TIME VALUE: PROBLEMS & SOLUTIONS *(copyright © 2024 Joseph W. Trefzger)*

This problem set covers all of our basic time value of money applications, with a general progression in degree of difficulty as we proceed from problem 1 to problem 24 (note that 22 through 24 are for FIL 404 only). A full understanding of all steps in these problems should indicate solid knowledge of our basic time value of money ideas. Be sure you have mastered the easier problems before moving ahead, because the more difficult examples tend to expand on the ideas from the easier ones. Opportunities for additional practice are provided in Problem Set B, which is organized with the same ordering as this main set, and in the largely open-ended Problem Set C.

1. Connie expects to earn a 4.45% average annual interest rate on her savings account. If she makes a $2,000 deposit today, and then makes no more out-of-pocket deposits, what should her account balance be after six years?

Type: Non-Annuity; Ending Amount Unknown. This problem is not an annuity problem; our depositor is not putting money into the account each year. She deposits money out-of-pocket only once, and then simply lets it grow with interest over time. So we are concerned only with a beginning value and an ending value. Here is what is expected to happen year-by-year:

Year 1 Account grows from $2,000.00 to $2,000.00 x (1.0445) = $ 2,089.00
Year 2 Account grows from $2,089.00 to $2,089.00 x (1.0445) = $ 2,181.96
Year 3 Account grows from $2,181.96 to $2,181.96 x (1.0445) = $ 2,279.06
Year 4 Account grows from $2,279.06 to $2,279.06 x (1.0445) = $ 2,380.48
Year 5 Account grows from $2,380.48 to $2,380.48 x (1.0445) = $ 2,486.41
Year 6 Account grows from $2,486.41 to $2,486.41 x (1.0445) = $ <u>2,597.05</u>

She starts with $2,000 and it grows at a 4.45% average interest rate each year over six years, to an ending amount of $2,597.05. But there is no need to use the cumbersome year-by-year method shown above. With our general equation for solving "non-annuity" problems, we can say:

$$\text{Beginning Amount } (1 + r)^n = \text{Ending Amount} \quad \text{or}$$
$$\text{BAMT } (1 + r)^n = \text{EAMT}$$
$$\$2{,}000\ (1.0445)^6 = \text{EAMT}$$
$$\$2{,}000\ (1.298526) = \$\underline{2{,}597.05}$$

2. What would Amanda pay for an investment that provides no cash flows in years 1 through 9, but will provide a single payment of $18,750 at the end of year 10, if commitments of similar risk generate a 9% average annual rate of return?

Type: Non-Annuity; Beginning Amount Unknown. Again, we have a non-annuity situation, although here we are solving for the beginning amount rather than the ending amount. But the "skeleton" of the problem is the same; it is just a question of which unknown in the non-annuity problem format we are solving for. Here we must find the beginning amount that the individual would be willing to pay (or would have to deposit) today such that it would grow, at a 9% compounded average interest rate per year for 10 years, to reach $18,750 (in other words, we want to compute the present value of her right to collect $18,750 in 10 years if the annual discount rate is 9%):

$$\text{BAMT } (1 + r)^n = \text{EAMT}$$
$$\text{BAMT } (1.09)^{10} = \$18{,}750$$
$$\$18{,}750 \div (1.09)^{10} = \$18{,}750 \div (2.367364) = \text{BAMT} \quad \text{or}$$
$$\$18{,}750 \left(\frac{1}{1.09}\right)^{10} = \$18{,}750\ (.422411) = \text{BAMT} = \$\underline{7{,}920.20}$$

So if \$7,920.20 were deposited today, and the money were left on deposit for 10 years, the balance would grow to \$18,750 by the end of the tenth year if the account's growing balance earned a 9% average annual compounded rate of return.

3. Ronald, age 15, just inherited \$100,000 from his late great-uncle Hironymus. His mother, Ronette, says that if Ronald invests the money in a mutual fund he will "probably have a few million dollars" by the time he is 65. What average annual rate of return would Ronald have to earn for his \$100,000 is to grow to \$3,000,000 over 50 years?

Type: Non-Annuity; Rate of Return Unknown. This problem is yet another non-annuity example. There is not a series of cash flows (repeated payments into or out of some account), but rather a \$100,000 beginning amount and a \$3,000,000 desired ending amount, with no withdrawals or out-of-pocket deposits to be made in between. Here the average annual rate of return earned is the unknown, but the basic skeleton remains the same as for any non-annuity problem:

$$\text{BAMT}\,(1 + r)^n = \text{EAMT}$$
$$\$100{,}000\,(1 + r)^{50} = \$3{,}000{,}000$$
$$(1 + r)^{50} = 30$$

We could proceed here with "trial and error," but, there is a more systematic approach. We can cancel out an exponent by taking the corresponding root (e.g., something that has been squared can be "un-squared" with the square root). But we must always remember to do the same thing to both sides of the equation, to keep the equality intact. Here we will take the 50th root to undo the 50th power.

$$(1 + r)^{50} = 30$$
$$\sqrt[50]{(1+r)^{50}} = \sqrt[50]{30}$$

The square root is the ½ power; by the same token, the fiftieth root is the 1/50 power, so we will compute to the power of 1/50, which we accomplish by using the decimal equivalent, here 1/50 = .02:

$$1 + r = \sqrt[50]{30} = 30^{1/50} = 30^{.02} = 1.070391$$
$$\text{If } 1 + r = 1.070391 \text{ then } r = .070391 \text{ or } \underline{\underline{7.0391\%}}$$

Since, historically speaking, it is not unreasonable for someone investing in stocks and bonds (here, indirectly through a mutual fund) to expect an average annual compounded rate of return slightly above 7%, it is quite plausible that Ronald will have "a few million dollars" by the time he is 65.

4. B.W. makes a \$10,000 investment today. How long will it take for his money to grow to \$40,000 if he can earn a 5% average annual after-tax compounded rate of return on any balance in the account?

Type: Non-Annuity; Number of Periods Unknown. Again we have a non-annuity application, with a beginning amount and an ending amount but no deposits or withdrawals to be made in between. Here, however, we want to find the number of time periods, n, that makes the future value factor equal to 4 (a quadrupling of the money) if the annual rate of return r averages 5%. The basic "skeleton" remains the same; we set the problem up with our general non-annuity equation:

$$\text{BAMT}\,(1 + r)^n = \text{EAMT}$$
$$\$10{,}000\,(1.05)^n = \$40{,}000$$
$$(1.05)^n = 4$$

One way to solve at this stage would be to use "trial and error;" just plug in different values for n until we find the n for which the equation holds true. But a more systematic approach to solving is to use logarithms:

$$\text{If } (1.05)^n = 4$$
$$\text{Then } \ln [(1.05)^n] = \ln 4$$

And because a logarithm is an exponent, when we take the logarithm of something with an exponent the exponent factors out as a multiplier:

$$n \times \ln 1.05 = \ln 4$$
$$n\,(.048790) = 1.386294$$
$$n = \underline{28.413398},$$

or a little more than 28 years for the $10,000 initial value to grow to $40,000. Let's double-check by putting in 28.413398 as our exponent:

$$\$10{,}000\,(1.05)^{28.413398} = \$10{,}000\,(4.000000) = \$40{,}000 \ \checkmark$$

Here we have used "natural" logarithms, based on the irrational number e = 2.7182818… and shown as "ln" on scientific and financial calculators. But it also is fine to use base-10 logarithms (shown as "log" on scientific calculators). We're simply finding a proportional relationship, so logarithms based on any number will work – just be sure not to mix natural logs and base-10 logs in the same problem. [Using base-10 logarithms, shown as "log" on a scientific calculator, we would solve as n x log 1.05 = log 4; n (.021189) = .602060; n = .602060 ÷.021189 = the exact same 28.413398 years.]

5. If Gladys can earn a 3.75% annual rate of return on her account's growing balance from year to year, how much will she have by the end of year 6 if she makes the series of beginning-of-year deposits described in each of the situations listed below? (Another way to word this problem is: what is the future value of each of the following cash flow streams, with beginning-of-year cash flows and a 3.75% annual compounding rate?)

a) $500 in year 1, $900 in year 2, $400 in year 3, $800 in year 4, $100 in year 5, $300 in year 6

Type: Future Value of a Series of Payments. Because the amounts to be deposited differ, with no pattern, from year to year we can not lump them together and use the distributive property – we must perform a series of non-annuity computations. With deposits at the beginning of each year, the first deposit will earn interest six times (while the last deposit will be made at the beginning of year 6, and thus will earn interest for one year before the account's final balance is tabulated). The amount she will have at the end of year 6 if she deposits $500 at the beginning of year 1 is

$$\text{BAMT}\,(1 + r)^n = \text{EAMT}$$
$$\$500\,(1.0375)^6 = \text{EAMT}$$
$$\$500 \times 1.247179 = \$\underline{623.59}$$

The amounts she would have by the end of year 6 if she deposited $900 at the start of year 2, $400 at the start of year 3, $800 at the start of year 4, $100 at the start of year 5, and $300 at the start of year 6 are as follows:

$$\$900\,(1.0375)^5 = \text{EAMT}$$
$$\$900 \times 1.202100 = \$\underline{1{,}081.89}$$

$$\$400\ (1.0375)^4 = \text{E}_{\text{AMT}}$$
$$\$400 \times 1.158650 = \$\underline{463.46}$$

$$\$800\ (1.0375)^3 = \text{E}_{\text{AMT}}$$
$$\$800 \times 1.116771 = \$\underline{893.42}$$

$$\$100\ (1.0375)^2 = \text{E}_{\text{AMT}}$$
$$\$100 \times 1.076406 = \$\underline{107.64}$$

$$\$300\ (1.0375)^1 = \text{E}_{\text{AMT}}$$
$$\$300 \times 1.0375 = \$\underline{311.25}$$

The total amount she should have by the end of year 6 is the sum of the six underlined amounts, or $\$\underline{\underline{3{,}481.25}}$ (it could be in six separate accounts, or could all be in one account at the same bank). She deposits \$500 + \$900 + \$400 + \$800 + \$100 + \$300 = \$3,000 over 6 years, and ends up with \$3,481.25, with the difference consisting of the 3.75% annual interest earned on the account's growing balance from year to year.

