEF on a zero-coupon basis for 4 years, so they discount the $1,000 par value at a higher rate (10%, not 7%) and get a lower indicated value ($683.01, not

$762.90). Paying a lower price causes the single expected $1,000 receipt to represent a higher periodic rate of return.

5. EFG Company just issued two billion dollars' worth of zero-coupon bonds, each with a $1,000 par (or maturity) value and a 10-year maturity. Investors buying these bonds today require an 11.35% effective yield to maturity. You believe that investors will continue to expect an 11.35% effective yield to maturity as the bonds approach their maturity date. At what price should each of these bonds sell today, and at the start of each of the next 10 years? How much interest will the bond holder be earning each year, in the eyes of U.S. federal income tax authorities?

Type: Zero-Coupon Bond Valuation. As seen in the previous problem, with no regular interest payments the Coupon Payment term drops out of our bond value equation

$$V_B = \begin{matrix}\text{Coupon}\\ \text{Payment}\end{matrix} \left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right) + \begin{matrix}\text{Ending}\\ \text{Amount}\end{matrix} \left(\frac{1}{1+r}\right)^n$$

and we can compute the zero-coupon bond's value simply as the present value of the Ending Amount (the $1,000 or other par value to be received at maturity). Here we find that present value by discounting at an 11.35% yearly rate for the number of years remaining until maturity or, if we want to think in semiannual terms, at a $\sqrt[2]{1.1135}$ – 1 = 5.5225% rate for the number of half-years remaining until maturity (if there are no regular annual or semiannual payments, but rather just one large payment at the end, we can think of the waiting period in either annual or semiannual terms). Today each bond should sell for

$$\$1{,}000 \left(\frac{1}{1.1135}\right)^{10} = \$1{,}000 \left(\frac{1}{1.055225}\right)^{20} = \$\underline{341.27} \text{ with 10 years remaining until maturity.}$$

[TI BA II Plus financial calculator: enter $0 PMT; $1,000 FV; 11.35 I/Y; 10 N; Compute PV; or else $0 PMT; $1,000 FV; 1.1135 $\sqrt[2]{x}$ – 1 = x 100 = I/Y; 10 x 2 = N; CPT PV; either way we get –$341.27.] As we move toward maturity, the bond's value should rise, because all the investor has is the right to collect the $1,000 at maturity — and that value rises as we get closer to the maturity date. (If all you are going to get is an unadjusted $1,000, you would be happier having to wait only 4 years to collect it than when you had to wait 7 years to collect it.) As we move from 9 years remaining until maturity to 1 year remaining (while holding investors' required periodic rate of return constant), the value of each one of these zero-coupon bonds should systematically increase as shown:

$$\$1{,}000 \left(\frac{1}{1.1135}\right)^{9} = \$1{,}000 \left(\frac{1}{1.055225}\right)^{18} = \$\underline{380.00} \text{ with 9 years remaining until maturity.}$$

$$\$1{,}000 \left(\frac{1}{1.1135}\right)^{8} = \$1{,}000 \left(\frac{1}{1.055225}\right)^{16} = \$\underline{423.13} \text{ with 8 years remaining until maturity.}$$

$$\$1{,}000 \left(\frac{1}{1.1135}\right)^{7} = \$1{,}000 \left(\frac{1}{1.055225}\right)^{14} = \$\underline{471.16} \text{ with 7 years remaining until maturity.}$$

$$\$1{,}000 \left(\frac{1}{1.1135}\right)^{6} = \$1{,}000 \left(\frac{1}{1.055225}\right)^{12} = \$\underline{524.64} \text{ with 6 years remaining until maturity.}$$

$$\$1{,}000 \left(\frac{1}{1.1135}\right)^{5} = \$1{,}000 \left(\frac{1}{1.055225}\right)^{10} = \$\underline{584.18} \text{ with 5 years remaining until maturity.}$$

$\$1{,}000\left(\frac{1}{1.1135}\right)^4 = \$1{,}000\left(\frac{1}{1.055225}\right)^8 = \$\underline{650.49}$ with 4 years remaining until maturity.

$\$1{,}000\left(\frac{1}{1.1135}\right)^3 = \$1{,}000\left(\frac{1}{1.055225}\right)^6 = \$\underline{724.32}$ with 3 years remaining until maturity.

$\$1{,}000\left(\frac{1}{1.1135}\right)^2 = \$1{,}000\left(\frac{1}{1.055225}\right)^4 = \$\underline{806.53}$ with 2 years remaining until maturity.

$\$1{,}000\left(\frac{1}{1.1135}\right)^1 = \$1{,}000\left(\frac{1}{1.055225}\right)^2 = \$\underline{898.07}$ with 1 year remaining until maturity.

$\$1{,}000\left(\frac{1}{1.1135}\right)^0 = \$1{,}000\left(\frac{1}{1.055225}\right)^0 = \$\underline{1{,}000.00}$ with 0 years remaining (maturity date).

[On the BA II Plus financial calculator, the rate and the $0 PMT and the $1,000 FV do not change as time passes, so as you move toward the maturity date just keep entering the new, shorter time period remaining and hit CPT PV.] At maturity, of course, the bond will be worth its $1,000 par/maturity value.

The $1,000 maturity value minus $341.27 initial purchase price = $<u>658.73</u> is deemed to constitute the total interest earned over the bond's life. The annual portion of this total is the amount by which the bond increases in value over the previous year (based on the original yield to maturity). [U.S. income tax law used to allow interest to be treated as a straight-line $658.73 ÷ 10 = $65.87 per year, but it no longer does.] So even though the bond holder receives no cash until the zero-coupon bond matures, the interest "earned" each year by the bond holder (i.e. the increase in the bond's value, which is the amount on which the zero-coupon lender has to pay income tax and also the amount that can be deducted each year by the zero-coupon borrower as an interest expense) is deemed to be:

$ 380.00	–	$341.27	=	$ 38.73	in year 1;
$ 423.13	–	$380.00	=	$ 43.13	in year 2;
$ 471.16	–	$423.13	=	$ 48.03	in year 3;
$ 524.64	–	$471.16	=	$ 53.48	in year 4;
$ 584.18	–	$524.64	=	$ 59.54	in year 5;
$ 650.49	–	$584.18	=	$ 66.31	in year 6;
$ 724.32	–	$650.49	=	$ 73.83	in year 7;
$ 806.53	–	$724.32	=	$ 82.21	in year 8;
$ 898.07	–	$806.53	=	$ 91.54	in year 9; and
$1,000.00	–	$898.07	=	$<u>101.93</u>	in year 10;
		Total Interest		$<u>658.73</u>	

for the same $658.73 total we found above. Note that the amount of interest deemed to be earned by the bond buyer (lender)/deductible by the bond issuer (borrower) grows each year, as the 11.35% annual yield is earned/paid on a progressively higher principal value.

6. [Similar to problem 1, for extra practice.] FGH Company issued several hundred million dollars' worth of bonds seven years ago. These bonds had 20-year original maturities; therefore they will mature in thirteen years. Each individual bond has a $1,000 par value and an 8.5% annual coupon interest rate, with interest paid *annually*. If rational buyers of bonds with similar risk, 13-year remaining lives, and annual interest payments today require a

9.5% effective yield to maturity (YTM), what should they pay for each of these FGH bonds? What if such buyers require a 7.5% effective YTM? What if they require an 8.5% effective YTM?

Type: Bond Valuation; Annual Coupon Payments. Recall that we can analyze bonds with the equation

$$V_B = \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right) + \begin{matrix}\text{Ending}\\\text{Amount}\end{matrix}\left(\frac{1}{1+r}\right)^n$$

Here the Coupon Payments are made annually, so n is the number of years that remain until the bond matures, and r is the required annual rate of return. (With annual cash flows there is no APR/EAR distinction, so the yield to maturity is simply the required annual rate of return.) When these bonds were issued, lenders required an 8.5% annual rate of return for lending to a firm like FGH for 20 years. Thus the coupon interest rate was set at 8.5%, and the periodic (here, annual) interest payment throughout the bond's 20-year life was to be 8.5% of $1,000, or $85.00. That way FGH knew that its ongoing cost of using a lender's $1,000 would be $85.00 each year for 20 years, regardless of changes in the economy or the company. But the value to an investor of the right to collect the series of remaining $85.00 interest payments, plus a return of the $1,000 principal at maturity, depends on changes in the economy and/or the company.

Let's assume that today either the overall economy or the company has changed, such that potential lenders require an annual rate of return higher than 8.5%; parties lending money to companies like FGH for 13 years now require a 9.5% annual rate of return. That is, if a company like FGH were to issue new 13-year bonds today, they would pay a 9.5% annual coupon rate. (Either interest rates across the economy have risen, or FGH has become a riskier company to lend money to.) Discounting the series of $85.00 interest payments to be received at the end of each of years 1 – 12, and the $1,085 of interest-plus-principal to be received at the end of year 13 when the bond matures, to present values at a 9.5% periodic (annual) discount rate gives

$$\begin{aligned}V_B &= \$85.00\left(\frac{1}{1.095}\right)^1 + \$85.00\left(\frac{1}{1.095}\right)^2 + \cdots + \$85.00\left(\frac{1}{1.095}\right)^{11} + \$85.00\left(\frac{1}{1.095}\right)^{12} + \$1{,}085\left(\frac{1}{1.095}\right)^{13}\\ &= \$85.00\left(\frac{1-\left(\frac{1}{1.095}\right)^{12}}{.095}\right) + \$1{,}085\left(\frac{1}{1.095}\right)^{13}\\ &= \$85.00\,(6.983839) + \$1{,}085\,(.307338)\\ &= \$593.63 + \$333.46 = \$\underline{\underline{927.09}}\end{aligned}$$

Recall that when required rates of return in the market rise, the values of existing fixed-income investments (such as bonds) fall. It should be intuitively clear that the unchanging series of 8.5% cash flows can represent a 9.5% annual rate of return only if the buyer pays a price less than the $1,000 par value. A more typical way to group the cash flows would be to show

$$\begin{aligned}V_B &= \$85.00\left(\frac{1}{1.095}\right)^1 + \$85.00\left(\frac{1}{1.095}\right)^2 + \cdots + \$85.00\left(\frac{1}{1.095}\right)^{12} + \$85.00\left(\frac{1}{1.095}\right)^{13} + \$1{,}000\left(\frac{1}{1.095}\right)^{13}\\ &= \$85.00\left(\frac{1-\left(\frac{1}{1.095}\right)^{13}}{.095}\right) + \$1{,}000\left(\frac{1}{1.095}\right)^{13}\\ &= \$85.00\,(7.291178) + \$1{,}000\,(.307338)\\ &= \$619.75 + \$307.34 = \$\underline{\underline{927.09}}\end{aligned}$$

We normally think of the last interest payment as part of the interest stream annuity, and look at the $619.75 and $307.34 as the relevant values if the bond were to be "stripped" into interest-only and principal-only pieces (as an investment firm might do, creating in the process a straight annuity that is attractive to an investor like a charitable organization needing regular payments, and a zero-coupon bond attractive to an investor like a life insurance company that prefers one large future receipt). The investor will receive $85 at the ends of each of years 1 – 12, and then another $85 along with $1,000 at the end of year 13; the total present value comes out the same no matter where we group the last $85 interest payment for computing purposes. [TI BA II Plus financial calculator: enter $85 PMT; $1,000 FV; 9.5 I/Y; 13 N; CPT PV; it shows –$927.09. In using the calculator we have to group the 13 interest payments together; if we entered 12 $85 payments and a $1,085 FV it will compute as though the bond holder is getting $85 plus an additional $1,085 all at the end of year 12.]

What if, instead, either the economy or company has changed, such that potential lenders require an annual return lower than 8.5%; parties lending money to firms like FGH for 13 years require only a 7.5% annual rate of return? That is, if a company like FGH were to issue new 13-year bonds today, it would pay a 7.5% coupon rate. (Either interest rates across the economy have fallen, or FGH has come to be seen as less risky to lend money to.) Discounting the series of $85.00 interest payments, and the $1,000 principal to be paid back when the bond matures, to present values at a 7.5% periodic (annual) discount rate gives

$$V_B = \$85.00\left(\frac{1-\left(\frac{1}{1.075}\right)^{13}}{.075}\right) + \$1{,}000\left(\frac{1}{1.075}\right)^{13}$$
$$= \$85.00\ (8.125840) + \$1{,}000\ (.390562)$$
$$= \$690.70 + \$390.56 = \$\underline{\underline{1{,}081.26}}\ .$$

[BA II Plus calculator: enter $85 PMT; $1,000 FV; 7.5 I/Y; 13 N; CPT PV; it shows –$1,081.26.] Recall that when required rates of return in the market fall, the values of existing fixed-income investments rise. It should be intuitively clear that the unchanging series of 8.5% cash flows can represent a 7.5% annual rate of return only if the buyer pays more than the $1,000 par value.

Finally, what if the state of the economy and company are such that parties lending money to firms like FGH for 13 years would still require an 8.5% annual rate of return? That is, if a company like FGH were to issue new 13-year bonds today, they would pay an 8.5% annual coupon interest rate. (Interest rates across the economy have remained stable, and/or FGH is seen as no more or less risky to lend to than it was seven years earlier.) Discounting the series of $85.00 interest payments to be received each year, and the $1,000 principal to be paid back when the bond matures, to present values at an 8.5% periodic (annual) discount rate gives

$$V_B = \$85.00\left(\frac{1-\left(\frac{1}{1.085}\right)^{13}}{.085}\right) + \$1{,}000\left(\frac{1}{1.085}\right)^{13}$$
$$= \$85.00\ (7.690955) + \$1{,}000\ (.346269)$$
$$= \$653.73 + \$346.27 = \$\underline{\underline{1{,}000.00}}\ .$$

[BA II Plus calculator: enter $85 PMT; $1,000 FV; 8.5 I/Y; 13 N; CPT PV; it shows –$1,000.00.] If investors' required periodic rate of return is equal to the bond's periodic coupon rate, then the bond's cash flows are equivalent to what an investor would receive on a newly issued bond, and the existing bond should be worth its par value. It should be intuitively clear that if a buyer pays the $1,000 par value, the unchanging series of 8.5% cash flows provides an 8.5% annual rate of return.

7. [Similar to problem 2, for extra practice.] GHI Company issued several hundred million dollars' worth of bonds seven years ago. These bonds had 20-year original maturities; therefore they will mature in thirteen years. Each individual bond has a $1,000 par value and an 8.5% annual coupon interest rate, with interest paid *semiannually*. If rational buyers of bonds with similar risk, 13-year remaining lives, and semiannual interest payments today require a 9.5% stated *annual percentage rate* (APR) of return, what should they pay for each of these GHI bonds? What if such buyers require a 7.5% APR? What if they require an 8.5% APR?

Type: Bond Valuation; Semiannual Coupon Payments. As always, we compute the value of any financial asset as the sum of the present values of the expected periodic cash flows, discounted for the appropriate number of periods and by the required periodic rate of return. And again, in the case of bonds, with the coupon payments all equal (sometimes at a $0 value), our valuation equation is

$$V_B = \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right) + \begin{matrix}\text{Ending}\\\text{Amount}\end{matrix}\left(\frac{1}{1+r}\right)^n$$

Here Coupon Payment is the amount of interest paid semiannually, n is the number of semiannual periods that remain until the bond matures, and r is the required semiannual periodic rate of return. With semiannual interest payments, we have to know whether the annual rate of return measure talked about in the discussion phase is an APR (which ignores intra-year compounding) or an EAR (which includes intra-year compounding's impact). When these bonds were issued, lenders required an 8.5% stated APR for lending to a firm like GHI for 20 years. Thus the coupon interest rate was set at 8.5%, and the periodic (here, semiannual) interest payment throughout the bond's 20-year life was to be (.085 ÷ 2) x $1,000, or $42.50. That way GHI knew that its ongoing cost of using the lender's $1,000 would be $85.00 each year for 20 years, broken into two equal $42.50 semiannual payments, regardless of changes in the economy or the company. But the value to an investor now of the right to collect the series of remaining $42.50 interest payments, plus a return of the $1,000 principal at maturity, depends on perceived changes in the economy or the company.

