Compute the payments for a $240,000 graduated payment mortgage (GPM) loan with payments growing by 7.5% per year for 5 years and then leveling off for years 6 – 30, if the interest rate is an APR of 7%.

The graduated payment mortgage (GPM) loan is based on the deferred annuity idea. First recall that, for any loan, the principal lent is the present value of the expected stream of payments. Thus the present value of all the payments to be received during years 1 – 30 on this GPM must be $240,000. If we had a 30-year fixed-rate, fixed-payment loan with $240,000 principal borrowed and a 7% APR (for a monthly working r of .07 ÷ 12= .005833), we compute each of the 360 equal payments to be

$$\$240{,}000 \div \left(\frac{1 - \left(\frac{1}{1.005833}\right)^{360}}{.005833} \right) = \$1{,}596.73\ .$$

Turning it around, we can state that

$$\$1{,}596.73 \times \left(\frac{1 - \left(\frac{1}{1.005833}\right)^{360}}{.005833} \right) = \$240{,}000\ .$$

In other words, at the loan's origination date the stream of future payments, discounted to a present value, equals the original principal amount lent. This statement is true for any loan, including a graduated payment mortgage (GPM) loan.

This particular GPM (with features popularized by an FHA program) calls for payments to start at a specified level and then to rise by 7.5% per year, leveling off in year 6 to remain the same through year 30. Here we want:

Year 1 payment stream, discounted back to a present value +
Year 2 payment stream, discounted back to a present value +
Year 3 payment stream, discounted back to a present value +
Year 4 payment stream, discounted back to a present value +
Year 5 payment stream, discounted back to a present value +
Year 6 – 30 payment stream, discounted back to a present value
= $240,000 principal originally lent.

Let's think of the as-yet-unknown first-year payment amount as PMT_1. With a monthly periodic rate of .07 ÷ 12 = .005833, we can represent the present value of year 1's stream of 12 payments, for the moment, as

$$PMT_1 \left(\frac{1 - \left(\frac{1}{1.005833}\right)^{12}}{.005833} \right) = PMT_1 \times 11.557120\ .$$

Now let's think of the as-yet-unknown second-year payment amount as PMT_2. But note that PMT_2 is simply $PMT_1 \times (1.075)^1$. Because we will receive 12 payments during year 2, but must wait until the 12 months of year 1 have passed before we start collecting the 12 year 2 payments, we can represent the present value of year 2's stream of 12 payments, for the moment, as

$$PMT_2 \left(\frac{1 - \left(\frac{1}{1.005833} \right)^{12}}{.005833} \right) \left(\frac{1}{1.005833} \right)^{12} =$$

$PMT_2 \times 11.557120 \times .932583 =$
$PMT_1 \times (1.075)^1 \times 11.557120 \times .932583 =$
$PMT_1 \times 11.586328$.

From this point we can just follow the pattern. We can think of the as-yet-unknown third-year payment amount as PMT_3. But PMT_3 is simply $PMT_1 \times (1.075)^2$. Because we will receive 12 payments during year 3, but must wait until the 24 months of years 1 and 2 have passed before we start collecting the 12 year-3 payments, we can represent the present value of year 3's stream of 12 payments, for the moment, as

$$PMT_3 \left(\frac{1 - \left(\frac{1}{1.005833} \right)^{12}}{.005833} \right) \left(\frac{1}{1.005833} \right)^{24} =$$

$PMT_3 \times 11.557120 \times .869712 =$
$PMT_1 \times (1.075)^2 \times 11.557120 \times .869712 =$
$PMT_1 \times 11.615609$.

The as-yet-unknown fourth-year payment amount, PMT_4, is simply $PMT_1 \times (1.075)^3$. Because we will receive 12 payments during year 4, but must wait until the 36 months of years 1 – 3 have passed before we start collecting the 12 year 4 payments, we can represent the present value of year 4's stream of 12 payments, for the moment, as

$$PMT_4 \left(\frac{1 - \left(\frac{1}{1.005833} \right)^{12}}{.005833} \right) \left(\frac{1}{1.005833} \right)^{36} =$$

$PMT_4 \times 11.557120 \times .811079 =$
$PMT_1 \times (1.075)^3 \times 11.557120 \times .811079 =$
$PMT_1 \times 11.644964$.