b) \$500 in each of years 1 through 6 (compute both by compounding individual year payments to the end of year 6, and by grouping these equal payments together with the distributive property and corresponding annuity factor)

Because the amounts to be deposited are the same each year, we can lump them together and use the distributive property. But we don't have to; let's begin by doing a series of non-annuity computations. If she deposits \$500 at the start of year 1, at the end of year 6 she will have

$$\text{B}_{\text{AMT}}\ (1 + r)^n = \text{E}_{\text{AMT}}$$
$$\$500\ (1.0375)^6 = \text{E}_{\text{AMT}}$$
$$\$500 \times 1.247179 = \$\underline{623.59}$$

The amounts she would have by the end of year 6 if she deposited \$500 at the start of each of years 2 through 6 are as follows:

$$\$500\ (1.0375)^5 = \text{E}_{\text{AMT}}$$
$$\$500 \times 1.202100 = \$\underline{601.05}$$

$$\$500\ (1.0375)^4 = \text{E}_{\text{AMT}}$$
$$\$500 \times 1.158650 = \$\underline{579.32}$$

$$\$500\ (1.0375)^3 = \text{E}_{\text{AMT}}$$
$$\$500 \times 1.116771 = \$\underline{558.39}$$

$$\$500\ (1.0375)^2 = \text{E}_{\text{AMT}}$$
$$\$500 \times 1.076406 = \$\underline{538.20}$$

$$\$500\ (1.0375)^1 = \text{E}_{\text{AMT}}$$
$$\$500 \times 1.0375 = \$\underline{518.75}$$

The total amount the saver should have by the end of year 6 is the sum of the six underlined amounts, or $\$\underline{\underline{3{,}419.30}}$ (it could be in six separate accounts, or could all be in one account). She deposits \$500 x 6 = \$3,000 over 6 years, and ends up with \$3,419.30, with the difference consisting of the 3.75% annual interest earned on the account's growing balance from year to year.

So having equal cash flows from year to year does not prevent us from using the non-annuity approach to our computations; we can think of a series of deposits that we deal with individually. However, when the cash flows are projected to be equal, we can use the annuity formula shortcut, based on the distributive property. Here we have the future value of an annuity (with beginning-of-year deposits it is an annuity due), because the equal cash flows correspond to a lump sum of money that will not exist until a future date:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$500\,[(1.0375)^6 + (1.0375)^5 + (1.0375)^4 + (1.0375)^3 + (1.0375)^2 + (1.0375)^1] = \text{TOT}$$
$$\$500\left[\left(\frac{(1.0375)^6-1}{.0375}\right)(1.0375)\right] = \text{TOT}$$
$$\$500 \times 6.838607 = \$\underline{3,419.30}$$

(just as we found above when treating the deposits separately). So here we see that the future value of an annuity is just the sum of the future values of a series of individual cash flows compounded at the same rate over the same number of time periods. Another way to state the issue is that the future value of a level annuity due factor is the sum of the future value of a dollar factors for the same number of time periods (here the last of the future value of a dollar factors has an exponent of 1, because the last beginning-of-year deposit does earn interest for a year) and the same expected annual rate of return. Here, 1.247179 + 1.202100 + 1.158650 + 1.116771 + 1.076406 + 1.0375 = 6.838607, the future value of a level annuity due factor.

c) $500 in year 1, $506.25 in year 2, $512.58 in year 3, $518.99 in year 4, $525.47 in year 5, and $532.04 in year 6 (amounts that increase by 1.25% from year to year; compute both by compounding individual year payments to the end of year 6, and by grouping these related payments together with the distributive property and corresponding annuity factor)

The amounts deposited are not equal, but because they follow a convenient pattern (changing from period to period by a constant percentage) we can lump them together and use the distributive property. But we don't have to; let's begin by doing a series of non-annuity computations. The amount our saver will have at the end of year 6 if she deposits $500 at the beginning of year 1 is, as in all the earlier examples,

$$\text{BAMT}\,(1 + r)^n = \text{EAMT}$$
$$\$500\,(1.0125)^0\,(1.0375)^6 = \$500\,(1.0375)^6 = \text{EAMT}$$
$$\$500 \times 1.247179 = \$\underline{623.59}$$

Amounts she would have by the end of year 6 if she made the indicated deposits at the starts of each of years 2 through 6 are as follows:

$$\$500\,(1.0125)^1\,(1.0375)^5 = \$506.25\,(1.0375)^5 = \text{EAMT}$$
$$\$506.25 \times 1.202100 = \$\underline{608.56}$$

$$\$500\,(1.0125)^2\,(1.0375)^4 = \$512.58\,(1.0375)^4 = \text{EAMT}$$
$$\$512.58 \times 1.158650 = \$\underline{593.90}$$

$$\$500\,(1.0125)^3\,(1.0375)^3 = \$518.99\,(1.0375)^3 = \text{EAMT}$$
$$\$518.99 \times 1.116771 = \$\underline{579.59}$$

$$\$500\,(1.0125)^4\,(1.0375)^2 = \$525.47\,(1.0375)^2 = \text{EAMT}$$
$$\$525.47 \times 1.076406 = \$\underline{565.62}$$

$$\$500\ (1.0125)^5\ (1.0375)^1 = \$532.04\ (1.0375)^1 = \text{EAMT}$$
$$\$532.04 \times 1.0375 = \$\underline{551.99}$$

Thus the total amount she will have by the end of year 6 is the sum of the six underlined amounts, or $\underline{3,523.25}$ (it could be in six separate accounts, or could be all in one account). She will deposit \$3,095.33 during the six years, and then have a total of \$3,523.25 by the end of year 6, with the difference consisting of 3.75% interest earned on the account's growing balance from year to year.

Having cash flows that vary by a constant percentage from year to year clearly does not prevent us from using the non-annuity approach to computing; we can think of a series of deposits that we deal with individually. However, FIL 404 students should note that when the cash flows are projected to change by a constant percentage we can use the annuity formula shortcut, based on the distributive property. Here we have the future value of a changing annuity due, because the constantly changing beginning-of-year deposits correspond to a lump sum total of money that will not exist until a future date:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$500\ [(1.0125)^0 (1.0375)^6 + (1.0125)^1 (1.0375)^5 + (1.0125)^2 (1.0375)^4 + (1.0125)^3 (1.0375)^3$$
$$+ (1.0125)^4 (1.0375)^2 + (1.0125)^5 (1.0375)^1] = \text{TOT}$$
$$\$500 \left[\left(\frac{(1.0375)^6 - (1.0125)^6}{.0375 - .0125}\right)(1.0375)\right] = \text{TOT}$$
$$\$500 \times 7.046508 = \$\underline{3,523.25}$$

(just as we found above when treating the cash flows separately).

6. If Charles can earn a 3.75% annual rate of return on amounts remaining in his account from year to year, how much must he have on deposit today to make the series of year-end withdrawals described in each of the situations listed below? (Another way to word this problem is: what is the present value of each of the following cash flow streams, with year-end cash flows and a 3.75% annual discount rate? Another way is to ask how large a loan a borrower could repay with the payments described?)

a) \$500 in year 1, \$900 in year 2, \$400 in year 3, \$800 in year 4, \$100 in year 5, \$300 in year 6

Type: Present Value of a Series of Payments. Because the amounts to be received differ without a pattern from year to year, we can not lump them together and use the distributive property; we must proceed with a series of non-annuity computations. The amount he must have on deposit today in order to withdraw \$500 at the end of year 1 is

$$\text{BAMT}\ (1 + r)^n = \text{EAMT}$$
$$\text{BAMT}\ (1.0375)^1 = \$500$$
$$\text{BAMT} = \$500 \div (1.0375)^1 \quad \text{or}$$
$$\text{BAMT} = \$500 \left(\frac{1}{1.0375}\right)^1 = \$\underline{481.93}$$

The amounts he would deposit today to be able to withdraw (or would pay today for the right to collect) \$900 at the end of year 2, \$400 at the end of year 3, \$800 at the end of year 4, \$100 at the end of year 5, and \$300 at the end of year 6 are as follows:

$$\text{BAMT} = \$900 \div (1.0375)^2 \quad \text{or}$$

$$\text{BAMT} = \$900 \left(\frac{1}{1.0375}\right)^2 = \$\underline{836.12}$$

$$\text{BAMT} = \$400 \div (1.0375)^3 \quad \text{or}$$

$$\text{BAMT} = \$400 \left(\frac{1}{1.0375}\right)^3 = \$\underline{358.18}$$

$$\text{BAMT} = \$800 \div (1.0375)^4 \quad \text{or}$$

$$\text{BAMT} = \$800 \left(\frac{1}{1.0375}\right)^4 = \$\underline{690.46}$$

$$\text{BAMT} = \$100 \div (1.0375)^5 \quad \text{or}$$

$$\text{BAMT} = \$100 \left(\frac{1}{1.0375}\right)^5 = \$\underline{83.19}$$

$$\text{BAMT} = \$300 \div (1.0375)^6 \quad \text{or}$$

$$\text{BAMT} = \$300 \left(\frac{1}{1.0375}\right)^6 = \$\underline{240.54}$$

Thus the total that should be on hand today is the sum of the six underlined amounts, or $\$\underline{\underline{2,690.42}}$ (it could be in six separate accounts, or all in one account). He starts with $2,690.42 today, and can take out a total of $500 + $900 + $400 + $800 + $100 + $300 = $3,000 over the ensuing 6 years, with the difference consisting of interest earned on the remaining balance from year to year.

b) $500 in each of years 1 through 6 (compute both by discounting individual year payments to present values, and by grouping these equal payments together with the distributive property and corresponding annuity factor)

Because the amounts to be received are the same each year, we can lump them together and use the distributive property. But we don't have to; let's begin with a series of non-annuity computations. The amount that must be on deposit today if the account holder wants to withdraw $500 at the end of year 1 is, as in part a,

$$\text{BAMT}\ (1 + r)^n = \text{EAMT}$$

$$\text{BAMT}\ (1.0375)^1 = \$500$$

$$\text{BAMT} = \$500 \div (1.0375)^1 = \$500 \quad \text{or}$$

$$\text{BAMT} = \$500 \left(\frac{1}{1.0375}\right)^1 = \$\underline{481.93}$$

The amounts he would deposit today to be able to withdraw (or would pay today for the right to collect) $500 at the ends of each of years 2 through 6 are as follows:

$$\text{BAMT} = \$500 \div (1.0375)^2 \quad \text{or}$$

$$\text{BAMT} = \$500 \left(\frac{1}{1.0375}\right)^2 = \$\underline{464.51}$$

$$\text{BAMT} = \$500 \div (1.0375)^3 \quad \text{or}$$

$$\text{BAMT} = \$500 \left(\frac{1}{1.0375}\right)^3 = \$\underline{447.72}$$

$$\text{BAMT} = \$500 \div (1.0375)^4 \quad \text{or}$$

$$\text{BAMT} = \$500 \left(\frac{1}{1.0375}\right)^4 = \$\underline{431.54}$$

$$\text{BAMT} = \$500 \div (1.0375)^5 \quad \text{or}$$
$$\text{BAMT} = \$500 \left(\frac{1}{1.0375}\right)^5 = \$\underline{415.94}$$

$$\text{BAMT} = \$500 \div (1.0375)^6 \quad \text{or}$$
$$\text{BAMT} = \$500 \left(\frac{1}{1.0375}\right)^6 = \$\underline{400.90}$$

Thus the total amount he would need to have on deposit today is the sum of the six underlined amounts, or $\$\underline{\underline{2,642.54}}$ (spread over six separate accounts, or all in one account). Charles deposits \$2,642.54 today, and then can withdraw a total of \$500 x 6 = \$3,000 over the ensuing six years, with the difference consisting of the 3.75% interest earned on the account's remaining balance from year to year.