Let's assume that today either the overall economy or the company has changed, such that potential lenders require an APR higher than 8.5%; parties lending money to companies like GHI for 13 years now require a 9.5% APR. That is, if a company like GHI were to issue new 13-year bonds today, they would pay a 9.5% annual coupon rate; recall that a coupon rate is an APR measure that does not take into account intra-year compounding. (Either interest rates across the economy have risen, or GHI has become a riskier company to lend to.) Discounting the series of 13 x 2 = twenty-six $42.50 interest payments to be received each half-year, and the $1,000 principal to be paid back when the bond matures, to present values at a 9.5% ÷ 2 = 4.75% required periodic (semiannual) rate gives

$$V_B = \$42.50\left(\frac{1}{1.0475}\right)^1 + \cdots + \$42.50\left(\frac{1}{1.0475}\right)^{25} + \$42.50\left(\frac{1}{1.0475}\right)^{26} + \$1{,}000\left(\frac{1}{1.0475}\right)^{26}$$

$$= \$42.50\left(\frac{1-\left(\frac{1}{1.0475}\right)^{26}}{.0475}\right) + \$1{,}000\left(\frac{1}{1.0475}\right)^{26}$$

$$= \$42.50\,(14.753197) + \$1{,}000\,(.299223)$$

$$= \$627.01 + \$299.22 = \$\underline{\underline{926.23}}$$

(a slightly different answer from the $927.09 computed with annual payments in question 6; there is a difference between getting paid annually and getting an equal total paid semiannually). [BA II Plus financial calculator: enter $85 ÷ 2 = PMT; $1,000 FV; 9.5 ÷ 2 = I/Y; 13 x 2 = N; CPT PV; it shows –$926.23.] Recall that when market interest rates rise, the values of existing fixed-income investments fall.

It should be intuitively clear that the unchanging series of 4.25% periodic cash flows can represent a 4.75% periodic rate of return only if the buyer pays a price less than the $1,000 par value.

What if, instead, either the overall economy or the company has changed such that potential lenders will accept an APR lower than 8.5%; parties lending to companies like GHI for 13 years require only a 7.5% APR? That is, if a firm like GHI were to issue new 13-year bonds today, they would pay a 7.5% annual coupon interest rate (coupon rates are quoted in APR terms). [Either interest rates across the economy have fallen, or else GHI has come to be seen as less risky to lend to.] Discounting the series of 13 x 2 = twenty-six $42.50 interest payments to be received each half-year, and the $1,000 principal to be paid back when the bond matures, to present values at a 7.5% ÷ 2 = 3.75% required periodic (semiannual) rate gives

$$V_B = \$42.50\left(\frac{1-\left(\frac{1}{1.0375}\right)^{26}}{.0375}\right) + \$1{,}000\left(\frac{1}{1.0375}\right)^{26}$$

$$= \$42.50\ (16.427185) + \$1{,}000\ (.383981)$$

$$= \$698.16 + \$383.98 = \$\underline{\underline{1{,}082.14}}$$

(a slightly different answer from the $1,081.26 computed with annual payments in question 6; again, there is a difference between annual payments and an equal total paid semiannually). [BA II Plus calculator: enter $85 ÷ 2 = PMT; $1,000 FV; 7.5 ÷ 2 = I/Y; 13 x 2 = N; CPT PV; it shows –$1,082.14.] Recall that when market interest rates fall, the values of existing fixed-income investments rise.

It should be intuitively clear that the unchanging series of 4.25% periodic cash flows will represent a 3.75% periodic rate of return only if the buyer pays a price higher than the $1,000 par value.

Finally, what if the state of the economy and the company are such that parties lending to GHI for 13 years would still require an 8.5% APR? That is, if a company like GHI were to issue new 13-year bonds today, they would pay an 8.5% annual coupon rate. (Interest rates across the economy have remained stable, and/or GHI is no more or less risky to lend to than it was seven years earlier.) Discounting the series of twenty-six $42.50 interest payments to be received each half-year, and the $1,000 principal to be paid back when the bond matures, to present values at a 4.25% periodic (semiannual) discount rate gives

$$V_B = \$42.50\left(\frac{1-\left(\frac{1}{1.0425}\right)^{26}}{.0425}\right) + \$1{,}000\left(\frac{1}{1.0425}\right)^{26}$$

$$= \$42.50\ (15.556198) + \$1{,}000\ (.338862)$$

$$= \$661.14 + \$338.86 = \$\underline{\underline{1{,}000.00}}$$

[Texas Instruments BA II Plus financial calculator: enter $85 ÷ 2 = PMT; $1,000 FV; 8.5 ÷ 2 = I/Y; 13 x 2 = N; CPT PV; it shows –$1,000.00.] If investors' required periodic rate of return is equal to the periodic rate indicated by the bond's annual coupon rate – here, the 4.25% required semiannual rate of return equals the 8.5% ÷ 2 = 4.25% periodic rate indicated by the 8.5% annual coupon rate –

then the bond's cash flows are equivalent to what an investor would receive on a newly issued bond, and the existing bond should be worth its par value. It should be intuitively clear that if the buyer pays the $1,000 par value, the unchanging series of 4.25% periodic cash flows provides a 4.25% periodic rate of return.

8. [Similar to problem 3, for extra practice.] HIJ Company issued three billion dollars' worth of bonds seven years ago. These bonds had 20-year original maturities; therefore they will mature in thirteen years. Each individual bond has a $1,000 par value and an 8.5% annual coupon interest rate, with interest paid *semiannually*. If rational buyers of bonds with similar risk, 13-year remaining lives, and semiannual interest payments today require a 9.5% *effective annual rate* (EAR) of return, or yield to maturity (sometimes specified as the effective yield to maturity), what should they pay for each of these HIJ bonds? What if such buyers require a 7.5% EAR? What if they require an 8.6806% EAR?

Type: Bond Valuation; Semiannual Coupon Payments. This problem is quite similar to #7 above, with $42.50 in interest expected every six months for 13 years (26 times) and a return of the $1,000 in principal at the end of half-year 26. The difference here is that we know investors' required effective annual rates (EAR) of return, not their required annual percentage rates (APR) of return. The EAR for a bond investment is known as the yield to maturity, or YTM (we treat it that way; some sources treat the YTM as an APR and might give the EAR measure the name effective yield to maturity). Recall that we compute a bond's value by discounting the expected periodic cash flows at a periodic required rate of return. Because we interpret the YTM as a compounded EAR measure, we must "un-compound" it to find the corresponding periodic rate of return. If the YTM is 9.5%, then the semiannual periodic rate is $\sqrt[2]{1.095}$ – 1 = 4.6422%.

[If you know the APR, as in question 7 above, just divide it by n to get the periodic rate. But if you know only the EAR, you must take the nth root of (1 + r), and then subtract 1, to get the periodic rate. Because the EAR is the correct measure to use in comparing different investments, in our examples involving bonds with semiannual interest payments we are likely to be told the YTM (EAR), and thus must un-compound it to get the periodic rate needed in our bond valuation formula

$$V_B = \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1}{1+r}\right)^1 + \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1}{1+r}\right)^2 + \cdots + \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1}{1+r}\right)^{n-1} + \left[\begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix} + \begin{matrix}\text{Ending}\\\text{Amount}\end{matrix}\right]\left(\frac{1}{1+r}\right)^n$$

$$= \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right) + \begin{matrix}\text{Ending}\\\text{Amount}\end{matrix}\left(\frac{1}{1+r}\right)^n .]$$

With a YTM of 9.5% and an accompanying $\sqrt[2]{1.095}$ – 1 = 4.6422% required semiannual periodic rate of return, the bond's value should be

$$V_B = \$42.50\left(\frac{1}{1.046422}\right)^1 + \cdots + \$42.50\left(\frac{1}{1.046422}\right)^{25} + \$42.50\left(\frac{1}{1.046422}\right)^{26} + \$1{,}000\left(\frac{1}{1.046422}\right)^{26}$$

$$= \$42.50\left(\frac{1-\left(\frac{1}{1.046422}\right)^{26}}{.046422}\right) + \$1{,}000\left(\frac{1}{1.046422}\right)^{26}$$

$$= \$42.50\ (14.920830) + \$1{,}000\ (.307338)$$

$$= \$634.14 + \$307.34 = \$\underline{\underline{941.48}}$$

[Texas Instruments BA II Plus financial calculator: enter $42.50 PMT; $1,000 FV; 1.095 $\sqrt[2]{x}$ – 1 = x 100 = I/Y; 13 x 2 = N; CPT PV; we get –$941.48.] Let's make sure all these numbers make sense. Buyers of these bonds require a 4.6422% semiannual rate of return. Thus if a company like HIJ were to issue new 13-year bonds today, they would pay a coupon rate (an APR measure, which does not take into account intra-year compounding) of 4.6422% x 2 = 9.2845%. With semiannual compounding, the investor's compounded annual return would end up being $(1.046422)^2 - 1 = 9.5\%$. So if we knew the 9.2845% APR we would simply divide it by 2 to get the 4.6422% semiannual periodic rate needed in computing the bond's value; but here we know the 9.5% YTM/EAR, so we take $\sqrt[2]{1.095} - 1$ to get that 4.6422% semiannual periodic discount rate. Note also that when these bonds were issued investors expected an 8.5% ÷ 2 = 4.25% semiannual rate of return and a $(1.0425)^2 - 1 = 8.6806\%$ YTM/EAR. Since the required EAR has risen from 8.6806% to 9.5%, the bond's value should be less than the $1,000 par.

With a required YTM of 7.5% and an accompanying $\sqrt[2]{1.075} - 1 = 3.6822\%$ required semiannual periodic rate of return, the bond's value should be

$$V_B = \$42.50 \left(\frac{1 - \left(\frac{1}{1.036822}\right)^{26}}{.036822} \right) + \$1{,}000 \left(\frac{1}{1.036822} \right)^{26}$$
$$= \$42.50\ (16.550891) + \$1{,}000\ (.390562)$$
$$= \$703.41 + \$390.56 = \$\underline{\underline{1{,}093.97}}$$

[BA II Plus financial calculator: enter $42.50 PMT; $1,000 FV; 1.075 $\sqrt[2]{x}$ – 1 = x 100 = I/Y; 13 x 2 = N; CPT PV; we get –$1,093.97.] Buyers of these bonds now require a 3.6822% semiannual rate of return. Thus a company like HIJ issuing new 13-year bonds today would pay a coupon rate (an APR measure, which does not take into account intra-year compounding) of 3.6822% x 2 = 7.3644%. With semiannual compounding, the investor's compounded return each year would be $(1.036822)^2 - 1 = 7.5\%$. So if we knew the 7.3644% APR we would simply divide it by 2 to get the 3.6822% semiannual periodic rate needed in computing the bond's value; but here we know the 7.5% YTM/EAR, so we have to take $\sqrt[2]{1.075} - 1$ to get that 3.6822% semiannual periodic discount rate. Recall that investors expected an 8.5% ÷ 2 = 4.25% semiannual rate of return and a $(1.0425)^2 - 1 =$ 8.6806% EAR, or YTM, when these bonds were issued, so with the required EAR/YTM now at a lower 7.5%, the bond's value should be more than the $1,000 par.

Finally, with a YTM of 8.6806% and an accompanying $\sqrt[2]{1.086806} - 1 = 4.25\%$ required semiannual periodic rate of return, the value should be

$$V_B = \$42.50 \left(\frac{1 - \left(\frac{1}{1.0425}\right)^{26}}{.0425} \right) + \$1{,}000 \left(\frac{1}{1.0425} \right)^{26}$$
$$= \$42.50\ (15.556198) + \$1{,}000\ (.338862)$$
$$= \$661.14 + \$338.86 = \$\underline{\underline{1{,}000.00}}$$

[BA II Plus calculator: enter $42.50 PMT; $1,000 FV; 1.086806 $\sqrt[2]{x}$ – 1 = x 100 = I/Y; 13 x 2 = N; CPT PV; we get –$1,000.00.] With buyers of these bonds now requiring 4.25% semiannually, if a company like HIJ were to issue new 13-year bonds today, they would pay a coupon rate (an APR, with no accounting for intra-year compounding) of .0425 x 2 = 8.5%. With semiannual compounding, the investor's compounded effective annual rate of return each year would be $(1.0425)^2 - 1 =$ 8.6806%. So if we knew the 8.5% required APR we would simply divide it by 2 to get the 4.25% semiannual periodic rate for computing the bond's value; but here we know the 8.6806% required YTM/EAR, so we take $\sqrt[2]{1.086806}$ - 1 to get that 4.25% periodic discount rate. Investors expected an 8.5% ÷ 2 = 4.25% semiannual return and a $(1.0425)^2 - 1 = 8.6806\%$ EAR, or YTM, when the bonds were issued, so with the required EAR now also 8.6806% each bond should be worth its $1,000 par value.

9. IJK Company has issued more than one billion dollars' worth of bonds. Each individual bond has a $1,000 par value and a 10.6% annual coupon interest rate, with interest paid semiannually. Today rational buyers of bonds with similar features require an 8.785% yield to maturity (which we interpret as an effective annual rate of return). What price should each of these bonds sell for today if 20 years remain until the maturity date? 10 years? 2 years? What if rational investors instead required a 13% yield to maturity? A 10.88% yield to maturity?

Type: Bond Valuation; Semiannual Coupon Payments. The point of this problem is to show how long-term bonds' values are more subject to change, as required rates of return change, than are otherwise-similar short-term bonds. As always, our bond valuation equation is

$$V_B = \begin{matrix}\text{Coupon}\\ \text{Payment}\end{matrix} \left(\frac{1 - \left(\frac{1}{1+r}\right)^n}{r} \right) + \begin{matrix}\text{Ending}\\ \text{Amount}\end{matrix} \left(\frac{1}{1+r}\right)^n$$

Here we have a semiannual Coupon Payment of (10.6% x $1,000) ÷ 2 = $53.00; an Ending Amount of $1,000; and an initial semiannual r of $\sqrt[2]{1.08785}$ – 1 = 4.3%. So the indicated values are

$$V_B = \$53 \left(\frac{1 - \left(\frac{1}{1.043}\right)^{40}}{.043} \right) + \$1{,}000 \left(\frac{1}{1.043}\right)^{40} = \$\underline{1{,}189.39}\ ,$$

$$V_B = \$53 \left(\frac{1 - \left(\frac{1}{1.043}\right)^{20}}{.043} \right) + \$1{,}000 \left(\frac{1}{1.043}\right)^{20} = \$\underline{1{,}132.36}\ , \text{ and}$$

$$V_B = \$53 \left(\frac{1 - \left(\frac{1}{1.043}\right)^{4}}{.043} \right) + \$1{,}000 \left(\frac{1}{1.043}\right)^{4} = \$\underline{1{,}036.04}$$

for the 20-year (40 half-year), 10-year, and 2-year cases, respectively. What we see is that when the required periodic rate of return (4.3%) is less than the periodic coupon rate (5.3%), the bond is worth more than the par value the holder might be thought of as having a "5.3% bond in a 4.3% world" – and the premium over par is higher if time remaining before maturity is longer. So if the required periodic rate of return remains the same as time passes [an unlikely occurrence actually, since movement along a stable but upward-sloping yield curve would lead to lower required returns as we move closer to the bond's maturity date], the market value of a bond that sells for more than the $1,000 par value moves closer to par as we move closer to the maturity date. If r were instead $\sqrt[2]{1.13}$ – 1 = 6.3%, the respective values would be

$$V_B = \$53 \left(\frac{1 - \left(\frac{1}{1.063}\right)^{40}}{.063} \right) + \$1{,}000 \left(\frac{1}{1.063}\right)^{40} = \$\underline{855.05}\ ,$$

$$V_B = \$53 \left(\frac{1 - \left(\frac{1}{1.063}\right)^{20}}{.063} \right) + \$1{,}000 \left(\frac{1}{1.063}\right)^{20} = \$\underline{888.04}\ , \text{ and}$$

$$V_B = \$53\left(\frac{1-\left(\frac{1}{1.063}\right)^4}{.063}\right) + \$1{,}000\left(\frac{1}{1.063}\right)^4 = \$\underline{965.59}$$

When the required periodic rate of return (6.3%) is greater than the periodic coupon rate (5.3%), the bond is worth less (a "5.3% bond in a 6.3% world") than the par value – and the price is progressively lower as the time remaining before maturity gets longer. So if the required periodic rate of return remains the same as time passes, the market value of a bond that sells for less than the $1,000 par value moves closer to par as we move closer to the maturity date.