The as-yet-unknown fifth-year payment amount, PMT_5, is simply $PMT_1 \times (1.075)^4$. Because we will receive 12 payments during year 5, but must wait until the 48 months of years 1 – 4 have passed before we start collecting the 12 year 5 payments, we can represent the present value of year 5's stream of 12 payments, for the moment, as

$$PMT_5 \left(\frac{1 - \left(\frac{1}{1.005833} \right)^{12}}{.005833} \right) \left(\frac{1}{1.005833} \right)^{48} =$$

$PMT_5 \times 11.557120 \times .756399 =$
$PMT_1 \times (1.075)^4 \times 11.374508 \times .756399 =$
$PMT_1 \times 11.674394$.

Finally, the as-yet-unknown payment applying to years 6 – 30, PMT_6, is simply $PMT_1 \times (1.075)^5$. Because we will receive 300 payments during years 6 – 30, but must wait until the 60 months of years 1 – 5 have passed before we start collecting the 300 year 6 – 30 payments, we can represent the present value of year 6 – 30's stream of 300 payments, for the moment, as

$$PMT_6 \left(\frac{1-\left(\frac{1}{1.005833}\right)^{300}}{.005833} \right) \left(\frac{1}{1.005833} \right)^{60} =$$

$$PMT_6 \times 141.486903 \times .705405 =$$
$$PMT_1 \times (1.075)^5 \times 141.486903 \times .705405 =$$
$$PMT_1 \times 143.283810 .$$

Look at what we have: PMT_1 times a series of different factors. Using the distributive property, we can simply multiply PMT_1 by the sum of (11.557120 + 11.586328 + 11.615609 + 11.644964 + 11.674394 + 143.283810) = 201.362224. Because the present value of the payment stream must be \$240,000, we can simply state that

$$PMT_1 \times 201.362224 = \$240{,}000 \quad SO$$
$$PMT_1 = \$240{,}000 \div 201.362224 = \$\underline{1{,}191.88} .$$

Now we know the amount of each of the first year's 12 payments. Then the payment stream will appear as follows:

PMT_1 = \$1,191.88 (months 1 – 12)
PMT_2 = \$1,191.88 $(1.075)^1$ = \$1,281.27 (months 13 – 24)
PMT_3 = \$1,191.88 $(1.075)^2$ = \$1,377.37 (months 25 – 36)
PMT_4 = \$1,191.88 $(1.075)^3$ = \$1,480.67 (months 37 – 48)
PMT_5 = \$1,191.88 $(1.075)^4$ = \$1,591.72 (months 49 – 60)
PMT_6 = \$1,191.88 $(1.075)^5$ = \$1,711.10 (months 61 – 360)

(vs. the FRM's \$1,596.73 payment for each of months 1 – 360). Let's double check:

$$\$1{,}191.88\left(\frac{1-\left(\frac{1}{1.005833}\right)^{12}}{.005833} \right) + \$1{,}281.27\left(\frac{1-\left(\frac{1}{1.005833}\right)^{12}}{.005833} \right)\left(\frac{1}{1.005833} \right)^{12}$$
$$+ \$1{,}377.37\left(\frac{1-\left(\frac{1}{1.005833}\right)^{12}}{.005833} \right)\left(\frac{1}{1.005833} \right)^{24} + \$1{,}480.67\left(\frac{1-\left(\frac{1}{1.005833}\right)^{12}}{.005833} \right)\left(\frac{1}{1.005833} \right)^{36}$$
$$+ \$1{,}591.72\left(\frac{1-\left(\frac{1}{1.005833}\right)^{12}}{.005833} \right)\left(\frac{1}{1.005833} \right)^{48} + \$1{,}711.10\left(\frac{1-\left(\frac{1}{1.005833}\right)^{300}}{.005833} \right)\left(\frac{1}{1.005833} \right)^{60}$$

= \$1,191.88 (11.557120)+ \$1,281.27 (10.777979) + \$1,377.37 (10.051365) + \$1,408.67 (9.373737) + \$1,591.72 (7.637209) + \$1,711.10 (66.885451) =

\$13,774.72 + \$13,809.53 + \$13,844.43 + \$13,879.42 +\$13,914.50 + \$170,777.38 = \$240,000.00 . ✓