So having equal cash flows from year to year does not prevent us from using the non-annuity approach to our computations; we can think of a series of cash flows to deal with individually. However, when payments in or out of some plan are projected to be equal we can use the annuity formula shortcut, based on the distributive property. Here we have the present value of an annuity, because the equal cash flows correspond to a lump sum of money that exists in the present (the amount that would have to be on deposit or on hand today so he could make the indicated withdrawals, or the amount he would pay today for the right to receive the indicated payments, or the amount he could borrow today in return for agreeing to make the indicated payments):

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$500 \left[\left(\frac{1}{1.0375}\right)^1 + \left(\frac{1}{1.0375}\right)^2 + \left(\frac{1}{1.0375}\right)^3 + \left(\frac{1}{1.0375}\right)^4 + \left(\frac{1}{1.0375}\right)^5 + \left(\frac{1}{1.0375}\right)^6\right] = \text{TOT}$$
$$\$500 \left(\frac{1-\left(\frac{1}{1.0375}\right)^6}{.0375}\right) = \text{TOT}$$
$$\$500 \times 5.285072 = \$\underline{\underline{2,642.54}}$$

(just as we found above when treating the cash flows separately). So here we see that the present value of an annuity is just the sum of the PVs of a series of individual cash flows discounted at the same rate over the same number of time periods. Another way to state the issue is that the present value of a level ordinary annuity factor is the sum of the present value of a dollar factors for the same number of periods and the same discount rate. Here, .963855 + .929017 + .895438 + .863073 + .831878 + .801810 = 5.285072, the present value of a level ordinary annuity factor. (The first exponent in the series is 1 because a year passes before the first \$500 is taken out.)

c) \$500 in year 1, \$506.25 in year 2, \$512.58 in year 3, \$518.99 in year 4, \$525.47 in year 5, and \$532.04 in year 6 (amounts that increase by 1.25% from year to year; compute both by discounting individual year payments to present values, and by grouping these related payments together with the distributive property and corresponding annuity factor)

Because the amounts to be received follow a convenient pattern (changing from period to period by a constant percentage), we can lump them together and use the distributive property. But we don't have to; let's begin by doing a series of non-annuity computations. The amount Charles must have on deposit today in order to withdraw \$500 at the end of year 1 is, as in all the earlier examples:

$$\text{BAMT}\ (1 + r)^n = \text{EAMT}$$
$$\text{BAMT}\ (1.0375)^1 = \$500$$
$$\text{BAMT} = \$500\ (1.0125)^0 \div (1.0375)^1 = \$500 \div (1.0375)^1 \quad \text{or}$$
$$\text{BAMT} = \$500\ (1.0125)^0 \left(\frac{1}{1.0375}\right)^1 = \$\underline{481.93}$$

The amounts he would deposit today to be able to withdraw (or would pay today for the right to collect, or could borrow today in return for the obligation to pay) the indicated amounts at the ends of each of years 2 through 6 are as follows:

$$\text{BAMT} = \$500\ (1.0125)^1 \div (1.0375)^2 = \$506.25 \div (1.0375)^2 \quad \text{or}$$
$$\text{BAMT} = \$500\ (1.0125)^1 \left(\frac{1}{1.0375}\right)^2 = \$506.25 \left(\frac{1}{1.0375}\right)^2 = \$\underline{470.31}$$

$$\text{BAMT} = \$500\ (1.0125)^2 \div (1.0375)^3 = \$512.58 \div (1.0375)^3 \quad \text{or}$$
$$\text{BAMT} = \$500\ (1.0125)^2 \left(\frac{1}{1.0375}\right)^3 = \$512.58 \left(\frac{1}{1.0375}\right)^3 = \$\underline{458.98}$$

$$\text{BAMT} = \$500\ (1.0125)^3 \div (1.0375)^4 = \$518.99 \div (1.0375)^4 \quad \text{or}$$
$$\text{BAMT} = \$500\ (1.0125)^3 \left(\frac{1}{1.0375}\right)^4 = \$518.99 \left(\frac{1}{1.0375}\right)^4 = \$\underline{447.93}$$

$$\text{BAMT} = \$500\ (1.0125)^4 \div (1.0375)^5 = \$525.47 \div (1.0375)^5 \quad \text{or}$$
$$\text{BAMT} = \$500\ (1.0125)^4 \left(\frac{1}{1.0375}\right)^5 = \$525.47 \left(\frac{1}{1.0375}\right)^5 = \$\underline{437.13}$$

$$\text{BAMT} = \$500\ (1.0125)^5 \div (1.0375)^6 = \$532.04 \div (1.0375)^6 \quad \text{or}$$
$$\text{BAMT} = \$500\ (1.0125)^5 \left(\frac{1}{1.0375}\right)^6 = \$532.04 \left(\frac{1}{1.0375}\right)^6 = \$\underline{426.59}$$

Thus the total Charles would want to have available today is the sum of the six underlined amounts, or $\$\underline{\underline{2,722.87}}$ (it could be in six separate accounts, or could be all in one account). He deposits that amount today, and then can withdraw \$3,095.33 in total over the ensuing 6 years, with the difference consisting of the 3.75% interest earned on the remaining balance from year to year.

So cash flows that vary by a constant percentage from year to year do not prevent using the non-annuity approach to computing; we can think of a series of cash flows that we deal with individually. However, FIL 404 students should note that when the withdrawals are projected to change by a constant percentage from period to period (here, 1.25%), we can use an annuity formula shortcut, based on the distributive property. Here we have the PV of a changing ordinary annuity, because constantly changing end-of-period cash flows correspond to a lump sum of money that exists in the present (the amount he would have to deposit today so he could make the indicated withdrawals, or would pay/borrow today for the right to receive/obligation to make the indicated payments):

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$500\left[(1.0125)^0\left(\frac{1}{1.0375}\right)^1 + (1.0125)^1\left(\frac{1}{1.0375}\right)^2 + (1.0125)^2\left(\frac{1}{1.0375}\right)^3 + (1.0125)^3\left(\frac{1}{1.0375}\right)^4 + (1.0125)^4\left(\frac{1}{1.0375}\right)^5 + (1.0125)^5\left(\frac{1}{1.0375}\right)^6\right] = \text{TOT}$$
$$\$500\left(\frac{1-\left(\frac{1.0125}{1.0375}\right)^6}{.0375-.0125}\right) = \text{TOT}$$
$$\$500 \times 5.4457437 = \$\underline{\underline{2,722.87}}$$

(just as found above when we treated the cash flows separately).

7. TLM Mutual Funds receives $4,000 from Sharon, a fund investor, at the end of each year. TLM expects to credit every investor's account with a 6.5% compounded average annual rate of return. How much should TLM expect to owe Sharon after five years? What if instead she makes her $4,000 contribution at the beginning of each year? Compute using individual year payment factors, annuity factors, and year-by-year breakdowns of all payments.

Type: FV of Annuity; Total Unknown. This problem is an annuity example, with a series of equal or related payments into or out of an account, and payments equally spaced in time. We could compound the cash flows individually, but with equal or related payments we can make use of the distributive property to facilitate the computations. Our equation for handling annuity situations is

Payment x Factor = Total or PMT x FAC = TOT

The payment PMT is the $4,000 invested by the saver and received by the fund manager each year. This series of level cash flows corresponds in time value-adjusted terms, as every series of deposits or withdrawals must correspond in time value-adjusted terms, to a large lump sum of money (large in comparison to the payments). Here, this lump sum (TOT) will not exist intact until a future date, the end of year 5, so we have a future value of a level annuity problem, and our factor must be an FV of a level annuity factor (sum of a group of FV of $1 factors). If the investment is made at the end of each year it is a level ordinary annuity and we use the FV of a level ordinary annuity factor:

PMT x FAC = TOT

$$\$4{,}000\ [(1.065)^4 + (1.065)^3 + (1.065)^2 + (1.065)^1 + (1.065)^0] = \$4{,}000 \left(\frac{(1.065)^5 - 1}{.065}\right) = \text{TOT}$$

$4,000 x 5.693641 = $22,774.56

If the fund company receives the investment at the beginning of each year, it is a level annuity due and therefore we use the future value of a level annuity due factor:

$$\$4{,}000\ [(1.065)^5 + (1.065)^4 + (1.065)^3 + (1.065)^2 + (1.065)^1]$$

$$= \$4{,}000\ [(1.065)^4 + (1.065)^3 + (1.065)^2 + (1.065)^1 + (1.065)^0]\ [(1.065)]$$

$$= \$4{,}000 \left[\left(\frac{(1.065)^5 - 1}{.065}\right)(1.065)\right] = \text{TOT}$$

$4,000 x 6.063728 = $24,254.91

Two points we might want to note here. First, people sometimes say "if you pay money into an account it is FV of annuity, and if you take money out it is PV of annuity." But remember there are two sides to every annuity situation, one side paying in and the other side receiving or taking out, and the numbers are the same for both sides of the relationship. Saving up for retirement might be viewed as the classic FV of annuity example. Here someone is saving for retirement, but here we analyze primarily from the viewpoint of the investment manager that receives the regular payments along the way and will have to hand over the big amount when the account matures in five years.