Notice, in the previous examples, the percentage changes in the bond's value for a given maturity. If the required semiannual return were to rise from 4.3% to 6.3%: the 20-year bond would fall in value by $1,189.39 – $855.05 = $334.34, which is $334.34 ÷ $1,189.39 = 28%; while the 10-year bond's value would fall by $1,132.36 – $888.04 = $244.32 ($244.32 ÷ $1,132.36 = 22%); but the 2-year bond's value would fall by $1,036.04 – $965.59 = $70.45 ($70.45 ÷ $1,036.04 = only 8%). However, with a semiannual required return r of $\sqrt[2]{1.1088}$ - 1 = 5.3%, the respective values would be

$$V_B = \$53\left(\frac{1-\left(\frac{1}{1.053}\right)^{40}}{.053}\right) + \$1{,}000\left(\frac{1}{1.053}\right)^{40} = \$873.27 + \$126.73 = \$\underline{1{,}000.00}$$

$$V_B = \$53\left(\frac{1-\left(\frac{1}{1.053}\right)^{20}}{.053}\right) + \$1{,}000\left(\frac{1}{1.053}\right)^{20} = \$644.01 + \$355.99 = \$\underline{1{,}000.00} \text{ , and}$$

$$V_B = \$53\left(\frac{1-\left(\frac{1}{1.053}\right)^{4}}{.053}\right) + \$1{,}000\left(\frac{1}{1.053}\right)^{4} = \$186.63 + \$813.37 = \$\underline{1{,}000.00}$$

If the required periodic rate of return (5.3%) is equal to the periodic coupon rate (5.3%), the bond is worth its par value (a "5.3% bond in a 5.3% world") no matter how much time remains until the maturity date. And once again, we see that a change in the required return causes a longer-term bond's value to change by a greater percentage than a shorter-term bond's value. If the required semiannual return were to rise from 4.3% to 5.3%: the 20-year bond would fall in value by $1,189.39 – $1,000.00 = $189.33, which is $189.39 ÷ $1,189.39 = 16%; while the 10-year bond would fall in value by $1,132.36 – $1,000.00 = $132.36, which is $132.36 ÷ $1,132.36 = 12%; but the 2-year bond would fall in value by $1,036.04 – $1,000.00 = $36.04, which is $36.04 ÷ $1,036.04 = only 3.5%.

Finally, note in the three examples above that as we get closer and closer to the maturity date, the present value of the stream of coupon payments constitutes a progressively smaller portion of the bond's total value ($873.27 vs. $644.01 vs. $186.63, since as time passes there are fewer coupon interest payments yet to be collected); while the present value of the par value to be received at maturity constitutes a progressively larger portion ($126.73 vs. $355.99 vs. $813.37, since as time passes there is less time to wait before receiving that large single payment).

10. Eleven years ago, JKL Company issued several hundred million dollars' worth of bonds with 30-year maturities. Each individual bond has a $1,000 par value and a 7.75% annual coupon interest rate, with interest payments made *annually*. What yield to maturity does an investor receive if she buys one of these bonds today for $892.23?

Type: Yield to Maturity; Annual Coupon Payments. In the earlier problems we used our multi-purpose bond value formula

$$V_B = \begin{matrix}\text{Coupon}\\ \text{Payment}\end{matrix} \left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right) + \begin{matrix}\text{Ending}\\ \text{Amount}\end{matrix} \left(\frac{1}{1+r}\right)^n$$

to compute a bond's theoretical value given the effective annual rate of return that investors currently require, based on the risks they perceive in lending to the company in question for the remainder of the bond's life. That required rate of return is known as the yield to maturity (YTM). Here we instead know the price informed people are paying today to buy each of the bonds, and must determine what effective annual rate of return (YTM) they are earning if they pay today's market price and then collect the promised interest and principal payments until the maturity date.

Here, V_B is each bond's \$892.23 current market value, and with annual payments the Coupon Payment should be 7.75% x \$1,000 = \$77.50 every year. Because the bonds were issued 11 years ago with 30-year lives, they should mature in 19 more years. Plugging these figures into our bond valuation equation, we find

$$\$892.23 = \$77.50 \left(\frac{1-\left(\frac{1}{1+r}\right)^{19}}{r}\right) + \$1{,}000 \left(\frac{1}{1+r}\right)^{19}$$

and must solve for the unknown r. For exam purposes, you should be able to set up, or at least recognize, this equation (you will not be asked to solve such an equation, because the trial-and-error procedure needed in solving for r could eat up a lot of time). An investor would pay \$892.23, and then get back \$77.50 nineteen times, plus \$1,000 at the end of the 19th year, for a total of (\$77.50 x 19) + \$1,000 = \$2,472.50 in total. Paying \$892.23 and then getting back \$2,472.50 (albeit in bits and pieces over many years) has to represent a positive rate of return; we find that return using trial and error (with both r and r^{19} in the above single equation, we can not solve for r directly).

The figure that ends up solving for r is 8.95% [we knew that it had to be greater than 7.75%, since the bond's value would fall below the \$1,000 par value only if the required periodic (here, annual) rate of return were to increase above the 7.75% periodic (here, annual) coupon rate]; double-check:

$$\$77.50 \left(\frac{1-\left(\frac{1}{1.0895}\right)^{19}}{.0895}\right) + \$1{,}000 \left(\frac{1}{1.0895}\right)^{19}$$
$$= \$77.50\ (8.981089) + \$1{,}000\ (.196193) = \$892.23\ \checkmark$$

[Financial calculators like the Texas Instruments BA II Plus are programmed to solve trial-and-error exercises quickly; for this question we would enter \$892.23 and then the +/- key and then PV (negative because that is what an investor pays out to buy the bond today); \$77.50 PMT; \$1,000 FV; 19 N; CPT I/Y. The screen will go blank for a couple of seconds as the calculator does its trial-and-error work; ultimately it should show 8.95.]

So what is the yield to maturity? Here it is simply <u>8.95%</u>; with annual payments we do not have to convert from a semiannual r we work with to an annual rate we talk about, or worry about APR vs. EAR distinctions. But if the interest payments were made semiannually the r that solved the equation would be a required semiannual rate of return; we would have to compound it out as $(1 + r)^2 - 1$ to get the computed effective annual rate (EAR) of return that we know as the yield to maturity in our coverage (see the following problem).

11. Eleven years ago, KLM Company issued several hundred million dollars' worth of bonds with 30-year maturities. Each individual bond has a $1,000 par value and a 7.75% annual coupon interest rate, with interest payments made *semiannually*. What yield to maturity does an investor receive if he buys one of these bonds today for $1,035.41?

Type: Yield to Maturity; Semiannual Coupon Payments. As in the prior problem, we know what intelligent people are paying today to buy each of these bonds, and must determine what effective annual rate of return (which, when analyzing bonds, we call the yield to maturity) they earn if they pay today's market price and then collect the promised interest and principal payments until the maturity date. Here, V_B is each bond's $1,035.41 current market value, and the semiannual Coupon Payment should be (7.75% ÷ 2) x $1,000 = $38.75 every six months. Because the bonds were issued 11 years ago with 30-year lives, they should mature in 19 more years = 38 half-years. Plugging these figures into our bond valuation equation, we find

$$\$1{,}035.41 = \$38.75 \left(\frac{1-\left(\frac{1}{1+r}\right)^{38}}{r}\right) + \$1{,}000 \left(\frac{1}{1+r}\right)^{38}$$

and must solve for the unknown r. For exam purposes, you should be able to set up, or at least recognize, this equation (you will not be asked to solve such an equation, because the trial-and-error procedure needed in solving for r could eat up a lot of time). An investor would pay $1,035.41, and then get back $38.75 thirty-eight times, plus $1,000 at the end of the 38th half-year, for a total of ($38.75 x 38) + $1,000 = $2,472.50. Paying $1,035.41 and then getting back $2,472.50 (albeit in bits and pieces over many half-years) has to represent a positive rate of return; we find it using trial and error (with both r and r^{38} in the above single equation, we can not solve for r directly).

The value that ends up solving for r is 3.7% [we knew that it had to be less than 3.875%, since the bond's value would rise above the $1,000 par value only if the required periodic (here, semiannual) rate of return were to fall below the 3.875% periodic (here, semiannual) coupon rate]; double-check:

$$\$38.75 \left(\frac{1-\left(\frac{1}{1.037}\right)^{38}}{.037}\right) + \$1{,}000 \left(\frac{1}{1.037}\right)^{38}$$

$$= \$38.75\ (20.231776) + \$1{,}000\ (.251424) = \$1{,}035.41\ \checkmark$$

[Financial calculators like the BA II Plus are programmed to solve trial & error exercises quickly; for this question we would enter $1,035.41 and then the +/- key and then PV (negative because that is what an investor pays to buy the bond today); $77.50 ÷ 2 = PMT; $1,000 FV; 19 x 2 = N; CPT I/Y. The screen will go blank briefly as the trial-and-error work proceeds; ultimately it should show 3.7.]

So what is the yield to maturity? Here it is NOT simply 3.7%; with semiannual payments we must convert our computed r from a semiannual rate we work with to an annual rate we talk about. Because 3.7% is investors' required semiannual rate of return, we annualize it to our EAR-based yield to maturity by taking $(1.037)^2 - 1 = \underline{\underline{7.5369\%}}$. Recall that we treat the YTM as an effective annual rate (EAR) of return measure and a coupon rate as an annual percentage rate (APR) measure (though some reference sources treat the YTM as an APR and specify "effective yield to maturity" as an EAR measure). Taking .037 x 2 = 7.4% gives the APR that investors today require for lending to KLM for 38 half-years. So if KLM were to issue new 19-year bonds with semiannual interest today (which they likely have no intention of doing), they would offer a coupon rate of 7.4% which, with semiannual compounding, would give investors an EAR (which we call the YTM) of 7.5369%.

12. Eleven years ago, LMN Company issued several hundred million dollars' worth of bonds with 30-year maturities. Each individual bond will mature at a price of $1,000, but no periodic interest payments are to be made (thus we are looking at zero-coupon bonds). What yield to maturity does an investor receive if she buys one of these bonds today for $111.29?

Type: Yield to Maturity; Zero-Coupon Bond. Once again we know the price that intelligent people are paying today to buy each bond, and must determine what EAR (yield to maturity) they are earning if they pay today's market price and then collect the promised payments over the rest of the bond's life – which, here, is just one $1,000 inflow to be received at the end of year 19 (recall that the bonds were issued with 30-year lives 11 years ago, so 19 years remain). As always, we can use our general bond value equation:

$$V_B = \begin{matrix}\text{Coupon}\\ \text{Payment}\end{matrix} \left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right) + \begin{matrix}\text{Ending}\\ \text{Amount}\end{matrix} \left(\frac{1}{1+r}\right)^n$$

Here, V_B is each bond's $111.29 current market value, n is 19, the Ending Amount is the $1,000 the investor will receive when the bond matures, and Coupon Payment is $0 (there are no coupon payments; thus we have a zero-coupon bond). Plugging these figures into our bond valuation equation, we find

$$\$111.29 = \$0 \left(\frac{1-\left(\frac{1}{1+r}\right)^{19}}{r}\right) + \$1{,}000 \left(\frac{1}{1+r}\right)^{19}$$

$$\$111.29 = \$1{,}000 \left(\frac{1}{1+r}\right)^{19}$$

$$\$111.29 \div \left(\frac{1}{1+r}\right)^{19} = \$1{,}000 \text{ , THUS}$$

$$\$111.29\ (1 + r)^{19} = \$1{,}000$$

$$(1 + r)^{19} = 8.985533$$

$$\sqrt[19]{(1\ +\ r)^{19}} = \sqrt[19]{8.985533}$$

$$(1 + r) = 8.985533^{1/19} \quad \text{OR} \quad 8.985533^{.052632}$$

$$(1 + r) = 1.1225, \text{ so } r = .1225 \text{ or } \underline{\underline{12.25\%}}$$

[BA II Plus financial calculator: enter $111.29 and then the +/- key and then PV (negative because that is what an investor pays out to buy the bond today); $0 PMT; $1,000 FV; 19 N; CPT I/Y; it should show 12.25.] The yield to maturity is 12.25%; this is the compounded effective annual rate of return the investor will receive if she buys this zero-coupon bond today for $111.29 and then holds it until it matures and she gets $1,000.

Note that a zero-coupon bond's yield to maturity can be solved for directly; we do not have to resort to trial and error. (Here we do not have both r and r^{19}; we have only r^{19}, so we can solve directly with just one equation by taking the 19^{th} roots of both sides of that equation.) Because a zero-coupon bond has no periodic interest payments, it does not matter whether we think in terms of annual or semiannual periods – so we might as well think annual, so that we do not have to convert the solved-for r to an annual measure and deal with the APR vs. EAR issue. But what if someone wants to think of all bonds in semiannual terms? Then we simply restate the equation above as

$$\$111.29 = \$0\left(\frac{1-\left(\frac{1}{1+r}\right)^{38}}{r}\right) + \$1{,}000\left(\frac{1}{1+r}\right)^{38}$$

$$\$111.29 = \$1{,}000\left(\frac{1}{1+r}\right)^{38}$$

$$\$111.29 \div \left(\frac{1}{1+r}\right)^{38} = \$1{,}000\text{ , THUS}$$

$$\$111.29\,(1+r)^{38} = \$1{,}000$$

$$(1+r)^{38} = 8.985533$$

$$\sqrt[38]{(1+r)^{38}} = \sqrt[38]{8.985533}$$

$$(1+r) = 8.985533^{1/38}\quad\text{OR}\quad 8.985533^{.026316}$$

$$(1+r) = 1.059481,\ r = .059481\text{ or }\underline{5.9481\%}$$

[BA II Plus financial calculator: enter $111.29 and the +/- key and then PV; $0 PMT; $1,000 FV; 19 x 2 = N; CPT I/Y; it should show 5.9481.] But now we are dealing with semiannual periods, so r is investors' required semiannual return; we annualize this semiannual rate we work with to an EAR, or yield to maturity, that we can talk about as $(1.059481)^2 - 1 = \underline{12.25\%}$.

13. Two years ago, MNO Company issued several hundred million dollars' worth of bonds with 30-year original maturities and an 8.25% annual coupon interest rate, with interest payments to be made semiannually. Today Ms. Investor, a wealthy individual, buys some of these $1,000 par value bonds at the market price of $1,068.33 each. What is her yield to maturity on this investment?

Type: Yield to Maturity; Semiannual Coupon Payments. It is always helpful to deal with bond problems by setting up our general bond value equation:

$$V_B = \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1-\left(\frac{1}{1+r}\right)^{n}}{r}\right) + \begin{matrix}\text{Ending}\\\text{Amount}\end{matrix}\left(\frac{1}{1+r}\right)^{n}$$

As in some earlier examples, we know the price that well-informed people are paying today to buy each bond, and must determine what yield to maturity (EAR) they earn by paying today's market price and then collecting the promised interest and principal payments until the maturity date. In this case, V_B is the bond's $1,068.33 current market value, and with interest paid semiannually the Coupon Payment should be (8.25% x $1,000) ÷ 2 = $41.25 every six months. Because each of these bonds was issued two years ago with a 30-year life, each should mature in 28 more years = 56 half-years. Plugging into our bond valuation equation, we find

$$\$1{,}068.33 = \$41.25\left(\frac{1-\left(\frac{1}{1+r}\right)^{56}}{r}\right) + \$1{,}000\left(\frac{1}{1+r}\right)^{56}\ ,$$

and must solve for the unknown r. For exam purposes, you should be able to set up, or at least recognize, this equation (you will not be asked to solve such an equation, because the trial-and-error procedure needed in solving for r could eat up a lot of time). An investor would pay $1,068.33, and then get back $41.25 fifty-six times, plus $1,000 at the end of the 56th half-year, for a total of ($41.25 x 56) + $1,000 = $3,310.00. Paying $1,068.33 and then getting back $3,310.00 (albeit in smaller bits and pieces over many half-years) has to represent a positive average periodic rate of return; we find that return using trial and error (with both r and r^{56} in the single equation above, we can not solve for r directly).

The value that ends up solving for r is 3.827133% [we knew that it had to be less than 4.125%, since the bond's value would rise above the $1,000 par value only if the required periodic (here, semiannual) rate of return were to fall below the 4.125% periodic (here, semiannual) coupon rate]; double-check:

$$\$41.25 \left(\frac{1-\left(\frac{1}{1.03827133}\right)^{56}}{.03827133}\right) + \$1{,}000 \left(\frac{1}{1.03827133}\right)^{56}$$

$$= \$41.25\ (22.939753) + \$1{,}000\ (.122065) = \$1{,}068.33\ \checkmark$$

[Financial calculators like the BA II Plus are programmed to solve trial-and-error exercises quickly; for this question we would enter $1,068.33 and then the +/- key and then PV (negative because that is what an investor pays out to buy the bond today); $82.50 ÷ 2 = PMT; $1,000 FV; 28 x 2 = N; CPT I/Y. The screen will go blank for a couple of seconds as the trial & error work proceeds; ultimately it should show 3.827133.]

But recall that the yield to maturity is not simply 3.827133%; with semiannual payments we must annualize our computed r, which is investors' required semiannual rate of return, by taking $(1.03827133)^2 - 1$ = $\underline{\underline{7.8007}}$%. Recall that what we refer to in our class as the yield to maturity is an effective annual rate (EAR) of return measure, while a coupon rate is an annual percentage rate (APR) measure – and if the cash flows do not occur annually we must distinguish between the two (whereas the APR and EAR are equal if payments are annual). Taking .03827133 x 2 = 7.6543% gives us the APR that investors today would require for lending to MNO for 56 half-years. So if MNO were to issue new 28-year bonds with semiannual interest payments today (which they are not planning to do), its managers would offer a coupon interest rate of 7.6543% which, with semiannual compounding, would give investors an EAR (YTM) of $(1.03827133)^2 - 1$ = 7.8007%.

14. Now Ms. Investor (see the previous question) realizes that the MNO bonds she bought contain a *call provision*. Specifically, the company can *call* the bonds (buy them back from Ms. Investor and other bond holders, even if those holders do not wish to sell) as soon as ten years have passed from the date of issue, which was two years ago. If the bonds are called, MNO will pay a price of $1,082.50 (the par value plus an extra year's worth of interest) for each. If Ms. Investor buys each bond today for $1,068.33, and then MNO calls the entire bond issue at the earliest possible date – which Ms. Investor fears they may do, since the roughly 7.8% APR required by the market today is lower than the 8.25% annual coupon rate on the callable bonds she holds – what will be her yield to first call?

Type: Yield to First Call; Semiannual Coupon Payments. Again we want to think in terms of our general bond value equation:

$$V_B = \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix} \left(\frac{1-\left(\frac{1}{1+r}\right)^{n}}{r}\right) + \begin{matrix}\text{Ending}\\\text{Amount}\end{matrix} \left(\frac{1}{1+r}\right)^{n}$$

And again we know the price that rational people are paying today to buy the bond, and must determine what effective annual rate (EAR) of return they earn by paying today's market price of V_B = $1,068.33 and then collecting the subsequent interest and principal payments. However, in this case we are not thinking about holding the bond until it matures; here the investor fears she may have to sell the bond back to MNO as early as 8 years from today (the bonds were issued with a

10-year call protection period, and 2 of those 10 years already are gone, so 8 years = 16 half-years remain before the bond holders could be forced to sell back to MNO). If that occurs, Ms. Investor will receive (8.25% x $1,000) ÷ 2 = $41.25 every six months for just 8 years, and then will get the call price of $1,082.50, along with her final interest payment, at the end of half-year 16. Plugging into our bond valuation equation, we find

$$\$1{,}068.33 = \$41.25\left(\frac{1-\left(\frac{1}{1+r}\right)^{16}}{r}\right) + \$1{,}082.50\left(\frac{1}{1+r}\right)^{16},$$

and must solve for the unknown r. For exam purposes, you should be able to set up, or at least recognize, this equation. An investor would pay $1,068.33, and then get back $41.25 sixteen times, plus $1,082.50 at the end of the 16^{th} half-year, for a total ($41.25 x 16) + $1,082.50 = $1,742.50. Paying $1,068.33 and then getting back $1,742.50 (albeit in bits and pieces over 16 half-years) has to represent a positive average periodic rate of return; we find that return using trial and error (with both r and r^{16} in the single equation above, we can not solve for r directly).

The value that ends up solving for r is 3.922320%. Let's double-check:

$$\$41.25\left(\frac{1-\left(\frac{1}{1.03922320}\right)^{16}}{.03922320}\right) + \$1{,}082.50\left(\frac{1}{1.03922320}\right)^{16}$$

$$= \$41.25\ (11.719353) + \$1{,}082.50\ (.540330) = \$1{,}068.33\ \checkmark$$

[Financial calculators like the TI BA II Plus are programmed to solve trial-and-error exercises quickly; for this question we would enter $1,068.33 and the +/- key and then PV; $82.50 ÷ 2 = PMT; $1,082.50 FV; 8 x 2 = N; CPT I/Y. The screen will go blank very briefly as the trial-and-error work proceeds; ultimately it should show 3.922320.] But recall that the yield to first call is not simply 3.922320%; with semiannual payments we must annualize our computed r, which is investors' required semiannual rate of return, by taking $(1.03922320)^2 - 1$ = <u>7.9985</u>%. In our coverage we treat the yield to first call, as we do the yield to maturity, as an effective annual rate (EAR) of return measure.

Here bond holders' required rate of return has fallen in the period since these bonds were issued. Thus there are concerns that MNO may refinance by calling the existing bonds after year 8 and issuing new bonds (borrowing new money to replace the existing debt) at a lower interest rate, so the yield to first call may be a more valid measure of investors' expected return than is the yield to maturity. However, if required returns had risen since the issue date there would be less concern that MNO would feel motivated to call these bonds (it would have little motivation to replace low-interest rate debt with more expensive debt), and the YTM would likely be the more valid measure of investors' expected annual rate of return (remember that we talk about rates in annual terms).

15. Now let's move forward in time eleven years with Ms. Investor (see the previous two questions). It turns out that the MNO bonds she bought were not called 8 years after she bought them; in fact, market interest rates began rising such that MNO ultimately had no incentive to call the bonds. However, Ms. Investor also did not hold the bonds until they matured. Instead, eleven years after she bought her bonds for $1,068.33 each, she sold them to Mr. Financial for $952.84 each. Looking back, what has been her average annual holding period yield for the 11 years she held her MNO bonds?

Type: Holding Period Yield; Semiannual Coupon Payments. It should not be surprising that once again we look to our general bond value equation:

$$V_B = \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right) + \begin{matrix}\text{Ending}\\\text{Amount}\end{matrix}\left(\frac{1}{1+r}\right)^n$$

Here we know the price that the investor originally paid for each bond (V_B = \$1,068.33), and the price she sold for at the end of her chosen holding period (Ending Amount = \$952.84). We want to compute the effective annual rate (EAR) of return she earned by paying the purchase-date market price and later selling for the sale-date market price, while receiving the periodic interest payments in between. Note that Ms. Investor received (8.25% x \$1,000) ÷ 2 = \$41.25 every six months for eleven years = 22 half-years, and then received \$952.84, along with her final interest payment, at the end of year 11 = half-year 22. Plugging into our bond valuation equation, we find

$$\$1{,}068.33 = \$41.25\left(\frac{1-\left(\frac{1}{1+r}\right)^{22}}{r}\right) + \$952.84\left(\frac{1}{1+r}\right)^{22},$$

and must solve for the unknown r. For exam purposes, you should be able to set up, or at least recognize, this equation. Ms. Investor paid \$1,068.33, and then got back \$41.25 twenty-two times, plus \$952.84 at the end of the 22nd half-year, for a total of (\$41.25 x 18) + \$952.84 = \$1,695.44. Paying \$1,068.33 and then getting back \$952.84 (albeit in bits and pieces over many half-years) has to represent a positive average periodic rate of return; we find that return using trial and error (with both r and r^{22} in the above equation, we can not solve for r directly). The value that ends up solving for r is 3.527847%. Double-check:

$$\$41.25\left(\frac{1-\left(\frac{1}{1.03527847}\right)^{22}}{.03527847}\right) + \$952.84\left(\frac{1}{1.03527847}\right)^{22}$$

$$= \$41.25\ (15.125876) + \$952.84\ (.466382) = \$1{,}068.33\ .✓$$

[BA II Plus financial calculator: enter \$1,068.33 and the +/- key and then PV; \$82.50 ÷ 2 = PMT; \$952.84 FV; 11 x 2 = N; CPT I/Y; ultimately it should show 3.527847.] But recall that the holding period yield is not simply 3.527847%; with semiannual payments we must annualize our computed average periodic r, which is this particular investor's realized semiannual rate of return, by taking $(1.03527847)^2 - 1$ = <u>7.1802%</u>. We treat the holding period yield, as we do the yield to maturity or yield to first call, as an effective annual rate (EAR) of return measure. Note that Ms. Investor got a fairly low rate of return because she made an unfavorable bet on interest rates, paying a price above \$1,000 for the bond (so the coupon payments represented a low percentage relative to the high price paid), and later selling it after market interest rates rose for a price below \$1,000.

16. NOP Company issued several hundred million dollars' worth of bonds eighteen years ago. These bonds had 35-year original maturities; therefore they will mature in 17 years. Each individual bond has a \$1,000 par value, and interest is paid semiannually. Today, rational buyers of bonds with similar risk, 4-year remaining lives, and semiannual interest payments require an 8.6% effective annual rate (EAR) of return, or yield to maturity, and they are paying a price of \$1,074.07 per bond. What annual coupon interest rate is NOP paying on this bond issue?

Type: Computing the Coupon Rate. This is, admittedly, an unusual problem, in that analyzing bonds typically calls for us to know the coupon rate while solving for either the appropriate price to pay or the yield to maturity (or other yield) indicated by a given price – but our bond valuation formula

$$V_B = \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right) + \begin{matrix}\text{Ending}\\\text{Amount}\end{matrix}\left(\frac{1}{1+r}\right)^n$$

serves us well in solving for any possible unknown. Here, V_B is the bond's $1,074.07 current market value and, with semiannual coupon payments, n represents the 17 x 2 = 34 half-years that will pass before the bond matures (when the $1,000 principal will be repaid, along with the final payment of interest). We also know that bond investors require an 8.6% yield to maturity (which, in our coverage, we treat as an effective annual rate of return, or EAR), which takes into account intra-year compounding. So we must un-compound it to get the semiannual required rate of return r for use in our computations: $\sqrt[2]{1.086}$ – 1 = .042113, or 4.2113% (such that the semiannual rate compounds out to $[1.042113]^2$ – 1 = 8.6%). Plugging these figures into our bond valuation equation, we find

$$\$1{,}074.07 = \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1-\left(\frac{1}{1.042113}\right)^{34}}{.042113}\right) + \$1{,}000\left(\frac{1}{1.042113}\right)^{34}$$

$1,074.07 = Coupon Payment (17.904676) + $1,000 (.245976)
$1,074.07 = Coupon Payment (17.904676) + $245.98
$828.09 = Coupon Payment (17.904676)
Coupon Payment = $828.09 ÷ 17.904676 = $46.25 .

[TI BA II Plus financial calculator: enter $1,074.07 and the +/- key and then PV; $1,000 FV; 17 x 2 = N; 1.086 $\sqrt[2]{x}$ – 1 = x 100 = I/Y; CPT PMT; it should show $46.25.] Every six months NOP pays $46.25 in interest, such that the annual payment of interest is $46.25 x 2 = $92.50. Thus the annual coupon interest rate is $92.50 ÷ $1,000 = 9.25%.

17. OPQ Company issued several hundred million dollars' worth of bonds 7½ years ago. Each individual bond has a $1,000 par value, and the annual coupon interest rate is 11.25%, with interest paid semiannually. Today, rational buyers of bonds with similar risk and semiannual interest payments, and with remaining lives equal to the remaining life of this OPQ bond issue, require a 12.75% effective annual rate (EAR) of return, or yield to maturity. If rational investors are currently paying $911.46 for each of these OPQ bonds, how many years will pass before they mature? What was the original life of each of these bonds?

Type: Computing Number of Time Periods. This is another admittedly unusual problem, in that analyzing bonds typically calls for us to know the remaining life while solving for either the appropriate price to pay or the yield to maturity indicated by a given price – but our bond valuation formula

$$V_B = \begin{matrix}\text{Coupon}\\\text{Payment}\end{matrix}\left(\frac{1-\left(\frac{1}{1+r}\right)^n}{r}\right) + \begin{matrix}\text{Ending}\\\text{Amount}\end{matrix}\left(\frac{1}{1+r}\right)^n$$

serves us well in solving for any possible unknown. Here, V_B is the bond's $911.46 current market value, and the Coupon Payment should be (11.25% x $1,000) ÷ 2 = $56.25 every six months. We also know that bond investors require a 12.75% yield to maturity (which we view as an effective annual rate of return, or EAR), which takes into account intra-year compounding. So we must un-compound it to get the semiannual required rate of return r for use in our computations: $\sqrt[2]{1.1275}$ – 1 = .061838, or 6.1838% (such that the semiannual rate compounds out to $[1.061838]^2$ – 1 = 12.75%). Plugging these figures into our bond valuation equation, we find

$$\$911.46 = \$56.25\left(\frac{1-\left(\frac{1}{1.061838}\right)^{n}}{.061838}\right) + \$1{,}000\left(\frac{1}{1.061838}\right)^{n}$$

$$\$911.46 = \$56.25\left(\frac{1-(.941763)^{n}}{.061838}\right) + \$1{,}000\,(.941763)^{n}$$

$$\$911.46 = \$909.634425\,[1 - (.941763)^{n}] + \$1{,}000\,(.941763)^{n}$$

$$\$911.46 = \$909.634425 - \$909.634425\,(.941763)^{n} + \$1{,}000(.941763)^{n}$$

$$\$1.825575 = \$90.365575\,(.941763)^{n}$$

$$.020202 = (.941763)^{n}$$

$$\ln .020202 = \ln[(.941763)^{n}]$$

$$\ln .020202 = n\,(\ln .941763)$$

$$-3.901968 = n\,(-.060001)$$

$$n = \underline{65}$$

[Texas instruments BA II Plus financial calculator: enter $911.46 and the +/- key and then PV; $1,000 FV; .1125 ÷ 2 = x $1,000 = PMT; 1.1275 $\sqrt[2]{x}$ – 1 = x 100 = I/Y; CPT N; it should show 65.] Because this n is a number of half-year periods, this bond issue will mature in 65 ÷ 2 = 32.5 years (thus the original life was 7.5 + 32.5 = 40 years). Let's double-check to be sure:

$$\$56.25\left(\frac{1-\left(\frac{1}{1.061838}\right)^{65}}{.061838}\right) + \$1{,}000\left(\frac{1}{1.061838}\right)^{65}$$

$$= \$56.25\,(15.843971) + \$1{,}000\,(.020240) = \$911.46\ \checkmark$$

18. PQR Company issued one billion dollars' worth of fifty-year bonds twenty-three years ago. Each bond has a $1,000 par value and a 6.75% annual coupon interest rate, with interest paid semiannually. Mr. and Mrs. Front O'Line just purchased some of these bonds at the market price of $1,091.39 each. They plan to hold each bond until their oldest child begins college in 12½ years, and then sell for the prevailing market price. If they expect their holding period yield to be 6.25%, at what price do they expect to sell each bond? What yield to maturity would that price represent for whoever buys the bonds from them?

Type: Computing Yield to Maturity, Expected Selling Price. This is another admittedly unusual problem, in that when we analyze bonds we typically are given the expected selling price (or know the actual selling price after the fact), and solve either for the appropriate price to pay today or for some rate of return (yield) measure indicated by a given price. But again, our bond valuation formula

$$V_B = \begin{matrix}\text{Coupon}\\ \text{Payment}\end{matrix}\left(\frac{1-\left(\frac{1}{1+r}\right)^{n}}{r}\right) + \begin{matrix}\text{Ending}\\ \text{Amount}\end{matrix}\left(\frac{1}{1+r}\right)^{n}$$

serves us well in solving for any possible unknown. Here, V_B is the bond's $1,091.39 current market value, and the Coupon Payment should be (6.75% x $1,000) ÷ 2 = $33.75 every six months. With 50 – 23 = 27 years (54 half-years) remaining until maturity, the O'Line family's yield to maturity is 6.15%, though we were not asked to compute that figure. [BA II Plus calculator: enter $1,091.39 and the +/- key and then PV; $1,000 FV; .0675 ÷ 2 = x $1,000 = PMT; 27 x 2 = N; CPT I/Y; it should ultimately show 3.029133 as the indicated semiannual rate of return. Enter ÷ 100 = + 1 = y^x 2 = to compound it to 1 + an annual figure and then subtract 1; it should show 6.15%.] The attendant APR is .03029133 x 2 = 6.058266%; thus the required annual rate of return in APR terms has fallen from 6.75% when the bonds were issued to 6.058% today, and that is why each of the bonds sells in the market for a price greater than its $1,000 par value.