Second, common sense can help us analyze this kind of problem. Let's say we somehow misidentified the situation as present value of an annuity, and computed $4,000 x 4.155679 = TOT = $16,622.72. But the fund company is getting $4,000 x 5 = $20,000 from the saver, so even if it paid her a zero periodic rate of return it would end up owing her $20,000. With interest (or however we classify the rate of return), it should expect to owe her more than $20,000 after 5 years. And receiving the investment at the beginning of each year should lead to a greater total owed than we would see with year-end deposits. Recall that three characteristics always apply to an FV of annuity problem: the big amount that relates to the payments will not exist intact until a future date, the factor's value exceeds the number of payments, and the periodic interest or other return is applied to a growing balance over time. For year-end investments, year-by-year cash flows in this problem are:

Year	Beginning Balance	Plus 6.5% Interest	Balance Before Deposit	Plus Deposit	Ending Balance
1	$ 0	$ 0	$ 0	$4,000.00	$ 4,000.00
2	$ 4,000.00	$ 260.00	$ 4,260.00	$4,000.00	$ 8,260.00
3	$ 8,260.00	$ 536.90	$ 8,796.90	$4,000.00	$12,796.90
4	$12,796.90	$ 831.80	$13,628.70	$4,000.00	$17,628.70
5	$17,628.70	$1,145.86	$18,774.56	$4,000.00	$22,774.56

(notice there are 5 cash flows but only 4 applications of interest). For start-of-year investments:

Year	Beginning Balance	Plus Deposit	Balance Before Interest	Plus 6.5% Interest	Ending Balance
1	$ 0	$4,000.00	$ 4,000.00	$ 260.00	$ 4,260.00
2	$ 4,260.00	$4,000.00	$ 8,260.00	$ 536.90	$ 8,796.90
3	$ 8,796.90	$4,000.00	$12,796.90	$ 831.80	$13,628.70
4	$13,628.70	$4,000.00	$17,628.70	$1,145.87	$18,774.56
5	$18,774.56	$4,000.00	$22,774.56	$1,480.35	$24,254.91

(there are 5 cash flows and 5 applications of interest). Also note that we could have worked this problem as a series of individual non-annuity problems, which we would have to do if amounts received were expected to be different and unrelated from year to year. But with equal or related cash flows the distributive property comes into play, and we can use the annuity shortcut method.

8. Beth has been working as a paralegal while saving money to go to law school. She wonders if she now has enough money in the bank to meet her goals. She wants to be able to withdraw $9,000 each year for 5 years (law school plus two years clerking for a federal judge) to help pay her living costs. She expects her account's declining balance to earn a 7% average annual interest rate. How much money must Beth have on deposit toady if she plans to make her withdrawal at the end of each year? What if she plans to take the $9,000 out at the start of each year? Compute using individual year payment factors, annuity factors, and year-by-year breakdowns of all payments

Type: PV of Annuity; Total Unknown. This problem is also an annuity example: we have a series of equal or related payments into or out of an account, with the payments equally spaced in time. Our equation for handling annuity situations is

Payment x Factor = Total or PMT x FAC = TOT

The payment PMT is the $9,000 to be withdrawn each year. This series of equal cash flows corresponds, in time value-adjusted terms, to a large lump sum of money (TOT) that exists intact today (in the present), so we have a present value of a level annuity problem, and our factor must be a PV of a level annuity factor (sum of a group of PV of $1 factors). If the withdrawal is to be made at the end of each year it is a level ordinary annuity situation and we use the present value of a level ordinary annuity factor:

PMT x FAC = TOT

$$\$9{,}000\left[\left(\frac{1}{1.07}\right)^1+\left(\frac{1}{1.07}\right)^2+\left(\frac{1}{1.07}\right)^3+\left(\frac{1}{1.07}\right)^4+\left(\frac{1}{1.07}\right)^5\right]=\$9{,}000\left(\frac{1-\left(\frac{1}{1.07}\right)^5}{.07}\right)=\text{TOT}$$

$$\$9{,}000 \times 4.100197 = \$\underline{\underline{36{,}901.78}}$$

If the withdrawal is made at the beginning of each year it is a level annuity due, and we use the present value of a level annuity due factor (again the sum of a group of PV of $1 factors):

$$\$9{,}000\left[\left(\frac{1}{1.07}\right)^0+\left(\frac{1}{1.07}\right)^1+\left(\frac{1}{1.07}\right)^2+\left(\frac{1}{1.07}\right)^3+\left(\frac{1}{1.07}\right)^4\right]$$
$$=\$9{,}000\left[\left(\frac{1}{1.07}\right)^1+\left(\frac{1}{1.07}\right)^2+\left(\frac{1}{1.07}\right)^3+\left(\frac{1}{1.07}\right)^4+\left(\frac{1}{1.07}\right)^5\right][(1.07)]$$
$$=\$9{,}000\left[\left(\frac{1-\left(\frac{1}{1.07}\right)^5}{.07}\right)(1.07)\right]=\text{TOT}$$
$$\$9{,}000\times 4.387211=\$\underline{\underline{39{,}484.90}}$$

Common sense can help us; let's say we somehow misidentify the situation as future value of an annuity, and compute $9,000 x 5.750739 = TOT = $51,756.65 for the end-of-year payments case. But the plan is to take out a total of only $9,000 x 5 = $45,000. So even if the account earned a zero interest rate she could fund the planned withdrawal series with only $45,000 on deposit today.

Because interest will be earned on the declining balance from year to year, she can start with less than $45,000 and still withdraw $45,000 over time. Of course, taking withdrawals at the beginning of each year necessitates having more today than if she let interest accumulate for a year before making the first of the five end-of-year withdrawals. Recall that three characteristics always apply to a PV of annuity problem: the big amount that corresponds to the payments exists intact in the present (when the plan is made), the factor's value is smaller than the number of payments, and interest (or however we classify the rate of return) is applied to a declining balance over time. Working year-by-year through the cash flows in this problem, for year-end withdrawals, we have:

Year	Beginning Balance	Plus 7% Interest	Balance Before Withdrawal	Minus Withdrawal	Ending Balance
1	$36,901.78	$2,583.12	$39,484.90	$9,000.00	$30,484.90
2	$30,484.90	$2,133.94	$32,618.84	$9,000.00	$23,618.84
3	$23,618.84	$1,653.32	$25,272.16	$9,000.00	$16,272.16
4	$16,272.16	$1,139.06	$17,411.22	$9,000.00	$ 8,411.22
5	$ 8,411.22	$ 588.78	$ 9,000.00	$9,000.00	$ 0

For beginning of year withdrawals:

Year	Beginning Balance	Minus Withdrawal	Balance Before Interest	Plus 7% Interest	Ending Balance
1	$39,484.90	$9,000.00	$30,484.90	$2,133.94	$32,618.84
2	$32,618.84	$9,000.00	$23,618.84	$1,653.32	$25,272.16
3	$25,272.16	$9,000.00	$16,272.16	$1,139.06	$17,411.22
4	$17,411.22	$9,000.00	$ 8,411.22	$ 588.78	$ 9,000.00
5	$ 9,000.00	$9,000.00	$ 0	$ 0	$ 0

9. As Mike blew out the candles on his 31st birthday cake today, he made a wish: to be able to buy a new Mercedes on his 35th birthday. He expects that a typical Mercedes will cost $65,000 four years from now, and he currently has no money saved toward making that large purchase. If he can earn a 5.25% compounded average annual rate of return on his growing savings balance, how much must he deposit into his account at the end of each year to accumulate $65,000 over 4 years? What if he instead made his deposits at the beginning of each year?

Type: FV of Annuity; Payment Unknown. Making regular savings deposits is a very common future value of an annuity application; it is an annuity situation because of the series of equal or related cash flows (the deposits), and it is FV of an annuity because the lump sum (the amount the saver is trying to accumulate) will not exist intact until a future date. The level amount he would have to deposit at the end of each year is

PMT x FAC = TOT

PMT $[(1.0525)^3 + (1.0525)^2 + (1.0525)^1 + (1.0525)^0]$ = PMT $\left(\frac{(1.0525)^4-1}{.0525}\right)$ = $65,000

PMT x 4.326170 = $65,000

PMT = $65,000 ÷ 4.326170 = $15,024.84

He will deposit a total of 4 x $15,024.84 = $60,099.36, but his total balance will be a greater $65,000 because of the interest that will be earned along the way. If instead he added to his account at the beginning of each year, he could get by with making smaller annual deposits of

PMT x FAC = TOT

PMT $[(1.0525)^4 + (1.0525)^3 + (1.0525)^2 + (1.0525)^1]$ = PMT $\left[\left(\frac{(1.0525)^4-1}{.0525}\right)(1.0525)\right]$ = $65,000

PMT x 4.553294 = $65,000

PMT = $65,000 ÷ 4.553294 = $14,275.38

Here he would put in only 4 x $14,628.27 = $58,513.08, but with earlier deposits there would be a larger balance earning interest at any point in time, and with more in interest earnings he still would reach his $65,000 target. Looking at things year by year, for year-end deposits we have:

Year	Beginning Balance	Plus 5.25% Interest	Balance Before Deposit	Plus Deposit	Ending Balance
1	$ 0	$ 0	$ 0	$15,024.84	$15,024.84
2	$15,024.84	$ 788.80	$15,813.64	$15,024.84	$30,838.48
3	$30,838.48	$1,619.02	$32,457.50	$15,024.84	$47,482.34
4	$47,482.34	$2,492.82	$49,975.16	$15,024.84	$65,000.00

(there are 4 deposits but only 3 applications of interest.) For beginning of year deposits, we have:

Year	Beginning Balance	Plus Deposit	Balance Before Interest	Plus 5.25% Interest	Ending Balance
1	$ 0	$14,275.38	$14,275.38	$ 749.46	$15,024.84
2	$15,024.84	$14,275.38	$29,300.21	$1,538.26	$30,838.47
3	$30,838.47	$14,275.38	$45,113.85	$2,368.48	$47,482.33
4	$47,482.33	$14,275.38	$61,757.71	$3,242.28	$65,000.00

(there are 4 deposits, with interest earned 4 times). Again the 4.326170 or 4.553294 factor is bigger than the 4 payments, interest is applied to a growing balance as time passes, and the $65,000 desired total will not be intact until the end of future year 4. (Some slight rounding differences can result from showing payment figures in terms of whole, and not fractional, cents.)