Now we want to find the selling price that would give our bond holders a 6.25% holding period yield over a $12\frac{1}{2}$ year = 25 half-year holding period. Because the 6.25% expected HPY is an EAR, which takes into account intra-year compounding, we must undo the compounding ("un-compound") to find the semiannual rate of return to be used as our discount rate: $\sqrt[2]{1.0625}$ – 1 = .030776, or 3.0776%. Plugging into our bond analysis equation, we find:

$$\$1{,}091.39 = \$33.75 \left(\frac{1-\left(\frac{1}{1.030776}\right)^{25}}{.030776}\right) + \begin{matrix}\text{Ending}\\ \text{Amount}\end{matrix} \left(\frac{1}{1.030776}\right)^{25}$$

$$\$1{,}091.39 = \$33.75\ (17.263459) + \text{Ending Amount}\ (.468693)$$

$$\$1{,}091.39 = \$582.64 + \text{Ending Amount}\ (.468693)$$

$$\$508.75 = \text{Ending Amount}\ (.468693)$$

$$\$508.75 \div .468693 = \text{Ending Amount} = \$\underline{\underline{1{,}085.46}}$$

[BA II Plus financial calculator: enter $1,091.39 and the +/- key and then PV; .0675 ÷ 2 = x $1,000 = PMT; 1.0625 $\sqrt[2]{x}$ – 1 = x 100 = I/Y; 12.5 x 2 = N; CPT FV; it should show $1,085.46.] Because the couple's expected HPY (6.25%) slightly exceeds their YTM (6.15%), it makes sense that they expect to sell for a price slightly above the maturity value. Whoever pays that price will get a bond with a 14.5 year remaining life (original 50-year life minus 23 years that already have passed minus 12.5 years that the O'Lines plan to hold it). Thus we think the buyer will pay $1,085.46 for the expectation of receiving $33.75 in interest 14.5 x 2 = 29 times and $1,000 at the end of the 29th half-year. We use trial and error to solve for r in the equation:

$$\$1{,}085.46 = \$33.75 \left(\frac{1-\left(\frac{1}{1+r}\right)^{29}}{r}\right) + \$1{,}000 \left(\frac{1}{1+r}\right)^{29}$$

For exam purposes, you should be able to set up, or at least recognize, the equation we use in solving for a bond's yield. The value that ends up solving for r is 2.933353%. Double-check:

$$\$33.75 \left(\frac{1-\left(\frac{1}{1.02933353}\right)^{29}}{.02933353}\right) + \$1{,}000 \left(\frac{1}{1.02933353}\right)^{29}$$

$$= \$33.75\ (19.350316) + \$1{,}000\ (.432387) = \$1{,}085.46\ \checkmark$$

[BA II Plus financial calculator: enter $1,085.46 and the +/- key and then PV; .0675 ÷ 2 = x $1,000 = PMT; $1,000 FV; 14.5 x 2 = N; CPT I/Y; after going blank briefly it ultimately should show 2.933353.] But the yield to maturity is not simply 2.933353%; with semiannual payments we must annualize the computed semiannual rate of return r by taking $(1.02933353)^2$ – 1 = $\underline{\underline{5.9528\%}}$.

19. QRS Company issued $300 million worth of 50-year bonds 20 years ago; thus these bonds will mature in 30 years. They carry a 13.5% annual coupon interest rate, with coupon payments made semiannually. Today people who might consider lending money to QRS for 30 years would expect to earn a 5.25% semiannual rate of return, so the company's financial managers feel that they could issue $300 million of 30-year bonds with an annual coupon interest rate (an APR measure) of 5.25% x 2 = 10.5%. QRS's marginal federal + state income tax rate is 37%, and

the administrative cost of replacing the old bonds with new bonds (call premiums on the old bonds, investment banking fees on the new bonds) would total $28 million. Should QRS refund the old bonds by issuing new ones?

Type: Refunding a Bond Issue. We want to see if the present value of the after-tax savings (from paying interest at a lower rate) exceeds the cost of the refunding. Replacing $300 million of 13.5% (6.75% semiannually) debt with $300 million of 10.5% (5.25% semiannually) debt would reduce QRS's interest cost every 6 months from .0675 x $300,000,000 = $20,250,000 to .0525 x $300,000,000 = $15,750,000, for a $4,500,000 savings twice each year. Because QRS's marginal income tax rate is 37%, the after-tax savings is a lower ($4,500,000)(1 – .37) = $2,835,000 every six months. (Paying interest is a tax-deductible expense, so paying less interest also means giving up some income tax benefits.) What is the value of saving $2,835,000 every 6 months for thirty years (60 times)? [We discount at the 5.25% periodic rate that reflects QRS's cost of obtaining money from an alternative source today.]

PMT x FAC = TOT

$$\$2{,}835{,}000 \left(\frac{1 - \left(\frac{1}{1.0525}\right)^{60}}{.0525} \right) = \$51{,}493{,}501.27$$

[TI BA II Plus financial calculator: enter $2,835,000 PMT; 10.5 ÷ 2 = I/Y; 30 x 2 = N; $0 FV; CPT PV; it should show –$51,493,501.27. It shows as negative because that is the amount you willingly would pay out today to get the benefit of the savings stream from paying the lower interest rate.] Because the present value of the savings stream is about $51.5 million, whereas the administrative and related costs of the refunding activity are only $28 million, QRS would realize a net present value of $51.5 million – $28 million = $23.5 million by replacing the 13.5% coupon interest rate bonds with a new 10.5% coupon rate issue.

20. [FIL 404 only] RST Company issued hundreds of millions of dollars' worth of 10-year bonds several months ago. Specifically, these bonds will mature in 8 years and 137 days. Each individual bond has a $1,000 par value and an 11.5% annual coupon interest rate, with interest paid semiannually. If a rational investor buying a bond with the same life and similar risks requires a yield to maturity of 12.5%, what should a rational investor willingly pay for each of these RST bonds?

Type: Bond Valuation; Partial Year. Pricing bonds that do not mature a full number of periods (years, half-years) from today is complicated; one problem is that there are different traditions for different types of bonds (corporate vs. government, for example). In fact, a bond is said to sell for a "clean" price if the next coupon payment is a full 6 months away, whereas it sells for a "dirty" price if the next coupon payment will occur before the six-month anniversary. Here we will assume that interest is computed based on the actual number of days until interest is paid and a 365-day year (whereas the U.S. corporate bond market follows the custom of treating each month as 30 days and the year as 360 days). (So we are trying to get a flavor for how investors would value this bond based on time value of money logic, rather than showing how it actually would be valued based on idiosyncratic traditions/practices of the bond market.) We again use our bond valuation formula:

$$V_B = \begin{matrix}\text{Coupon}\\ \text{Payment}\end{matrix} \left(\frac{1 - \left(\frac{1}{1+r}\right)^{n}}{r} \right) + \begin{matrix}\text{Ending}\\ \text{Amount}\end{matrix} \left(\frac{1}{1+r}\right)^{n}$$

but must employ a two-step process. First, we find the value the bond will have 137 days from now, right after the next coupon payment, when 8 years = 16 half-years will remain until maturity. The coupon payment will be (11.5% x $1,000) ÷ 2 = $57.50 every 6 months, while the semiannual required rate of return is $\sqrt[2]{1.125}$ -1 = .060660, or 6.0660%. When 16 half-years remain until maturity, a rational buyer should pay

$$V_B = \$57.50 \left(\frac{1-\left(\frac{1}{1.060660}\right)^{16}}{.060660}\right) + \$1{,}000 \left(\frac{1}{1.060660}\right)^{16}$$

$$= \$57.50\ (10.060236) + \$1{,}000\ (.389744) = \$\underline{968.20}$$

In 137 days the bond will be worth $968.20, and the investor also will get a $57.50 coupon payment, for a total value on that day of $968.20 + $57.50 = $1,025.70. To see what the bond is worth today, we must discount the $1,025.70 to a present value at the daily periodic rate for 137 daily periods. With a required yield to maturity of 12.5%, the daily periodic rate is $\sqrt[365]{1.125}$ – 1 = .000323, or .0323%. With 137 days remaining until a $1,025.70 value prevails, a rational buyer today should pay

$$\$1{,}025.70 \left(\frac{1}{1.000323}\right)^{137} = \$\underline{\underline{981.34}}$$

As noted, this example gives a general idea of how a bond is valued between interest payment dates; specific practices in different bond market sectors might lead to slightly different answers.

21. A bond's yield to maturity (or yield to first call or holding period yield) is an internal rate of return measure, which does not explicitly take into account the bond holder's reinvestment of interest payments. Some analysts therefore prefer to compute a bond's *realized compound yield* (RCY, sometimes called total return; we called it modified internal rate of return in capital budgeting analysis) by looking explicitly at reinvesting the coupon payments received before the bond matures (or is called or sold). [With no coupon payments to reinvest, a zero-coupon bond has a RCY equal to its yield to maturity (YTM).] Assume that STU Company issued several hundred million dollars' worth of 35-year bonds 24 years ago. Each $1,000 par value bond has a 7% annual coupon interest rate, with semiannual interest payments, and is priced to provide a 7.1225% YTM. What will Mr. Lender's RCY be if he buys an STU bond today and then reinvests over the bond's remaining 11-year life at a 9.2025% effective annual rate (EAR) of return? What if his reinvestments instead earn a 5.0625% EAR? A 7.1225 EAR?

Type: Realized Compound Yield. Just as computing IRR required trial and error while computing MIRR did not in our capital budgeting coverage, computing YTM (the IRR on a bond investment) requires trial and error but computing realized compound yield (the MIRR on a bond investment) does not. Our first step is to compute the bond's current market price, with our bond valuation formula

$$V_B = \begin{matrix}\text{Coupon}\\ \text{Payment}\end{matrix} \left(\frac{1-\left(\frac{1}{1+r}\right)^{n}}{r}\right) + \begin{matrix}\text{Ending}\\ \text{Amount}\end{matrix} \left(\frac{1}{1+r}\right)^{n}$$

Here the Coupon Payment is (7% x $1,000) ÷ 2 = $35, the Ending Amount is the $1,000 par value, and n is the 11 years = 22 half-years remaining until the bond matures. We get the semiannual discount rate r by "un-compounding" the 7.1225% effective annual rate of return, or YTM, that investors currently expect: $\sqrt[2]{1.071225}$ – 1 = .035, or 3.5%. The bond should currently sell for a price of

$$V_B = \$35 \left(\frac{1-\left(\frac{1}{1.035}\right)^{22}}{.035}\right) + \$1{,}000 \left(\frac{1}{1.035}\right)^{22}$$

$$= \$35\ (15.167125) + \$1{,}000\ (.469151) = \$\underline{\underline{1{,}000.00}}$$

[TI BA II Plus financial calculator: enter .07 ÷ 2 = x $1,000 = PMT; $1,000 FV; 11 x 2 = N; 1.071225 $\sqrt[2]{x}$ – 1 = x 100 = I/Y; CPT PV; it should show -$1,000.] Investors today require the same 3.5% semiannual rate of return that was expected when the bonds were issued (as evidenced by the 3.5% semiannual coupon rate); thus the bonds should be expected to sell at par (a bond with a 3.5% periodic cash flow stream discounted at a 3.5% periodic rate should be worth its par value).

Our next step is to determine the amount to which the reinvested coupon payments will grow by the maturity date, a future value of an annuity computation. Twenty-four years into the bond issue's 35-year life there are 11 years = 22 half-years remaining until maturity, and each of the 22 semiannual coupon interest payments will be (7% x $1,000) ÷ 2 = $35. Assume that, right after Mr. Lender buys the bond interest rates in the market rise, leaving him to believe he will be able to reinvest all received coupon payments to earn a 9.2025% effective annual rate (EAR) of return, for a semiannual periodic rate of $\sqrt[2]{1.092025}$ -1 = .045, or 4.5%. If so then the $35 x 22 = $770 in total coupon receipts will grow, with a 4.5% periodic reinvestment rate over 22 semiannual periods, to

$$\$35\ (1.045)^{21} + \$35\ (1.045)^{20} + \$35\ (1.045)^{19} + \cdots + \$35\ (1.045)^{2} + \$35\ (1.045)^{1} + \$35\ (1.045)^{0}$$

$$= \$35\ [(1.045)^{21} + (1.045)^{20} + (1.045)^{19} + \cdots + (1.045)^{2} + (1.045)^{1} + (1.045)^{0}]$$

$$= \$35\left(\frac{(1.045)^{22}-1}{.045}\right) = \$35\ (36.303378) = \$\underline{1{,}270.62}$$

[BA II Plus financial calculator: enter $35 PMT; 1.092025 $\sqrt[2]{x}$ – 1 = x 100 = I/Y; 11 x 2 = N; $0 PV; CPT FV; it should show -$1,270.62 (negative because the party Mr. Lender invests the coupon payments with will have to pay him that amount in 11 years).] At maturity, Mr. Lender will have accumulated $1,270.62 through interest received and reinvested, and also will collect the $1,000 par (maturity) value, when there will be no opportunity to reinvest, of $1,000 $(1.045)^0$ = $1,000, for a total of

$$\$35\left(\frac{(1.045)^{22}-1}{.045}\right) + \$1{,}000\ (1.045)^{0} = \$35\ (36.303378) + \$1{,}000\ (1.00) = \$1{,}270.62 + \$1{,}000$$

= $<u>2,270.62</u> at the end of half-year 22. Investing $1,000 today and then having $2,270.62 after 22 periods corresponds to an average periodic rate of return computed as:

$$\text{BAMT}\ (1 + r)^{n} = \text{EAMT}$$
$$\$1{,}000\ (1 + r)^{22} = \$2{,}270.62$$
$$(1 + r)^{22} = 2.27062$$
$$\sqrt[22]{(1+r)^{22}} = \sqrt[22]{2.27062}$$
$$(1 + r) = 2.27062^{1/22} = 2.27062^{.045455} = 1.037979\ ,$$

for an r = .037979 semiannual return and an annualized RCY of $(1.037979)^2$ – 1 = <u>7.74%</u>. [BA II Plus calculator: without clearing your answer from above hit the +/- key and then enter + $1,000 = FV; $1,000 and then the +/- key and PV; 11 x 2 = N; $0 PMT; CPT I/Y; it should show 3.7979%.]

If the bond is to be held until maturity, then a 9.2025% reinvestment rate gives an RCY greater than the 7.1225% YTM (just as we saw in capital budgeting analysis that reinvesting at a rate above the IRR gives an MIRR greater than the IRR). Now let's assume instead that, just after Mr. Lender buys the bond, interest rates in the market decline such that he expects to be able to reinvest all received coupon interest payments at only a 5.0625% EAR (semiannual periodic rate of $\sqrt[2]{1.050625}$ – 1 = .025, or 2.5%). If this lower reinvestment rate prevails then the \$35 x 22 = \$770 in total coupon receipts will grow, with a 2.5% periodic reinvestment rate over 22 periods, to a much smaller amount, and with the \$1,000 maturity value added the total Mr. Lender has at maturity will be

$$\$35\ (1.025)^{21} + \$35\ (1.025)^{20} + \cdots + \$35\ (1.025)^{2} + \$35\ (1.025)^{1} + \$35\ (1.025)^{0} + \$1{,}000\ (1.025)^{0}$$

$$= \$35\ [(1.025)^{21} + (1.025)^{20} + (1.025)^{19} + \cdots + (1.025)^{2} + (1.025)^{1} + (1.025)^{0}] + \$1{,}000\ (1.025)^{0}$$

$$= \$35\left(\frac{(1.025)^{22} - 1}{.025}\right) + \$1{,}000\ (1.025)^{0} = \$35\ (28.862856) + \$1{,}000\ (1.00) = \$\underline{2{,}010.20}$$

At maturity he will have accumulated \$1,010.20 through interest received and reinvested, and also will collect the \$1,000 par (maturity) value, for a total \$1,010.20 + \$1,000 = \$2,010.20 value at the end of half-year 22. Investing \$1,000 today and then having \$2,010.20 after 22 semiannual periods corresponds to an average periodic rate of return computed as:

$$\$1{,}000\ (1 + r)^{22} = \$2{,}010.20$$
$$(1 + r)^{22} = 2.01020$$
$$\sqrt[22]{(1 + r)^{22}} = \sqrt[22]{2.01020}$$
$$(1 + r) = 2.01020^{1/22} = 2.01020^{.045455} = 1.032247\ ,$$

for an r = .032247 semiannual return and an annualized RCY of $(1.032247)^2$ – 1 = <u>6.5534%</u>. If the bond is to be held until it matures, a 5.0625% reinvestment rate gives an RCY below the 7.1225% YTM (just as in capital budgeting analysis a reinvestment rate less than the IRR gives an MIRR less than the IRR). Finally, we might assume that on the day he buys the bond Mr. Lender expects to reinvest any coupon interest payments received at a 7.1225% EAR (semiannual periodic rate of $\sqrt[2]{1.071225}$ -1 = .035, or 3.5%), because interest rates in the market have stayed stable and are expected to be largely unchanged until the bond's maturity date. In this scenario the \$35 x 22 = \$770 in total coupon receipts will grow, with a 3.5% periodic reinvestment rate over 22 periods, to an amount that combines with the \$1,000 par value to be received at maturity of

$$\$35\ (1.035)^{21} + \$35\ (1.035)^{20} + \cdots + \$35\ (1.035)^{2} + \$35\ (1.035)^{1} + \$35\ (1.035)^{0} + \$1{,}000\ (1.035)^{0}$$

$$= \$35\ [(1.035)^{21} + (1.035)^{20} + (1.035)^{19} + \cdots + (1.035)^{2} + (1.035)^{1} + (1.035)^{0}] + \$1{,}000\ (1.035)^{0}$$

$$\$35\left(\frac{(1.035)^{22} - 1}{.035}\right) + \$1{,}000\ (1.035)^{0} = \$35\ (32.328902) + \$1{,}000\ (1.00) = \$\underline{2{,}131.51}$$

At maturity, Mr. Lender will have accumulated \$1,131.51 through interest received and reinvested, and also will collect the \$1,000 par (maturity) value, for a total \$1,131.51 + \$1,000 = \$2,131.51 value at the end of half-year 22. Investing \$1,000 today and then having \$2,131.51 after 22 periods corresponds to an average periodic rate of return computed as:

$$\$1{,}000\ (1 + r)^{22} = \$2{,}131.51$$
$$(1 + r)^{22} = 2.13151$$
$$\sqrt[22]{(1 + r)^{22}} = \sqrt[22]{2.13151}$$
$$(1 + r) = 2.13151^{1/22} = 2.13151^{.045455} = 1.035$$

With an r = .035 semiannual return, we compute an annualized RCY of $(1.035)^2 - 1 = \underline{7.1225\%}$. If the bond is held to maturity, then a 7.1225% reinvestment rate gives an RCY = the 7.1225% YTM (just as in capital budgeting analysis MIRR = IRR only if the reinvestment rate is equal to the IRR itself).

Introductory finance texts too often leave students believing that bond investors always benefit when interest rates in the market plunge. The problem with such reasoning becomes clear when we consider reinvestment in a RCY framework. Clearly, falling interest rates present a "two-edged sword" dilemma: a bond's immediate resale value rises but the lower interest rates have a negative impact on returns generated through reinvesting any coupon interest payments received. Problem 23 below offers some insights into how bond investors can deal with the tradeoff between higher (lower) reinvestment rates and lower (higher) bond values.

22. TUV Corp. issued a large dollar volume of bonds with 50-year original maturities nineteen years ago. Each bond has a $1,000 par value and a 7.2% annual coupon interest rate, with interest paid semiannually. Today you buy some of those bonds for the current market price of $930.25 each. What will your realized compound yield on this investment be if you hold each bond until it matures and you can reinvest each semiannual coupon payment, from the day you get it until the maturity date, for a return expressed as an 8.3681% effective annual rate (EAR)?

Type: Realized Compound Yield. Steps 1 and 2 shown below provide a fairly brief and direct explanation of the RCY concept; a longer and more detailed discussion appears with problem 21 above. Here the bond holder will receive a coupon payment of (.072 x $1,000) ÷ 2 = $36 every 6 months on each bond held, and expects to receive those payments semiannually for the 50 – 19 = 31 years, or 31 x 2 = 62 half-years, until the issue matures. Just to provide some context (this step is not necessary for computing the RCY), the yield to maturity (an internal rate of return measure) on this investment would be computed as

$$\$930.25 = \$36.00\left(\frac{1-\left(\frac{1}{1+r}\right)^{62}}{r}\right) + \$1{,}000.00\left(\frac{1}{1+r}\right)^{62};$$

the solution for r found with trial and error, likely with the help of a spreadsheet or financial calculator, would be .039 or 3.9% semiannually, converted to a compounded annual measure (because we talk about rates of return or cost in annual terms) of $(1.039)^2 - 1 = .079521$, or 7.9521%. But recall that in computing an IRR measure we do not directly consider the impact of reinvesting received cash flows until the end of the investment period. With modified internal rate of return (MIRR) measures we do directly factor in the impact of expected reinvestment over the remaining holding period. Bond market analysts call MIRR on a bond investment the Realized Compound Yield.

Step 1: First we must compute the semiannual reinvestment rate at which each semiannual coupon interest payment of (.072 x $1,000) ÷ 2 = $36, as found above, is expected to grow. If the annual reinvestment rate is expected to be an 8.3681% EAR, the corresponding semiannual rate would be $\sqrt[2]{1.083681} - 1 = .041$. Then we find the total to which all of the sixty-two remaining semiannual $36 coupon interest payments are expected to have grown, along with the presumed 4.1% semiannual returns, by the end of year 31 or half-year 62 when the bond matures. Finally, because the $1,000 par value also would be received at maturity the total amount the bond holder expects to have at the maturity date 62 half-years from now, as a result of having made this investment, is:

$$\$36\,(1.041)^{61} + \$36\,(1.041)^{60} + \cdots + \$36\,(1.041)^{2} + \$36\,(1.041)^{1} + \$36\,(1.041)^{0} + \$1{,}000\,(1.041)^{0}$$

$$= \$36\,[(1.041)^{61} + (1.041)^{60} + (1.041)^{59} + \cdots + (1.041)^{2} + (1.041)^{1} + (1.041)^{0}] + \$1{,}000\,(1.041)^{0}$$

$$= \$36.00 \left(\frac{(1.041)^{62} - 1}{.041}\right) + \$1{,}000\ (1.041)^0 = \$36.00\ (270.161347) + \$1{,}000\ (1.00) = \$\underline{10{,}725.81}$$

This amount corresponds to what we called the "terminal value" in our MIRR discussion in capital budgeting. (A business investment example that required similar steps, because it had an expected salvage value that fills the same computational role as the expected return of principal on a bond, was shown in part c of problem 6 in Topic 6.)

Step 2: Paying \$930.25 today and then expecting to have accumulated \$10,725.81 sixty-two half-years later represents an average periodic rate of return of

$$\$930.25\ (1 + r)^{62} = \$10{,}725.81$$
$$(1 + r)^{62} = 11.530028$$
$$\sqrt[62]{(1+r)^{62}} = \sqrt[62]{11.530028}$$
$$1 + r = 11.530028^{1/62} = 11.530028^{.016129} = 1.040223$$
$$\text{so } r = .040223, \text{ or } \underline{4.0223\%}$$

Note that as with any MIRR measure, this value falls somewhere between the .039 semiannual IRR and .041 semiannual reinvestment rate. So the .040223 is a semiannual rate too of course; the corresponding annualized rate we would want to talk about as the RCY is $(1.040223)^2 - 1 = \underline{8.2063\%}$.

23. [FIL 404 only] UVW Company issued several hundred million dollars' worth of 30-year bonds 26 years ago. Each \$1,000 par value bond has an 8% annual coupon interest rate, with semiannual interest payments, and is priced today to provide a 9.2025% yield to maturity (YTM). What will Ms. Holder's realized compound yield (RCY) be if she buys one of these bonds today and holds it for only 3½ years of its 4-year remaining life, while reinvesting the coupon payments to earn an average 11.3025% effective annual rate (EAR) of return? What if her reinvestments instead earn, on average, a 7.1225 or 9.2025% EAR?

Type: Realized Compound Yield, Duration. Here our first step is to compute the price Ms. Holder should be willing to pay for the bond today, with 4 years remaining until it matures. With a Coupon Payment of (8% x \$1,000) ÷ 2 = \$40, a \$1,000 par value Ending Amount, an n of 8 half-years remaining until maturity, and a 9.2025% YTM for a semiannual discount rate r of $\sqrt[2]{1.092025}$ -1 = 4.5%, the bond currently should be worth

$$V_B = \$40 \left(\frac{1 - \left(\frac{1}{1.045}\right)^8}{.045}\right) + \$1{,}000 \left(\frac{1}{1.045}\right)^8$$
$$= \$40\ (6.595886) + \$1{,}000\ (.703185) = \$\underline{\underline{967.02}}$$

Because investors today require a 4.5% semiannual rate of return, whereas when the bonds were issued investors were content with a lower 4% semiannually (as evidenced by the 8% annual coupon rate and semiannual payments), the bonds should sell at a price below par (a 4% periodic cash flow stream discounted at a 4.5% periodic rate should be worth less than its par value). Our next step is to determine the amount to which the reinvested coupon payments will grow by the end of the expected 3½ year = 7 half-year holding period, a future value of an annuity computation. With an 11.3025% EAR (for a semiannual periodic rate of $\sqrt[2]{1.113025}$ -1 = .055, or 5.5%), the \$40 x 7 = \$280 in total interest receipts will grow, with a 5.5% periodic reinvestment rate over 7 periods, to

$$\$40\ (1.055)^6 + \$40\ (1.055)^5 + \$40\ (1.055)^4 + \$40\ (1.055)^3 + \$40\ (1.055)^2 + \$40\ (1.055)^1 + \$40\ (1.055)^0$$

$$= \$40\,[(1.055)^6 + (1.055)^5 + (1.055)^4 + (1.055)^3 + (1.055)^2 + (1.055)^1 + (1.055)^0]$$

$$= \$40\left(\frac{(1.055)^7 - 1}{.055}\right) = \$40\,(8.266894) = \$\underline{330.68}$$

After 7 half-years, Ms. Holder will have accumulated $330.68 through interest received and reinvested. She also should be able to sell the bond (if the required semiannual return is 5.5%), with one half-year remaining in the bond's life (7 of the 8 half-years will have passed), for a price of

$$V_B = \$40\left(\frac{1-\left(\frac{1}{1.055}\right)^1}{.055}\right) + \$1{,}000\left(\frac{1}{1.055}\right)^1$$
$$= \$40\,(.947867) + \$1{,}000\,(.947867) = \$\underline{985.78}$$

So by the end of her 3½ year holding period, Ms. Holder will have accumulated $330.68 through interest received and reinvested, and also will receive $985.78 by selling, for a total $330.68 + $985.78 = $<u>1,316.46</u> value at the end of half-year 7. Investing $967.02 today and then having $1,316.46 after 7 periods corresponds to an average periodic rate of return computed as:

$$\text{B}_{\text{AMT}}\,(1 + r)^n = \text{E}_{\text{AMT}}$$
$$\$967.02\,(1 + r)^7 = \$1{,}316.46$$
$$(1 + r)^7 = 1.361358$$
$$\sqrt[7]{(1 + r)^7} = \sqrt[7]{1.361358}$$
$$(1 + r) = 1.361358^{1/7} = 1.361358^{.142857} = 1.045\ ,$$

for an r = .045 semiannual return and an annualized RCY of $(1.045)^2 - 1 = \underline{9.2025\%}$. With a 7.1225% EAR (for a semiannual periodic rate of $\sqrt[2]{1.071225} - 1 = .035$, or 3.5%), the $40 x 7 = $280 in total coupon receipts will grow, with a 3.5% periodic reinvestment rate over 7 periods, to

$$\$40\,(1.035)^6 + \$40\,(1.035)^5 + \$40\,(1.035)^4 + \$40\,(1.035)^3 + \$40\,(1.035)^2 + \$40\,(1.035)^1 + \$40\,(1.035)^0$$

$$= \$40\,[(1.035)^6 + (1.035)^5 + (1.035)^4 + (1.035)^3 + (1.035)^2 + (1.035)^1 + (1.035)^0]$$

$$= \$40\left(\frac{(1.035)^7 - 1}{.035}\right) = \$40\,(7.779408) = \$\underline{311.18}$$

After 7 half-years, Ms. Holder will have accumulated $311.18 through interest received and reinvested. She also should be able to sell the bond (if the required semiannual return is 3.5%), with one half-year remaining in the bond's life, for

$$V_B = \$40\left(\frac{1-\left(\frac{1}{1.035}\right)^1}{.035}\right) + \$1{,}000\left(\frac{1}{1.035}\right)^1$$
$$= \$40\,(.966184) + \$1{,}000\,(.966184) = \$\underline{1{,}004.83}$$

So by the end of her 3½ year holding period, Ms. Holder will have accumulated $311.18 through interest received and reinvested, and also will receive $1,004.83 by selling, for a total $311.18 + $1,004.83 = $<u>1,316.01</u> value at the end of half-year 7. Investing $967.02 today and then having $1,316.01 after 7 periods corresponds to an average periodic rate of return computed as:

$$\$967.02\,(1 + r)^7 = \$1{,}316.01$$
$$(1 + r)^7 = 1.360893$$
$$\sqrt[7]{(1\ +\ r)^7} = \sqrt[7]{1.360893}$$
$$(1 + r) = 1.360893^{1/7} = 1.360893^{.142857} = 1.045\ ,$$

for an r = .045 semiannual return and an annualized RCY, again, of $(1.045)^2 - 1$ = 9.2025%. Finally, with a 9.2025% EAR (for a semiannual periodic rate of $\sqrt[2]{1.092025} - 1$ = .045, or 4.5%), the \$40 x 7 = \$280 in total coupon receipts will grow, with a 4.5% periodic reinvestment rate over 7 periods, to

$$\$40\,(1.045)^6 + \$40\,(1.045)^5 + \$40\,(1.045)^4 + \$40\,(1.045)^3 + \$40\,(1.045)^2 + \$40\,(1.045)^1 + \$40\,(1.045)^0$$

$$= \$40\,[(1.045)^6 + (1.045)^5 + (1.045)^4 + (1.045)^3 + (1.045)^2 + (1.045)^1 + (1.045)^0]$$

$$= \$40\left(\frac{(1.045)^7 - 1}{.045}\right) = \$40\,(8.019152) = \$\underline{320.77}$$

After 7 half-years, Ms. Holder will have accumulated \$320.77 through coupon interest received and reinvested, and also should be able to sell the bond (if the required semiannual return is 4.5%), with one half-year remaining in its life, for

$$V_B = \$40\left(\frac{1-\left(\frac{1}{1.045}\right)^1}{.045}\right) + \$1{,}000\left(\frac{1}{1.045}\right)^1$$
$$= \$40\,(.956938) + \$1{,}000\,(.956938) = \$\underline{995.22}$$

So by the end of her 3½ year holding period, Ms. Holder will have accumulated \$320.77 through interest received and reinvested, and should be able to sell for \$995.22, for a total \$320.77 + \$995.22 = \$<u>1,315.99</u> value at the end of half-year 7. Investing \$967.02 today and then having \$1,315.99 after 7 periods corresponds to an average periodic rate of return computed as:

$$\$967.02\,(1 + r)^7 = \$1{,}315.99$$
$$(1 + r)^7 = 1.360867$$
$$\sqrt[7]{(1\ +\ r)^7} = \sqrt[7]{1.360867}$$
$$(1 + r) = 1.360867^{1/7} = 1.360867^{.142857} = 1.045\ ,$$

for an r = .045 semiannual return and an annualized RCY, once again, of $(1.045)^2 - 1$ = 9.2025% . What's going on here; why is the RCY 9.2025% (with only some very tiny rounding differences) no matter what the reinvestment rate??? They key is a concept known as *duration*, the weighted average time remaining until the bond's cash flows are received. Recall our original bond equation

$$V_B = \frac{\text{Coupon}}{\text{Payment}}\left(\frac{1}{1+r}\right)^1 + \frac{\text{Coupon}}{\text{Payment}}\left(\frac{1}{1+r}\right)^2 + \cdots + \frac{\text{Coupon}}{\text{Payment}}\left(\frac{1}{1+r}\right)^{n-1} + \left[\frac{\text{Coupon}}{\text{Payment}} + \frac{\text{Ending}}{\text{Amount}}\right]\left(\frac{1}{1+r}\right)^n$$