10. Curt, the manager and bass player for a central Illinois country/rock group, wants to buy some new amplifying equipment. A bank is willing to lend $14,000 toward the purchase of the $18,500 worth of needed equipment. However, because people's tastes in music can change over time the loan officer views the loan as a fairly risky one, and thus quotes an 11.5% annual interest rate.

a. If the loan is to be fully amortized, with equal end-of-year annual payments over 6 years, what should the amount of each payment be? What if beginning-of-year payments instead were to be made?

Type: PV of Annuity; Payment Unknown. Repaying a loan with equal payments is a present value of an annuity application; it is an annuity situation because of the series of equal or related payments, and it is PV of an annuity because the lump sum (the amount being lent) exists intact today, in the present. We find that each of six year-end payments should be

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left[\left(\frac{1}{1.115}\right)^1+\left(\frac{1}{1.115}\right)^2+\left(\frac{1}{1.115}\right)^3+\left(\frac{1}{1.115}\right)^4+\left(\frac{1}{1.115}\right)^5+\left(\frac{1}{1.115}\right)^6\right] = \text{PMT}\left(\frac{1-\left(\frac{1}{1.115}\right)^6}{.115}\right) = \$14{,}000$$

$$\text{PMT} \times 4.170294 = \$14{,}000$$

$$\text{PMT} = \$14{,}000 \div 4.170294 = \$\underline{\underline{3{,}357.08}}$$

Curt borrows $14,000 and pays back $3,357.08 at the end of each year for six years, for a total of $3,357.08 x 6 = $20,142.48. The extra $6,142.48 represents the total amount of interest paid on unpaid principal over the loan's life. The unlikely start-of-year payment each year would be a lower

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left[\left(\frac{1}{1.115}\right)^0+\left(\frac{1}{1.115}\right)^1+\left(\frac{1}{1.115}\right)^2+\left(\frac{1}{1.115}\right)^3+\left(\frac{1}{1.115}\right)^4+\left(\frac{1}{1.115}\right)^5\right] = \text{PMT}\left[\left(\frac{1-\left(\frac{1}{1.115}\right)^6}{.115}\right)(1.115)\right] = \$14{,}000$$

$$\text{PMT} \times 4.649878 = \$14{,}000$$

$$\text{PMT} = \$14{,}000 \div 4.649878 = \$\underline{\underline{3{,}010.83}}$$

Here Curt borrows $14,000 and pays back $3,010.83 x 6 = $18,604.98. Less in total (and thus less total interest, since principal repaid is $14,000 in either case) would be paid than with end-of-year payments, because with beginning-of-year payments some principal would be repaid immediately, and thus there would be less principal remaining to charge interest on at any point. As we always see in PV of annuity cases the large $14,000 amount borrowed changes hands in the present, the 4.170294 or 4.649878 computed factor is smaller than the number of payments (6), and interest is applied to a declining balance as time passes. Year-by-year, for year-end payments we have:

Year	Beginning Balance	Plus 11.5% Interest	Balance Before Payment	Minus Payment	Ending Balance
1	$14,000.00	$1,610.00	$15,610.00	$3,357.08	$12,252.92
2	$12,252.92	$1,409.09	$13,662.01	$3,357.08	$10,304.93
3	$10,304.93	$1,185.07	$11,490.00	$3,357.08	$ 8,132.92
4	$ 8,132.92	$ 935.29	$ 9,068.21	$3,357.08	$ 5,711.13
5	$ 5,711.13	$ 656.78	$ 6,367.91	$3,357.08	$ 3,010.83
6	$ 3,010.83	$ 346.25	$ 3,357.08	$3,357.08	$ 0

For beginning of year payments, we have:

Year	Beginning Balance	Minus Payment	Balance Before Interest	Plus 11.5% Interest	Ending Balance
1	$14,000.00	$3,010.83	$10,989.17	$1,263.75	$12,252.92
2	$12,252.92	$3,010.83	$ 9,242.09	$1,062.84	$10,304.93
3	$10,304.93	$3,010.83	$ 7,294.10	$ 838.82	$ 8,132.92

4	$ 8,132.92	$3,010.83	$ 5,122.09	$ 589.04	$ 5,711.13
5	$ 5,711.13	$3,010.83	$ 2,700.30	$ 310.53	$ 3,010.83
6	$ 3,010.83	$3,010.83	$ 0	$ 0	$ 0

Note that the total payment made each year is unchanging, but the amount of interest paid in each successive year is declining, so the amount of principal repaid in each successive year increases correspondingly. For example, in with year-end payments principal repaid in year 2 is $12,252.92 – $10,304.93 = $1,947.99; whereas principal repaid in year 5 is $5,711.13 – $3,010.83 = $2,700.30.

[PARTS B AND C MAY BE OF INTEREST IF YOU LIKE WORKING THE NUMBERS, BUT ARE NOT ESSENTIAL TO OUR COVERAGE IF YOU WANT TO SKIP THEM.]

b. Assume that Curt inherits some money at the end of year 4, and wants to use it to pay off the loan's remaining balance. If year-end payments are made, how much of the $14,000 borrowed will still be owed at the end of year 4?

A loan's remaining principal owed, at any time, is the sum of the present values of the remaining payments. When this loan was originated the PV of the remaining payment stream was a full

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$3{,}357.08 \left(\frac{1-\left(\frac{1}{1.115}\right)^{6}}{.115}\right) = \text{TOT}$$

$$\$3{,}357.08 \times 4.170294 = \$\underline{14{,}000}$$

With only 2 years remaining, the present value of the remaining payment stream would be only

$$\$3{,}357.08 \left(\frac{1-\left(\frac{1}{1.115}\right)^{2}}{.115}\right) = \text{TOT}$$

$$\$3{,}357.08 \times 1.701221 = \$\underline{\underline{5{,}711.13}}$$

c. Now assume that Curt has saved $3,000 by the end of year 2, and wants to use it to pay off enough of the loan's principal so that he can make the remaining payments over a 3-year period instead of 4 years. If year-end payments are made, how much (in addition to the regular payment) should he pay the bank at the end of year 2?

Again we note that a loan's remaining principal owed is simply the sum of the PVs of the remaining payments. Here the amount of principal that Curt owes at the end of year 2 (4 years remaining) is

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$3{,}357.08 \left(\frac{1-\left(\frac{1}{1.115}\right)^{4}}{.115}\right) = \text{TOT}$$

$$\$3{,}357.08 \times 3.069614 = \$\underline{10{,}304.93}$$

The amount he wants to owe, however, is the smaller amount that can be repaid over 3 years:

$$\$3{,}357.08 \left(\frac{1-\left(\frac{1}{1.115}\right)^{3}}{.115}\right) = \text{TOT}$$

$$\$3{,}357.08 \times 2.422619 = \$\underline{8{,}132.93}$$

So the extra amount he wants to pay at the end of year 2 is $10,304.93 – $8,132.93 = $2,172.00.

The answers we just computed for parts b and c also can be seen in the amortization schedule computed for the end-of-year case in part a above. But it is useful to be able to compute such values without doing an entire amortization plan, especially for a loan with a very long life.

11. At his high school graduating class's 10th reunion, Leonard gets drunk and brags to former homecoming queen Ursula Hotbodde that he will be a millionaire by the class's 25th reunion. The next day, while too hung over to go to work, he tries to figure out whether he will be able to live up to that claim. He feels that if he buys clothes only at garage sales, drives a moped, lives rent-free with his elderly aunt, and eats only canned peas from Aldi for the next 15 years he will be able to invest $18,500 per year in a stock market investment account at Gopher Brokers. What average compounded after-tax annual rate of return must he earn to reach a $1 million total by the end of the 15th year if he invests $18,500 at the end of each year? What if instead he invests the $18,500 at the start of each year?

Type: FV of Annuity; Rate of Return Unknown. This is a future value of an annuity situation; a series of equal investments corresponds in time value-adjusted terms to a $1 million lump sum of money that is not expected to be intact until 15 years in the future. Note that our hero's deposits will total only 15 x $18,500 = $277,500, so he clearly must earn a positive average annual rate of return if his plan of reaching $1,000,000 is to work. How high a rate? For the case of year-end investing, we solve for r in our now-familiar future value of a level ordinary annuity format:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$18{,}500 \left(\frac{(1+\mathrm{r})^{15}-1}{\mathrm{r}}\right) = \$1{,}000{,}000$$

We can not solve directly for r in an annuity problem; because r and r^{15} are both in the factor shown above, it is impossible to isolate r on one side of the = sign and have only non-r terms on the other side. We would have to use trial and error, and ultimately would find the answer to be 16.5488%:

$$\$18{,}500 \left(\frac{(1.165488)^{15}-1}{.165488}\right) = \$18{,}500\ (54.054054) = \$1{,}000{,}000 \ \checkmark$$

As in all of our FV of annuity cases we again see a large amount that will not be intact until a future date, a factor value that exceeds the number of payments, and returns applied to a growing balance as time passes. If Leonard instead invested the $18,500 at the beginning of each year, we would solve for r in our now-familiar future value of a level annuity due format:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$18{,}500 \left[\left(\frac{(1+\mathrm{r})^{15}-1}{\mathrm{r}}\right)(1+\mathrm{r})\right] = \$1{,}000{,}000$$

With trial and error we ultimately would find the answer to be a lower 14.8655% (with earlier deposits he could earn a lower average yearly rate of return and still reach his $1,000,000 goal):

$$\$18{,}500 \left[\left(\frac{(1.148655)^{15}-1}{.148655}\right)(1.148655)\right] = \$18{,}500\ (54.054054) = \$1{,}000{,}000 \ \checkmark$$

Because the average annual return on the stock market, as measured over the past several decades, has been only in the 9 – 13% range, Leonard may end up calling in sick for the 25th reunion, as well.