With Coupon Payment = \$40, Ending Amount = the \$1,000 par value, n = 8 half-years remaining until the bond matures, and r = $\sqrt[2]{1.092025}$ -1 = 4.5%, the bond should currently be worth \$967.02. We compute this value and the duration as follows:

Period	Present Value of Cash Flow		PV as Proportion of Bond Value	Period x Proportion
1	$\$40 \times \left(\frac{1}{1.045}\right)^1 =$	\$ 38.2775	\$ 38.2275 ÷ \$967.0205 = .0395	1 x .0395 = .0395
2	$\$40 \times \left(\frac{1}{1.045}\right)^2 =$	\$ 36.6292	\$ 36.6292 ÷ \$967.0205 = .0379	2 x .0379 = .0758
3	$\$40 \times \left(\frac{1}{1.045}\right)^3 =$	\$ 35.0519	\$ 35.0519 ÷ \$967.0205 = .0362	3 x .0362 = .1086
4	$\$40 \times \left(\frac{1}{1.045}\right)^4 =$	\$ 33.5425	\$ 33.5425 ÷ \$967.0205 = .0347	4 x .0347 = .1388
5	$\$40 \times \left(\frac{1}{1.045}\right)^5 =$	\$ 32.0980	\$ 32.0980 ÷ \$967.0205 = .0332	5 x .0332 = .1660
6	$\$40 \times \left(\frac{1}{1.045}\right)^6 =$	\$ 30.7158	\$ 30.7158 ÷ \$967.0205 = .0318	6 x .0318 = .1908
7	$\$40 \times \left(\frac{1}{1.045}\right)^7 =$	\$ 29.3931	\$ 29.3931 ÷ \$967.0205 = .0304	7 x .0304 = .2128
8	$\$1{,}040 \times \left(\frac{1}{1.045}\right)^8 =$	\$731.3125	\$731.3125 ÷ \$967.0205 = .7563	8 x .7563 = 6.0504
	Total Bond Value	\$967.0205	1.0000	Duration 6.9827

Duration (more specifically a figure called Macaulay duration) is a weighted average measure of *time*: a summation of the time when each cash flow is to be received, weighted by the PV of the corresponding cash flow. Because our periods here are semiannual, the bond's duration in years is 6.9827 ÷ 2 = 3.4914, or about 3.5 years. The benefit of knowing this measure is that we can approximately *immunize* (there are some complications in doing so) – locking in a compounded return approximately equal to the yield to maturity, no matter what happens to the reinvestment rate – for a time period equal to the duration of the bond in question. Here Ms. Holder locked in an average compounded annual return equal to the 9.2025% yield to maturity over a 3.5-year period by buying a bond with a 3.5-year duration.

We might want to immunize if we must meet a 9.2025% financial obligation in 3.5 years, and we know that our commitment could be difficult to meet if our reinvestment return fell short of 9.2025% (the yield to maturity currently available in the market on bonds of similar risk). In the previous problem we saw an example of what happens with reinvestment if the holding period is not equal to the duration, and thus we are not immunized. In that problem the investor plans to hold the bond for the entire 11 years remaining until maturity, whereas the bond's duration is about 14.4 half-years or 7.2 years; you might compute it as an exercise. (A coupon bond's duration is always shorter than its remaining maturity; a zero-coupon bond's duration is equal to its maturity.)

The idea might seem simple if we could always say that a reinvestment rate greater than the YTM gives a RCY greater than the YTM, and a reinvestment rate less than the YTM gives a RCY less than the YTM. But that relationship is true only if the holding period is longer than the duration. With a holding period shorter than the duration, a reinvestment rate greater than the YTM gives a RCY less than the YTM; a reinvestment rate less than the YTM gives a RCY greater than the YTM. (With no coupon payments to reinvest for better or worse, a zero-coupon bond's RCY is equal to its YTM.) Thus an aggressive bond investor might try to predict interest rate movements, and then purchase bonds with durations greater than some targeted holding period (for a holding period less than the duration) if he believed that reinvestment rates would fall, OR purchase bonds with durations less than that targeted holding period (a holding period longer than the duration) if he believed that reinvestment rates would rise.

What occurs is that if the holding period is longer than the bond's duration and market interest rates rise (fall), the gain (loss) on reinvestment more than offsets the loss (gain) in resale price accompanying higher (lower) required rates of return in the market. If the holding period is shorter than the bond's duration and market rates rise (fall), the loss (gain) in the resale price more than offsets the gain (loss) on reinvestment. Duration helps measure a bond's price sensitivity to changing interest rates, so if you think interest rates are going to rise you want to have bonds with relatively shorter durations (so their values will fall only a little and you will benefit from reinvestment gains), whereas if you think rates will fall you want bonds with relatively long durations (so their values will rise by a lot to offset the unattractive reinvestment situation).

24. a. Mr. VWX wants his money to be invested in U.S. government bonds for the next three years. Broker Reginald von Redbird at Normal Securities notes that there is an active market for U.S. government bonds with a 3-year maturity. If Mr. VWX buys one of these 3-year bonds, he will be assured of earning a 4.25% annual interest rate in each of the three years. But von Redbird also sees that there is an active market for U.S. government bonds with a 2-year maturity. If Mr. VWX buys one of these 2-year bonds, he will be assured of earning a 4.75% annual interest rate, but only for two years. Then, at the end of year 2/start of year 3, he will have to buy a new U.S. government bond with a 1-year maturity to complete the third year of his planned investment period. Based on the Expectations Theory, what annual interest rate should he expect to earn on a 1-year U.S. government bond during year 3?

Type: Yield curve/implied forward interest rates. The relationship between short-term and long-term interest rates, called the term structure of interest rates, as observed on any given day, can be shown on a graph called the yield curve (showing periodic interest rate on the vertical axis, and periods to maturity on the horizontal axis). Over recent decades, the yield curve has most often been upward-sloping, meaning that people who have bought long-term bonds have received higher annual interest rates than have those who bought short-term bonds. (In this context, a "long-term" period is simply long enough to include more than one "short-term" period; typically we compare the annual interest rate paid on a "short-term" 1-year bond to the annual interest rates paid on "long-term" bonds with lives of 2 years, 3 years, etc.) But sometimes we see a downward-sloping (with short-term bonds providing annual interest rates higher than those paid on long-term bonds) or flat (with long-term and short-term annual interest rates essentially equal) yield curve. [It should not be surprising that longer-term bonds typically provide higher annual interest rates; recall from our earlier interest rate discussion that one of the building blocks of an interest rate is the maturity risk premium that can accompany bonds with longer maturities.]

One of the methods for analyzing the yield curve is the expectations theory, under which we theorize that the annual interest rate offered on a long-term bond is the average of the annual interest rates that investors expect to receive on a series of short-term bonds that encompass the same total time horizon. So for purposes of this example, our expectation before-the-fact is that Mr. VWX will receive the same total amount of interest no matter which path he chooses:

- a 3-year bond that locks him in to a known 4.25% interest rate for each of the 3 years, or
- a 2-year bond that locks him in to a known 4.75% interest rate for each of the first 2 years, and then a 1-year bond that will pay an as-yet unknown rate (no one really knows what interest rates will prevail two years from now, but we can compute the bond market's best guess).

In other words, the expectations theory tells us that Mr. VWX should expect to collect a total of 4.25% + 4.25% + 4.25% = <u>12.75%</u> in interest over the 3 years of his investment holding period, regardless of which combination of maturities he chooses in filling out the three years. Therefore, 4.75% + 4.75% + x% (if he goes with a 2-year bond followed by a 1-year bond) should also equal 12.75%. So x% must be 12.75% – 4.75% – 4.75% = <u><u>3.25%</u></u>. (With 4.75% + 4.75% = 9.5% accounted for from the first two years, the amount of interest he expects to earn in year 3 is the remaining 3.25%.) We can examine the total interest Mr. VWX would earn under each option with the following table:

	Interest Rate Received/Expected			
	Year 1	Year 2	Year 3	Total
Lock in to 3-year bond	4.25%	4.25%	4.25%	12.75%
Buy 2-yr bond, then 1-yr bond	4.75%	4.75%	3.25%	12.75%

A few explanatory points:

- First, a borrower (here, the U.S. government) issues a 3-year bond because it wants to know, with certainty ahead of time, what its annual cost of money will be in each of the 3 years. Thus, once this commitment (a legal contract) has been made, the lender (bond buyer) also knows with certainty what interest rate she will receive in each of the 3 years. And a borrower that issues a 2-year bond does so because it wants to know, with certainty, what its annual cost of money will be both years; thus the 2-year bond buyer knows with certainty what interest rate she will receive in year 1 and year 2. But no one can know with certainty what interest rate borrowers will pay/lenders will get when a new commitment is made at the start of year 3 (or any future date), so in considering what interest rate the buyer of a 2-year bond will get in year 3, all we can do is ferret out the market's best guess on what that future rate will be.
- Second, the reason we interpret 3.25% as the market's prediction/guess of the 1-year interest rate that will prevail during year 3: if investors tended to think the 1-year rate available in year 3 would be 6.5%, they would expect anyone who bought the 2-year bond followed by the 1-year bond to end up with an impressive 3-year total return of 4.75% + 4.75% + 6.5% = 16%, and there would be no active market for the 3-year bonds (why settle for 12.75% if you could just plan the paper work differently and end up with a higher total?). Or if investors tended to think the 1-year rate available in year 3 would be only 1.5%, they would expect anyone who bought the 2-year bond followed by the 1-year bond to end up with only 4.75% + 4.75% + 1.5% = 11%, and then there might not be much activity in the market for the 2-year bonds (anyone wanting to invest for 3 years would buy 3-year bonds with the higher 12.75% total). But since we see active markets for both the 2-year and 3-year bonds, we must theorize that investors, overall on average, think either of the two "roads" (3-year bond, or else 2-year bond followed by 1-year bond) will lead to the same 3-year total of 12.75%.
- Third, this 3.25% we compute as the 1-year rate expected for year 3 is just an expectation (the market's best guess of what the 1-year rate will be in year 3); with 20/20 hindsight in two years we may look back and marvel at how wrong that guess turned out to be (for example, if the 1-year rate paid during year 3 turns out to be 10%). But that market average expectation/ guess is useful to economists as a predictor of what interest rates will actually be at various future dates. We even have a name for the market's average guess on what a future interest rate will be: an implied forward rate. It is "implied" (not guaranteed, in any way) by the structure of known long- and short-term rates, and the logic of the expectations theory.
- Fourth, while the implied forward rate is the market's average guess on what a future rate will turn out to be, an individual investor's own forecast may differ from that average. And someone who thinks the market's average guess is wrong might try to profit from that differing view. For example, consider someone who thinks the 1-year interest rate available in year 3 will actually be 4.5%, not the average, implied forward rate of 3.25%. This investor will buy a 2-year bond and, if she turns out to be right, will end up with 4.75% + 4.75% + 4.5% = 14% over the 3-year holding period, vs. the total 12.75% that buyers of 3-year bonds will end up getting.
- Fifth, here we have used an additive (arithmetic average) approach to computing the implied forward rate. It might be argued that a multiplicative (geometric average) approach is more

technically correct, because each year's interest is earned on top of the prior year's interest. In other words, here the holder of a 3-year bond ends up with [(1.0425) (1.0425) (1.0425) – 1] = 13.2996% more money than she started with (not a mere .0425 + .0425 + .0425 = 12.75%). So the implied forward rate should be such that [(1.0475) (1.0475) (1 + x)] – 1 = 13.2996%, not simply (.0475 + .0475 + x) = 12.75%. Thus we would compute 1.132996 ÷ [(1.0475) (1.0475)] – 1 = (1.132996 ÷ 1.097256) – 1 = .032571, or about 3.26%. But note two things here. First, this multiplicative approach is much less intuitive than the additive approach to analyzing the interest earned over a series of years. Second, the 3.26% found with the more complicated multiplicative approach is not much different from the 3.25% found with the more intuitive additive approach. Because we are trying to predict what will happen in an always-uncertain future, we will be happy (as many economists are) to use the simpler-but-reasonably-precise additive approach as our means of computing implied forward rates.

- Finally, with the annual interest rate available on the 2-year bond exceeding the annual rate available on a 3-year bond, we have an "inverted" yield curve, rather than the "normal" situation with longer-term bonds providing higher annual interest rates than shorter-terms bonds do. But this situation reflects only the conditions observed on a given day. If small changes occur each day, then over a period of weeks or months the relationship may reverse, such that we would see a normal yield curve again.

b. Before Mr. VWX can make his decision, von Redbird suggests yet another possibility. The U.S. government also issues bonds ("T-bills") with just a 1-year maturity. If Mr. VWX buys a 1-year bond, he will be assured of earning a 4.95% annual interest rate for one year. Then, at the end of year 1/start of year 2, he could buy a new U.S. government bond with a 1-year maturity to complete the second year of his planned investment period, following at the end of year 2/start of year 3 by buying another new bond with a 1-year maturity to complete the third year. Or, after earning 4.95% on the 1-year bond he could, at the end of year 1/start of year 2, buy a new U.S. government bond with a 2-year maturity to complete the second and third years of his planned investment period. According to the Expectations Theory, what annual interest rate should he expect to earn on a 1-year bond during year 3? On a 1-year bond during year 2? On a 2-year bond during years 2 and 3? (In other words, compute the applicable implied forward rates.)

Type: Yield curve/implied forward interest rates. We approach this question based on the same logic as in the above example: that the total interest expected over the 3 years should be 12.75%, regardless of the combination of bonds held over that period. The table (which here includes the 2-year followed by 1-year case described above) can be completed based on the following relationships:

	Interest Rate Received/Expected			
	Year 1	Year 2	Year 3	Total
Lock in to 3-year bond	4.25%	4.25%	4.25%	12.75%
Buy 2-yr bond, then 1-yr bond	4.75%	4.75%	xxx%	12.75%
Buy 1-yr bond, then 1-yr, then 1-yr	4.95%	yyy%	xxx%	12.75%
Buy 1-yr bond, then 2-yr bond	4.95%	zzz%	zzz%	12.75%

Once we compute the 1-year rate expected for year 3 (found above to be 3.25%), we can plug it into any line where we need the 1-year rate expected for year 3. (It does not matter whether we come into year 3 having held one 2-year bond or two 1-year bonds. Just like: say you drive from Normal to Joliet to Chicago, and find that the Joliet-to-Chicago leg of the trip takes 45 minutes. Then the next day you drive from Moline to Joliet to Chicago. The Joliet-to-Chicago leg of the trip is again expected to take 45 minutes; it doesn't matter what route you take in getting to the edge of Joliet. And here, a 1-year investment made in year 3 is expected to deliver an xxx% = 3.25% annual return; it doesn't matter what route we take in getting to the start of year 3.)