12. High-tech office supplier Normal Equipment Retail and Distribution (NERD) buys a brand new Copytron Super 5000 photocopier for $19,799. Gridley Electronic Engraving and Kopies, Inc. (GEEK) agrees to lease the machine for $3,750 per year for 8 years. GEEK will handle all maintenance, and NERD expects that the machine will have no resale value at the end of the 8-year lease period. What is NERD's expected average annual rate of return on its investment if the lease payment is received at the end of each year? What if it is received at the start of each year?

Type: PV of Annuity; Rate of Return Unknown. This is a present value of an annuity situation; the series of equal $3,750 lease payments corresponds in time value-adjusted terms to a lump sum of money that is intact in the present (the $19,799 the machine buyer pays today). Because the copier's owner pays $19,799 and then receives a total of $3,750 x 8 = $30,000 in lease payments, it is clear that a positive annual rate of return, on average, is being earned. How high a rate? For the case of year-end lease payments, we solve for r in the now-familiar present value of a level ordinary annuity format:

$$\text{PMT x FAC = TOT}$$

$$\$3{,}750\left(\frac{1-\left(\frac{1}{1+\mathrm{r}}\right)^{8}}{\mathrm{r}}\right) = \$19{,}799$$

With both r and r^8 in the factor, we can not isolate r on one side of the = sign and have only non-r terms on the other side. Trial and error ultimately gives an answer of 10.2865%; double-check:

$$\$3{,}750\left(\frac{1-\left(\frac{1}{1.102865}\right)^{8}}{.102865}\right) = \$3{,}750\ (5.279733) = \$19{,}799\ \checkmark$$

For the more likely case of beginning-of-year lease payments (who lets you pay rent at the end of each period??), we solve for r in our now-familiar present value of a level annuity due format:

$$\text{PMT x FAC = TOT}$$

$$\$3{,}750\left[\left(\frac{1-\left(\frac{1}{1+\mathrm{r}}\right)^{8}}{\mathrm{r}}\right)(1+\mathrm{r})\right] = \$19{,}799$$

With trial and error we ultimately would find the answer to be a higher 14.0655% (getting money sooner is more desirable for the recipient and thus represents a better investment return to the recipient or, alternatively, a higher cost to the payer); let's double check:

$$\$3{,}750\left[\left(\frac{1-\left(\frac{1}{1.140655}\right)^{8}}{.140655}\right)(1.140655)\right] = \$3{,}750\ (5.279733) = \$19{,}799\ \checkmark$$

As always seen in PV of annuity cases we again have the large $19,799 total existing intact in the present, the 5-ish factor smaller than the 8 payments, and returns applied to a declining balance.

13. Unlike the typical mutual fund company, which allows investors to open accounts with a few thousand dollars, the Snobby Mutual Fund group wants to deal only with the wealthy, and thus requires a $400,000 initial investment. Beverly, a highly-paid corporate lawyer, wants to open a Snobby Funds account. If she can make annual year-end deposits of $22,500 into a savings plan that earns a 7.5% average annual compounded return, how many years will it take for her to amass the needed $400,000? What if she instead makes her deposit at the beginning of each year?

Type: FV of Annuity; Number of Periods Unknown. Again we have an annuity application, with a series of equal or related cash flows corresponding to a lump sum of money. Because that lump sum (the $400,000 target) will not exist intact until a future date (though we do not know exactly when and thus are solving for n), it is a future value of an annuity situation. For year-end deposits, we solve for n in the future value of a level ordinary annuity format:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$22{,}500\left(\frac{(1.075)^{n}-1}{.075}\right) = \$400{,}000$$
$$\frac{(1.075)^{n}-1}{.075} = 17.777778$$
$$(1.075)^{n} - 1 = 1.333333$$
$$(1.075)^{n} = 2.333333$$

We could solve for n with trial and error, plugging in different values for n until we find the n for which the equation holds true. (Note that $400,000 ÷ $22,500 = 17.777778 is the number of years it would take Beverly to amass $400,000 with contributions of $22,500 each year if she earned no interest on her account's growing balance. However, because interest earnings will make up part of the $400,000, it will not take her that long.) Working instead with logarithms we find:

$$\ln[(1.075)^{n}] = \ln 2.333333$$
$$n \times \ln 1.075 = \ln 2.333333$$
$$n\,(.072321) = .847298$$
$$n = \underline{\underline{11.715850}}, \text{ or a little less than 12 years}$$

For beginning-of-year deposits, we solve for n in the future value of a level annuity due format:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$22{,}500\left[\left(\frac{(1.075)^{n}-1}{.075}\right)(1.075)\right] = \$400{,}000$$
$$\left[\left(\frac{(1.075)^{n}-1}{.075}\right)(1.075)\right] = 17.777778$$
$$\frac{(1.075)^{n}-1}{.075} = 16.537468$$
$$(1.075)^{n} - 1 = 1.240310$$
$$(1.075)^{n} = 2.240310$$
$$\ln[(1.075)^{n}] = \ln 2.240310$$
$$n \times \ln 1.075 = \ln 2.240310$$
$$n\,(.072321) = .806614$$
$$n = \underline{\underline{11.153251}}, \text{ just a little over 11 years}$$

(less time needed to meet the savings goal with start-of-year deposits).

14. Ne'er-do-well Ben wants to borrow $13,000 from his more responsible brother Glen. Glen reluctantly agrees, but wants to have a formal, written agreement. Ben agrees to sign a note ("IOU") that calls for a 6% annual interest rate, but states that he will be able to budget only $850 each year for payments. If Glen agrees to accept $850 at the end of each year, how many years will it take for Ben to repay him all principal plus applicable interest on the remaining unpaid principal? What if Ben makes his $850 payment at the beginning of each year?

Type: *PV of Annuity; Number of Periods Unknown*. The series of equal or related cash flows ($850 per year) tells us that we have an annuity situation, and the fact that the lump sum (the $13,000 lent) exists intact in the present makes it a present value of an annuity. In this case, the number of time periods n is the unknown we must solve for. For year-end payments, we solve for n in the PV of a level ordinary annuity format:

PMT x FAC = TOT

$$\$850\left(\frac{1-\left(\frac{1}{1.06}\right)^{n}}{.06}\right) = \$13{,}000$$

$$\left(\frac{1-\left(\frac{1}{1.06}\right)^{n}}{.06}\right) = 15.294118$$

$$1-\left(\frac{1}{1.06}\right)^{n} = .917647$$

$$1 - (.943396)^{n} = .917647$$

$$-(.943396)^{n} = -.082353$$

$$(.943396)^{n} = .082353$$

We could solve for n with trial and error, plugging in different values until we find the n for which the equation holds true. (Note that \$13,000 ÷ \$850 = 15.294118 is the number of years it would take to settle the loan if the borrower needed only to repay principal, but since he must pay interest on top of the remaining outstanding principal it will take longer. In fact, \$13,000 x .06 = \$780 of the first year's payment will go toward paying interest, so only the other \$70 will reduce the principal owed; thus it's going to take a lot longer than 15 years.) With logarithms we find:

$$\ln[(.943396)^{n}] = \ln .082353$$

$$n \times \ln .943396 = \ln .082353$$

$$n[-(.058269)] = -2.496740$$

$$n = \underline{\underline{42.848411}},$$

or almost 43 years. For beginning-of-year payments, we solve for n in the present value of a level annuity due format:

PMT x FAC = TOT

$$\$850\left[\left(\frac{1-\left(\frac{1}{1.06}\right)^{n}}{.06}\right)(1.06)\right] = \$13{,}000$$

$$\left[\left(\frac{1-\left(\frac{1}{1.06}\right)^{n}}{.06}\right)(1.06)\right] = 15.294118$$

$$\frac{1-\left(\frac{1}{1.06}\right)^{n}}{.06} = 14.428413$$

$$1-\left(\frac{1}{1.06}\right)^{n} = .865705$$

$$1 - (.943396)^{n} = .865705$$

$$-(.943396)^{n} = -.134295$$

$$(.943396)^{n} = .134295$$

$$\ln[(.943396)^{n}] = \ln .134295$$

$$n \times \ln .943396 = \ln .134295$$

$$n[-(.058269)] = -2.007716$$

$$n = \underline{\underline{34.455996}},$$

or between 34 and 35 years (Glen would be repaid in less time if Ben made an immediate payment that included some principal, but loans typically call for end-of-period payments precisely because the borrower needs the use of all of the principal and thus can not afford to pay part of it back right away). Glen had better hope that he and Ben both live to ripe old ages.

15. The Federal Ear, Nose, and Back Hair Commission, a government agency dealing with issues affecting America's aging male population, issues bonds. Each bond is an agreement to pay the investor who holds it $1,000 per year forever. (Such true perpetual bonds probably would not actually be created in today's world, but if they were the issuer likely would be a government agency, not a private company.) If the risk of this investment caused rational people to expect a 6.35% average annual rate of return, what should someone willingly pay for each bond if the $1,000 were to be received at the end of each year? What if it were to be received at the beginning of each year? What compounded average annual rate of return would the investor earn if she paid $18,000 for one of these bonds?

Type: PV of Perpetuity. This problem is a special present value of a level annuity application. A series of equal cash flows (the $1,000 to be received each year) corresponds in time value-adjusted terms to a large amount of money that exists intact in the present (the price paid for each bond). The interesting feature is that the series of cash flows is expected to last perpetually – a perpetuity. When the stream of equal cash flows is perpetual, the principal stays intact forever and the periodic cash flow consists only of "interest" (or however we would classify the financial return). So the PV of a level ordinary annuity factor becomes simply 1/r, and we find the value of each bond to be:

PMT x FAC = TOT

$$\$1{,}000\left(\frac{1-\left(\frac{1}{1.0635}\right)^{\infty}}{.0635}\right) = \$1{,}000\left(\frac{1-0}{.0635}\right) = \$1{,}000\left(\frac{1}{.0635}\right) \quad \text{or} \quad \frac{\$1{,}000}{.0635} = \text{TOT} = \$\underline{\underline{15{,}748.03}}$$

if the $1,000 per bond is to be received at the end of each year (a level ordinary perpetuity) and

PMT x FAC = TOT

$$\$1{,}000\left[\left(\frac{1}{.0635}\right)(1.0635)\right] = \text{TOT} = \$\underline{\underline{16{,}748.03}}$$

if the $1,000 is to be received at the beginning of each year (a level perpetuity due; the price paid by the buyer would have to be $1,000 more so the borrower could hand over an immediate $1,000 and leave the remaining $15,748.03 to earn interest of $1,000 per year to fund the subsequent perpetual stream of $1,000 payments). Notice that only the interest is paid out or received each year, with the initial principal remaining intact to repeat the process in each subsequent year:

Year	Beginning Balance	Plus 6.35% Interest	Balance Before Payment	Minus Payment	Ending Balance
1	$15,748.03	$1,000.00	$16,748.03	$1,000.00	$15,748.03
2	$15,748.03	$1,000.00	$16,748.03	$1,000.00	$15,748.03
3	$15,748.03	$1,000.00	$16,748.03	$1,000.00	$15,748.03
	⋮	⋮	⋮	⋮	⋮
25	$15,748.03	$1,000.00	$16,748.03	$1,000.00	$15,748.03
	⋮	⋮	⋮	⋮	⋮
250	$15,748.03	$1,000.00	$16,748.03	$1,000.00	$15,748.03

for end of year payments (above), while for beginning of year payments (below) it is:

Year	Beginning Balance	Minus Payment	Balance Before Interest	Plus 6.35% Interest	Ending Balance
1	$16,748.03	$1,000.00	$15,748.03	$1,000.00	$16,748.03
2	$16,748.03	$1,000.00	$15,748.03	$1,000.00	$16,748.03
3	$16,748.03	$1,000.00	$15,748.03	$1,000.00	$16,748.03
	⋮	⋮	⋮	⋮	⋮
25	$16,748.03	$1,000.00	$15,748.03	$1,000.00	$16,748.03
	⋮	⋮	⋮	⋮	⋮
250	$16,748.03	$1,000.00	$15,748.03	$1,000.00	$16,748.03

An investor paying $18,000 for one of these bonds would earn an average annual return of only

PMT x FAC = TOT

$$\$1{,}000 \left(\frac{1}{r}\right) = \$18{,}000$$

$$\frac{1}{r} = 18$$

$$r = \frac{1}{18} = \underline{\underline{5.55556\%}}$$

if the $1,000 in interest on each bond is to be received at the end of each year or

PMT x FAC = TOT

$$\$1{,}000 \left[\left(\frac{1}{r}\right)(1+r)\right] = \$18{,}000$$

$$\left[\left(\frac{1}{r}\right)(1+r)\right] = \left[\left(\frac{1}{r}\right) + 1\right] = 18$$

$$\frac{1}{r} = 17$$

$$r = \frac{1}{17} = \underline{\underline{5.88235\%}}$$

if the $1,000 is to be received at the beginning of each year (it should make sense that someone who receives the same amount of money sooner is getting a higher average periodic rate of return). [Note: there is no useful interpretation of a "future value of a perpetuity;" any such situation would involve either an infinite sum of money or an infinite number of time periods.]

What if the payments made to the bond holders instead were to start at $1,000 in year 1 and then increase by a steady 2.15% with each passing year? Then we would be computing the PV of a growing or changing perpetuity; here the amount the buyer would be willing to pay for this unusual government bond would be

PMT x FAC = TOT

$$\$1{,}000 \left(\frac{1}{.0635-.0215}\right) = \$1{,}000\ (23.809524) = \$\underline{\underline{23{,}809.52}}$$

if the cash is to be received at the end of each year forever, or an even higher

$$\$1{,}000 \left[\left(\frac{1}{.0635-.0215}\right)(1.0635)\right] = \$1{,}000\ (25.321429) = \$\underline{\underline{25{,}321.43}}$$

in the case of an infinite series of growing payments to be received at the start of each year. It should make sense that a buyer would pay more today to receive an infinite series of payments that start at $1,000 per year and then grow by 2.15% annually than the $15,748.03 or $16,748.03 that would be paid to receive an infinite series of payments that start at $1,000 and stay at that level. The PV of a growing or changing ordinary perpetuity, with a factor FAC of $\left(\frac{1}{r-g}\right)$ instead of the $\left(\frac{1}{r}\right)$ factor for a level ordinary perpetuity, will be important as a tool for estimating the intrinsic value per share of company's common stock in later Topic 13.

16. Glenda wants to borrow money to start a business, but she expects that eight years will pass before the venture is successful enough for her to be able to make loan payments. After that time, however, she expects to be able to comfortably make payments of $35,000 per year for twelve years (years 9 through 20). She finds a non-traditional lender willing to extend a loan with a 10.5% annual interest rate, no payments in years 1 through 8, and then annual

payments in years 9 through 20. How much should this lender be willing to lend if the borrower agrees to make a \$35,000 annual payment at the end of each indicated year? What if she pays \$35,000 at the beginning of each year?

Type: PV of a Deferred Annuity. This problem is a special PV of an annuity application. A series of equal or related cash flows (the \$35,000 annual payments) corresponds in time value-adjusted terms to a large amount of money that exists intact in the present (the amount to be lent). The interesting feature is that the series of cash flows will not begin until some number of periods has passed – a deferred level annuity. When we have a deferred stream of equal or related cash flows, the lump-sum present value held or owed will grow each year during the deferral period, and then be depleted during the period when nonzero cash flows occur. Because the lender will receive payments in years 9 to 20, a "brute force" approach to the end-of-year payments case would be set up as

$$\$35{,}000\left[\left(\frac{1}{1.105}\right)^{9}+\left(\frac{1}{1.105}\right)^{10}+\left(\frac{1}{1.105}\right)^{11}+\cdots+\left(\frac{1}{1.105}\right)^{18}+\left(\frac{1}{1.105}\right)^{19}+\left(\frac{1}{1.105}\right)^{20}\right].$$

But we want to actually compute with one of two shortcut approaches that are more efficient because they are based on PV of annuity factors. Recall that the PV of a level ordinary annuity factor is the sum of the PV of a single dollar amount factors for the same discount rate and same number of time periods, e.g.

$$\left[\left(\frac{1}{1.105}\right)^{1}+\left(\frac{1}{1.105}\right)^{2}+\cdots+\left(\frac{1}{1.105}\right)^{8}+\left(\frac{1}{1.105}\right)^{9}+\cdots+\left(\frac{1}{1.105}\right)^{19}+\left(\frac{1}{1.105}\right)^{20}\right]=\left(\frac{1-\left(\frac{1}{1.105}\right)^{20}}{.105}\right)$$

The PV of an annuity factor in a deferred level annuity problem should be the sum of the present value of a dollar factors for the years with nonzero expected cash flows. But instead of the "brute force" method (summing the present values of expected cash flows in years 9 through 20) we can use the same logic with the "compute a factor" method, in which we find

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$35{,}000\left[\left(\frac{1-\left(\frac{1}{1.105}\right)^{20}}{.105}\right)-\left(\frac{1-\left(\frac{1}{1.105}\right)^{8}}{.105}\right)\right]=\text{TOT}$$

$$\$35{,}000\ (8.230909 - 5.239188) = \$35{,}000 \times 2.991721 = \$\underline{\underline{104{,}710.23}}$$

for end-of-year payments or a larger

$$\$35{,}000\left(\left[\left(\frac{1-\left(\frac{1}{1.105}\right)^{20}}{.105}\right)-\left(\frac{1-\left(\frac{1}{1.105}\right)^{8}}{.105}\right)\right](1.105)\right)=\text{TOT}$$

$$\$35{,}000\ [(8.230909 - 5.239188)(1.105)] = \$35{,}000 \times 3.305852 = \$\underline{\underline{115{,}704.81}}$$

if she would be willing and able to make beginning-of-year payments (typically not done in loan situations, because a borrower normally needs to use all the money borrowed and thus could not make an immediate repayment – although a first payment at the start of year 9 might be doable). In the "compute a factor" approach we directly find the sum of the year 9 through 20 present value of a dollar factors. In the second "shortcut" approach we "discount the annuity" and find

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$35{,}000 \left[\left(\frac{1-\left(\frac{1}{1.105}\right)^{12}}{.105}\right)\left(\frac{1}{1.105}\right)^{8}\right] = \text{TOT}$$

$$\$35{,}000\ [(6.649964)(.449885)] = \$35{,}000 \times 2.991721 = \$\underline{\underline{104{,}710.23}}$$

for end-of-year payments or a larger

$$\$35{,}000 \left[\left(\frac{1-\left(\frac{1}{1.105}\right)^{12}}{.105}\right)(1.105)\left(\frac{1}{1.105}\right)^{8}\right] = \text{TOT}$$

$$\$35{,}000\ [(6.649964)(1.105)(.449885)] = \$35{,}000 \times 3.305852 = \$\underline{\underline{115{,}704.81}}$$

if beginning-of-year payments were to be made. In the "discount the annuity" approach (which we might at some point come to prefer because it is the approach that must be used in some more advanced applications) we initially think of the cash flow stream as lasting for 12 years, but then discount for the fact that 8 years pass before the 12 payments begin (and multiply the resulting factor by 1 + r in the beginning-of-year payments case, just as we do in all annuity examples).

What happens in the year-end payments case is as follows. $104,710.23 is lent today. Borrower pays nothing back in the first 8 years, but the lender applies 10.5% interest to the unpaid balance each year, so by the end of year 8 $104,710.23 × $(1.105)^8$ = $232,748.74 is owed. Then over years 9 – 20 the borrower pays 12 × $35,000 = $420,000, a total that covers the $232,748.74 "principal" plus interest on the remaining unpaid principal as the repayment period progresses. [Note: there is no useful interpretation of a "future value of a deferred annuity;" any such situation would involve simply waiting for a number of periods to pass before starting to make or receive a deposit series.]

17. When the 200th anniversary of Reggieville's founding occurs in 19 years, part of the celebration will involve creating a new Bicentennial Museum. Town planners expect the facility to require a $400,000 annual budget in perpetuity, and they hope the town founder's wealthy descendant, Reginald Redbird VI, will create an account today to provide for the museum's ongoing operations. If any money invested for the museum can be expected to earn a 4.15% average annual rate of return, how much should Redbird contribute now to fund the future museum? How much of an endowment would the museum need to receive from Redbird today to fund an operating budget that would start at $400,000 in year 19 and then increase by 1.65%, on average, every year going out indefinitely?