	Interest Rate Received/Expected			
	Year 1	Year 2	Year 3	Total
Lock in to 3-year bond	4.25%	4.25%	4.25%	12.75%
Buy 2-yr bond, then 1-yr bond	4.75%	4.75%	3.25%	12.75%
Buy 1-yr bond, then 1-yr, then 1-yr	4.95%	yyy%	3.25%	12.75%
Buy 1-yr bond, then 2-yr bond	4.95%	zzz%	zzz%	12.75%

Now we can see that the 1-year rate expected to prevail in year 2 must be a value yyy% such that 4.95% + yyy% + 3.25% = 12.75%, so 12.75% – 3.25% – 4.95% = yyy% = 4.55%. So now we can show

	Interest Rate Received/Expected			
	Year 1	Year 2	Year 3	Total
Lock in to 3-year bond	4.25%	4.25%	4.25%	12.75%
Buy 2-yr bond, then 1-yr bond	4.75%	4.75%	3.25%	12.75%
Buy 1-yr bond, then 1-yr, then 1-yr	4.95%	4.55%	3.25%	12.75%
Buy 1-yr bond, then 2-yr bond	4.95%	zzz%	zzz%	12.75%

[We also could compute yyy% by noting that the longer-term 2-year interest rate 4.75% should be the average of year 1's actual 4.95% rate and year 2's implied forward rate of yyy%; (.0475 x 2) - .0495 = .0455.] Now we have all needed values except for the 2-year rate we would expect to be quoted at the end of year 1/start of year 2 (such that it would be received during each of years 2 and 3). It should be clear that the 2-year interest rate expected in year 2 (at the start of year 2, thus applying to years 2 and 3) must be zzz% such that

4.95% + zzz% + zzz% = 12.75%
4.95% + 2 (zzz%) = 12.75%
2 (zzz%) = 7.80%
zzz% = 3.90%

(Double-check: 4.95% + 3.90% + 3.90% = 12.75%.✓) And now we can show:

	Interest Rate Received/Expected			
	Year 1	Year 2	Year 3	Total
Lock in to 3-year bond	4.25%	4.25%	4.25%	12.75%
Buy 2-yr bond, then 1-yr bond	4.75%	4.75%	3.25%	12.75%
Buy 1-yr bond, then 1-yr, then 1-yr	4.95%	4.55%	3.25%	12.75%
Buy 1-yr bond, then 2-yr bond	4.95%	3.90%	3.90%	12.75%

So here is the final interpretation. Mr. VWX, who wants to be invested for 3 years, could buy a 3-year bond today and be guaranteed to receive 4.25% each year. Or he could buy a 2-year bond today, knowing that he will receive 4.75% in each of years 1 and 2, and expecting to receive 3.25% when he buys a 1-year bond at the start of year 3 (so 3.25% is the 1-year rate expected in year 3). Or he could buy a 1-year bond today, knowing that he will receive 4.95% in year 1, and expecting to receive 4.55% when he buys a 1-year bond at the start of year 2 (so 4.55% is the 1-year rate expected in year 2) and 3.25% when he buys a 1-year bond at the start of year 3 (so 3.25% is, as we already knew, the 1-year rate expected in year 3). And finally, he could buy a 1-year bond today, knowing that he will receive 4.95% in year 1, and expecting to receive 3.90% per year when he buys a 2-year bond at the start of year 2 (so 3.90% is the 2-year rate expected in year 2, meaning at the start of year 2, such that it would apply to both of years 2 and 3).

25. WXY Securities has announced that the following interest rates are being paid on U.S. government bonds purchased today:

One-year maturity: pays 6.15% annual interest for one year
Two-year maturity: pays 6.45% annual interest each year for two years
Three-year maturity: pays 6.85% annual interest each year for three years

Compute and identify the applicable implied forward rates, based on our additive approximation approach.

Type: Yield curve/implied forward interest rates. Again we are dealing with the expectations theory, under which we theorize that over three years we should expect to receive the same total amount of interest (here, 3 x 6.85% = 20.55%) no matter which combination of bonds we hold over that period. Our initial table would appear as

	Interest Rate Received/Expected			
	Year 1	Year 2	Year 3	Total
Lock in to 3-year bond	6.85%	6.85%	6.85%	20.55%
Buy 2-yr bond, then 1-yr bond	6.45%	6.45%	xxx%	20.55%
Buy 1-yr bond, then 1-yr, then 1-yr	6.15%	yyy%	xxx%	20.55%
Buy 1-yr bond, then 2-yr bond	6.15%	zzz%	zzz%	20.55%

The 1-year interest rate expected in year 3 should be 20.55% – 6.45% – 6.45% = xxx% = 7.65%. Then the 1-year rate expected in year 2 should be 20.55% – 6.15% – 7.65% = yyy% = 6.75%, or else (6.45% x 2) – 6.15% = 6.75%. And the 2-year interest rate expected in year 2 is 20.55% – 6.15% = 2 (zzz%) $\Rightarrow$ 14.40% = 2 (zzz%) $\Rightarrow$ zzz% = 7.20%. Completing the table, we see

	Interest Rate Received/Expected			
	Year 1	Year 2	Year 3	Total
Lock in to 3-year bond	6.85%	6.85%	6.85%	20.55%
Buy 2-yr bond, then 1-yr bond	6.45%	6.45%	7.65%	20.55%
Buy 1-yr bond, then 1-yr, then 1-yr	6.15%	6.75%	7.65%	20.55%
Buy 1-yr bond, then 2-yr bond	6.15%	7.20%	7.20%	20.55%

Double-check: 6.85% + 6.85% + 6.85% = 20.55% ✓
6.45% + 6.45% + 7.65% = 20.55% ✓
6.15% + 6.75% + 7.65% = 20.55% ✓
6.15% + 7.20% + 7.20% = 20.55% ✓

Note that here we have an upward-sloping ("normal") yield curve, with the annual interest rate available on a long-term commitment (here the 3-year rate is 6.85% per year) exceeding the annual interest rate available on a short-term commitment (here the 1-year rate is just 6.15% per year).

26. XYZ Investments reports that the following interest rates are being paid on U.S. government bonds purchased today:

One-year maturity: pays 7.45% annual interest for one year
Two-year maturity: pays 7.55% annual interest each year for two years
Three-year maturity: pays 7.70% annual interest each year for three years
Four-year maturity: pays 7.90% annual interest each year for four years

Compute and identify the applicable implied forward rates, based on our additive approximation approach.

Type: Yield curve/implied forward interest rates. Once again we are dealing with the expectations theory, under which we theorize that over four years we should expect to receive the same total amount of interest (here, 4 x 7.90% = 31.60%) no matter which combination of bonds we hold over that period.

Our initial table would appear as

	Interest Rate Received/Expected				
	Year 1	Year 2	Year 3	Year 4	Total
Lock in to 4-year bond	7.90%	7.90%	7.90%	7.90%	31.60%
Buy 3-yr bond, then 1-yr bond	7.70%	7.70%	7.70%	aaa%	31.60%
Buy 2-yr bond, then 1-yr, then 1-yr	7.55%	7.55%	bbb%	aaa%	31.60%
Buy 1-yr bond, then 1-yr, then 1-yr, then 1-yr	7.45%	ccc%	bbb%	aaa%	31.60%
Buy 1-yr bond, then 1-yr, then 2-yr	7.45%	ccc%	ddd%	ddd%	31.60%
Buy 1-yr bond, then 2-yr, then 1-yr	7.45%	eee%	eee%	aaa%	31.60%
Buy 1-yr bond, then 3-yr bond	7.45%	fff%	fff%	fff%	31.60%
Buy 2-yr bond, then 2-yr bond	7.55%	7.55%	ddd%	ddd%	31.60%

The 1-year rate expected in year 4 should be 31.60% – 7.70% – 7.70% – 7.70% = aaa% = 8.50%.
The 1-year rate expected in year 3 should be 31.60% – 7.55% – 7.55% – 8.50% = bbb% = 8.00%.
The 1-year rate expected in year 2 should be 31.60% – 7.45% – 8.00% – 8.50% = ccc% = 7.65%.
The 2-year rate expected in year 3 should be (31.60% – 7.55% – 7.55%)/2 = ddd% = 8.25%.
The 2-year rate expected in year 2 should be (31.60% – 7.45% – 8.50%)/2 = eee% = 7.825%.
The 3-year rate expected in year 2 should be (31.60% – 7.45%)/3 = fff% = 8.05%.

Completing the table, we see	Interest Rate Received/Expected				
	Year 1	Year 2	Year 3	Year 4	Total
Lock in to 4-year bond	7.90%	7.90%	7.90%	7.90%	31.60%
Buy 3-yr bond, then 1-yr bond	7.70%	7.70%	7.70%	8.50%	31.60%
Buy 2-yr bond, then 1-yr, then 1-yr	7.55%	7.55%	8.00%	8.50%	31.60%
Buy 1-yr bond, then 1-yr, then 1-yr, then 1-yr	7.45%	7.65%	8.00%	8.50%	31.60%
Buy 1-yr bond, then 1-yr, then 2-yr	7.45%	7.65%	8.25%	8.25%	31.60%
Buy 1-yr bond, then 2-yr, then 1-yr	7.45%	7.825%	7.825%	8.50%	31.60%
Buy 1-yr bond, then 3-yr bond	7.45%	8.05%	8.05%	8.05%	31.60%
Buy 2-yr bond, then 2-yr bond	7.55%	7.55%	8.25%	8.25%	31.60%

Double-check: 7.90% + 7.90% + 7.90% + 7.90% = 31.60% ✓
7.70% + 7.70% + 7.70% + 8.50% = 31.60% ✓
7.55% + 7.55% + 8.00% + 8.50% = 31.60% ✓
7.45% + 7.65% + 8.00% + 8.50% = 31.60% ✓
7.45% + 7.65% + 8.25% + 8.25% = 31.60% ✓
7.45% + 7.825% + 7.825% + 8.50% = 31.60% ✓
7.45% + 8.05% + 8.05% + 8.05% = 31.60% ✓
7.55% + 7.55% + 8.25% + 8.25% = 31.60% ✓

27. [Similar to problems 3 and 8, for extra practice.] YZA Corporation issued several hundred million dollars' worth of bonds thirty-three years ago. These bonds had 50-year original maturities. Each individual bond has a $1,000 par value and a 5.8% annual coupon interest rate, with interest paid semiannually. Today, rational buyers of bonds with similar risk, similar remaining lives, and semiannual interest payments require a 5.3% *effective annual rate* (EAR) of return (what we call the bond's yield to maturity, although some sources call an EAR measure a bond's *effective* yield to maturity). What should a rational investor be willing to pay for one of these YZA bonds?

Type: Bond Valuation; Semiannual Coupon Payments. It is important to identify the number of interest payment periods remaining until the bonds mature. Here we do not spell that number out; you must compute it – a simple computation, but you have to tell yourself the story of what is going on. Each of the bonds described was issued 33 years ago with a 50-year life, so 17 years remain until the maturity date. Someone who buys one of these bonds today expects to collect interest twice each year for the remaining 17 years (so 17 x 2 = 34 times in total), and then get, along with the 34th interest payment on the maturity date, a return of the original lender's $1,000 principal.

With a 5.8% annual coupon interest rate the interest payment to be received at the end of each half-year period will be (.058 x $1,000) ÷ 2 = $29 every six months ($58 in total for each year). And here the annual return measure presented in the discussion stage (we talk about rates of return or cost in annual terms) is an EAR, the annual rate measure that reflects an adjustment for the impact of compounding that occurs within the year – so we must "un-compound" it to get the semiannual required rate of return r to use in our computations: $\sqrt[2]{1.053}$ – 1 = .026158, or 2.6158% (such that the semiannual rate compounds out to $[1.026158]^2 - 1 = 5.3\%$). [Note that we could have been told instead that the required annual return in APR terms is 5.2316%, such that we could find the semiannual periodic discount rate to work with as just .052316 ÷ 2 = .026158.] Plugging these figures into the various forms of our bond valuation equation (from the period-by-period "brute force" case to the more compact, and typically more useful, grouping of the interest payments as an annuity), we find

$$V_B = \$29\left(\frac{1}{1.026158}\right)^1 + \$29\left(\frac{1}{1.026158}\right)^2 + \$29\left(\frac{1}{1.026158}\right)^3 + \cdots + \$29\left(\frac{1}{1.026158}\right)^{33} + \$1{,}029\left(\frac{1}{1.026158}\right)^{34}$$

$$= \$29\left(\frac{1}{1.026158}\right)^1 + \$29\left(\frac{1}{1.026158}\right)^2 + \cdots + \$29\left(\frac{1}{1.026158}\right)^{33} + \$29\left(\frac{1}{1.026158}\right)^{34} + \$1{,}000\left(\frac{1}{1.026158}\right)^{34}$$

$$= \$29\left(\frac{1-\left(\frac{1}{1.026158}\right)^{34}}{.026158}\right) + \$1{,}000\left(\frac{1}{1.026158}\right)^{34}$$

$$= \$29\ (22.339716) + \$1{,}000\ (.415640)$$
$$= \$647.85 + \$415.64 = \$\underline{\underline{1{,}063.49}}\ ,$$

or 106.349% of the par value, as bond prices often are quoted. [On BA II Plus calculator we enter $58 ÷ 2 = PMT; $1,000 FV; 1.053 $\sqrt[2]{x}$ – 1 = x 100 = I/Y; 17 x 2 = N; CPT PV; we get – $1,063.49 (shown as negative because you would pay that amount today to buy the bond).] Here required rates of return are lower today than might have been anticipated when the bond was first issued; we have a 2.9% semiannual bond in a 2.6158% semiannual world – the periodic coupon rate is attractive compared to the periodic coupon rate someone would expect to get for making a 34-half-year loan to a borrower of similar risk today, so the bond is worth more than its $1,000 par value. Paying $1,063.49 and then getting thirty-four $29 semiannual interest payments and the receipt of the $1,000 principal value at the end of half-year 34 represents a 2.6158% average semiannual return on the $1,063.49 initially given up.

28. [Similar to problems 3, 8, and 27, for extra practice.] ZAB Corporation issued a large dollar amount of bonds sixteen years ago. These bonds had 55-year original maturities. Each individual bond has a $1,000 par value and an 8.2% annual coupon interest rate, with interest payments made semiannually. Today, rational buyers of bonds with similar risk (including the default risk that the borrower will not pay interest and repay the lender's principal on the agreed schedule), similar remaining lives, and semiannual interest payments require a 9.83% *effective annual rate* (EAR) of return (what we call the bond's yield to maturity; sometimes that EAR measure is more explicitly called a bond's *effective* yield to maturity). What should a rational investor be willing to pay for one of these ZAB bonds?

Type: Bond Valuation; Semiannual Coupon Payments. Again we have to determine the number of interest payment periods remaining until the bonds mature. Each of the bonds described was issued 16 years ago with a 55-year life, so 55 – 16 = 39 years remain until the maturity date. A buyer of one of these bonds today expects to collect interest twice each year for the remaining 39 years (so 39 x 2 = 78 times in total), and then get, along with the 78th interest payment on the day the bond matures, a return of the original lender's $1,000 principal.

With an 8.2% annual coupon interest rate the interest payment to be received at the end of each half-year period will be (.082 x $1,000) ÷ 2 = $41 ($41 every six months, or $82 in total every year). And here the annual return measure we talk about (rates of cost or return are discussed in annual terms, but then to do the computations we have to convert over to an r that corresponds to the timing of the payments and compounding) is an EAR, the annual rate measure that is adjusted for the impact of compounding that occurs within the year and thus is a more relevant figure to use in comparing investments. So we must "un-compound" it to get the semiannual required rate of return r to use in computing: $\sqrt[2]{1.0983}$ – 1 = .048, or 4.8% (such that the semiannual rate compounds out to $[1.048]^2$ – 1 = .0983 or 9.83%). [We could have been told instead that the required annual return in APR terms is 9.6%, such that we could find the semiannual periodic discount rate to work with as just .096 ÷ 2 = .048.] Plugging these figures into our bond valuation equation, we find

$$V_B = \$41\left(\frac{1}{1.048}\right)^1 + \$41\left(\frac{1}{1.048}\right)^2 + \cdots + \$41\left(\frac{1}{1.048}\right)^{77} + \$41\left(\frac{1}{1.048}\right)^{78} + \$1{,}000\left(\frac{1}{1.048}\right)^{78} \quad \text{OR}$$

$$V_B = \$41\left(\frac{1-\left(\frac{1}{1.048}\right)^{78}}{.048}\right) + \$1{,}000\left(\frac{1}{1.048}\right)^{78}$$

$$= \$41\,(20.295585) + \$1{,}000\,(.025812)$$
$$= \$832.12 + \$25.81 = \$\underline{\underline{857.93}}\ ,$$

or 85.793% of the par value, as bond prices often are quoted. [On BA II Plus calculator we enter $82 ÷ 2 = PMT; $1,000 FV; 1.0983 $\sqrt[2]{x}$ – 1 = x 100 = I/Y; 39 x 2 = N; CPT PV; we get – $857.96 (shown as negative because you would pay that amount today to buy the bond – and there is a slight 3¢ rounding difference because $\sqrt[2]{1.0983}$ – 1 is actually 4.7998% rather than exactly 4.8%).] Note two interesting aspects of the answer we found. First, required rates of return are higher today than might have been anticipated long ago when the bond was first issued; we have a 4.1% semiannual bond in a 4.8% semiannual world – the 4.1% periodic coupon rate is unattractive compared to the 4.8% periodic coupon rate someone would expect to get for making a 78-half-year loan to a borrower of similar risk today, so the bond is worth less than its $1,000 par value. Paying $857.93 and then getting seventy-eight $41 semiannual interest payments and the receipt of the $1,000 principal value at the end of half-year 78 represents a 4.8% semiannual return on the $857.93 initially given up. Second, this bond has a long remaining life, so its value is much lower than par – a 4.1% bond stuck in a 4.8% world for almost 8 decades is at a real disadvantage relative to a new $1,000 bond of similar risk that would be issued today, so its value is harmed considerably.