Type: PV of a Deferred Perpetuity. To provide for payments of $400,000 in each of years 19 through infinity (with 18 years of interest buildup before the museum opens), the donor today must provide

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$400{,}000 \left[\left(\frac{1}{.0415}\right)\left(\frac{1}{1.0415}\right)^{18}\right] = \$400{,}000\ (11.590041) = \$\underline{\underline{4{,}636{,}016.34}}$$

if the $400,000 will be paid out at the end of each of years 19 to infinity, or a higher

$$\$400{,}000 \left[\left(\frac{1}{.0415}\right)(1.0415)\left(\frac{1}{1.0415}\right)^{18}\right] = \$400{,}000\ (12.071028) = \$\underline{\underline{4{,}828{,}411.01}}$$

in the more realistic case of the money being needed to pay bills from the start of each year. In that more realistic case, between today and the end of year 18 the initial contribution grows to $4,828,411.01 $(1.0415)^{18}$ = $10,038,554.22. Then $400,000 is taken out at the start of year 19, leaving $10,038,554.22 – $400,000 = $9,638,554.22. That base will earn $9,638,554.22 x .0415 = $400,000 to be withdrawn to pay museum costs in every subsequent year.

To provide for an operating budget that starts at $400,000 in year 19 and then increases by 1.65% per year indefinitely, we compute the PV of a deferred changing perpetuity; here the amount the museum would need to receive today from the generous donor would be considerably higher, at

$$\$400{,}000 \left[\left(\frac{1}{.0415-.0165}\right)\left(\frac{1}{1.0415}\right)^{18}\right] = \$400{,}000\ (19.239468) = \$\underline{\underline{7{,}695{,}787.12}}$$

if the money is to be paid out at the end of each of years 19 to infinity, or an even higher

$$\$400{,}000 \left[\left(\frac{1}{.0415-.0165}\right)(1.0415)\left(\frac{1}{1.0415}\right)^{18}\right] = \$400{,}000\ (20.037906) = \$\underline{\underline{8{,}015{,}162.28}}$$

in the case of money being needed to pay increasing bills from the start of each year.

18. Every year for 11 years Raymond will deposit $2,200 in an account that earns a 3.85% average annual rate of return. But at the end of year 11 he will not close the account; rather, he will leave the accumulated balance to earn returns for an additional 6 years. What will the account balance be at the end of year 17 if the deposits are made at the end of each year? At the beginning of each year? What if instead he deposits $2,200 in each of years 1 through 11, then $3,200 in each of years 12 through 17, and then $4,200 in each of years 18 through 36?

Type: FV of a "Truncated" Annuity. Note that payouts in this situation, which for lack of a better term we might call a "truncated" annuity, are to be made in years 1 through 11, with interest to continue for an additional 6 years until the end of year 17. We can think through the example from a "brute force" angle (if deposit 1 occurs at the end of year 1 it will earn interest 16 times by the end of year 17, and if deposit 11 is made at the end of year 11 it will earn interest 6 times):

$$\$2{,}200\ [(1.0385)^{16} + (1.0385)^{15} + (1.0385)^{14} + \cdots + (1.0385)^{7} + (1.0385)^{6}],$$

A quicker computational technique is to "compute a factor," following the logic that the FV of an annuity factor is the sum of the FV of a single dollar amount factors for the same periodic discount rate and number of time periods:

PMT x FAC = TOT

$$\$2{,}200 \left[\left(\frac{(1.0385)^{17}-1}{.0385}\right) - \left(\frac{(1.0385)^{6}-1}{.0385}\right)\right]$$

$$= \$2{,}200\ (23.394457 - 6.608014) = \$2{,}200\ (16.786443) = \$\underline{\underline{36{,}930.17}}$$

But it seems a lot easier to use the alternative "compounding the annuity" technique (determine the balance that will be reached by the end of year 11 and then compound it out for six more years):

$$\$2{,}200 \left(\frac{(1.0385)^{11}-1}{.0385}\right)(1.0385)^{6}$$

$$= \$2{,}200\ (13.381958)(1.254409) = \$2{,}200\ (16.786443) = \$\underline{\underline{36{,}930.17}}$$

(One reason the "compute a factor" method may seem confusing in a truncated annuity problem is the exponents used; the FV of a 17-year level ordinary annuity factor is the sum of the 17 FV of a single dollar amount factors with exponents 0 to 16. "Computing a factor" in the PV of a deferred annuity case may seem easier because the n-year PV of a level ordinary annuity factor sums the PV of a single dollar amount factors with exponents 1 to n.) If deposits are made at the start of each year interest will work its way through the plan one additional time; again we can compute a factor

PMT x FAC = TOT

$$\$2{,}200\left(\left[\left(\frac{(1.0385)^{17}-1}{.0385}\right)-\left(\frac{(1.0385)^{6}-1}{.0385}\right)\right](1.0385)\right)$$

= $2,200 (23.394457 – 6.608014)(1.0385) = $2,200 (17.432721) = $38,351.99

but once more the easier approach seems to be compounding the annuity:

$$\$2{,}200\left(\frac{(1.0385)^{11}-1}{.0385}\right)(1.0385)\,(1.0385)^{6}$$

= $2,200 (13.381958) (1.0385) (1.254409) = $2,200 (17.432721) = $38,351.99

[Mathematical purists will note that (1.0385) $(1.0385)^6$ is $(1.0385)^7$, but the visual benefit of breaking it into two pieces is stressing that there are six subsequent earning periods combined with yet an added application of earnings through beginning-of-period cash flows – otherwise those who are not mathematical purists might ask, "where does the 7 come from?"]

With the 3-part sequence of deposits, the amount the saver will have by the end of year 36 will be

$$\$2{,}200\left(\frac{(1.0385)^{11}-1}{.0385}\right)(1.0385)^{25} + \$3{,}200\left(\frac{(1.0385)^{6}-1}{.0385}\right)(1.0385)^{19}$$

$$+\ \$4{,}200\left(\frac{(1.0385)^{19}-1}{.0385}\right)(1.0385)^{0}$$

= $2,200 (34.409804) + $3,200 (13.545483) + $4,200 (27.269007)
= $75,701.57 + $43,345.54 + $114,529.83 = $233,576.94

if deposits are to be made at the end of each year. Think of a saver who deposits $2,200 at Bank One 11 times, and then contributes nothing more to the account, but the accumulated balance earns interest for an additional 25 years until the end of year 36 (the deposits actually could all go into one account, but computing based on the logic of having 3 separate accounts sometimes is easier to understand). Then after those first 11 years $3,200 is deposited at Bank Two 6 times, and that accumulated balance earns interest for 19 years until the end of year 36 (= 11 + 6 + 19). Then after making deposits at Banks One & Two for 17 years (= 11 + 6), the saver puts $4,200 into Bank Three annually for 19 years, with the last deposit in year 36 (= 17 + 19). The factor $(1.0385)^0$ is just a visual place holder (anything taken to the 0 power is 1 so we are just multiplying by 1); it shows that after the final 19 deposits are made no time will remain for added interest to accrue before the grand total is computed. With start-of-year deposits the only difference is that interest is earned on each account one added time, so (1.0385) appears as an additional factor in each term:

$$\$2{,}200\left(\frac{(1.0385)^{11}-1}{.0385}\right)(1.0385)\,(1.0385)^{25} + \$3{,}200\left(\frac{(1.0385)^{6}-1}{.0385}\right)(1.0385)\,(1.0385)^{19}$$

$$+\ \$4{,}200\left(\frac{(1.0385)^{19}-1}{.0385}\right)(1.0385)\,(1.0385)^{0}$$

= $2,200 (35.734581) + $3,200 (14.066984) + $4,200 (28.318864)
= $78,616.08 + $45,014.35 + $118,939.23 = $242,569.66

19. Edith just finished medical school at age 28. She hopes to spend three years devoting her medical expertise for humanitarian purposes before she turns 35, but knows that she must amass some savings first. Her current plan is to work at a hospital for 4 years, and then spend the next 3 years volunteering with Doctors Without Borders. Because Doctors Without Borders can provide only a small salary, she will have to withdraw money from her savings in each of those 3 years to help meet her living costs.

a. If she can earn a 4.25% average annual rate of return on any money that remains in her savings account from year to year, and if she saves $12,000 at the end of each of her 4 working years, how much can she withdraw at the end of each of her 3 volunteer years? What if instead she makes her withdrawal at the beginning of each year?

Type: FV of Annuity and PV of Annuity Combined. This problem combines two types of applications: we amass a savings balance (future value of an annuity) and then deplete it (present value of an annuity). The amount she will have saved after 4 years of level year-end deposits will be

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$12{,}000\left(\frac{(1.0425)^4-1}{.0425}\right) = \text{TOT}$$
$$\$12{,}000 \times 4.262302 = \$\underline{51{,}147.62}$$

We initially think of the $51,147.62 as the future total to which the stream of 4 deposits grows. But then we have to move ourselves 4 years forward in time, and think of it as a present total, on that day, from which the stream of 3 subsequent level withdrawals can be taken. At the end of each of the 3 following years she can withdraw

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.0425}\right)^3}{.0425}\right) = \$51{,}147.62$$
$$\text{PMT} \times 2.761976 = \$51{,}147.62$$
$$\$51{,}147.62 \div 2.761976 = \text{PMT} = \$\underline{\underline{18{,}518.49}}$$

At the beginning of each of the 3 following years she could withdraw only a smaller

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1}{1.0425}\right)^3}{.0425}\right)(1.0425)\right] = \$51{,}147.62$$
$$\text{PMT} \times 2.879360 = \$51{,}147.62$$
$$\$51{,}147.62 \div 2.879360 = \text{PMT} = \$\underline{\underline{17{,}763.54}}$$

(a smaller annual withdrawal because no interest could have been earned between the last deposit and the first withdrawal). Let's look at what happens year-to-year with end-of-year withdrawals:

Year	Beginning Balance	Plus 4.25% Interest	Balance Before Deposit	Plus Deposit	Minus Withdrawal	Ending Balance
1	$0	$0	$0	$12,000.00	$ 0	$12,000.00
2	$12,000.00	$ 510.00	$12,510.00	$12,000.00	$ 0	$24,510.00
3	$24,510.00	$1,041.68	$25,551.68	$12,000.00	$ 0	$37,551.68
4	$37,551.68	$1,595.94	$39,147.62	$12,000.00	$ 0	$51,147.62
5	$51,147.62	$2,173.77	$53,321.39	$ 0	$18,518.49	$34,802.90
6	$34,802.90	$1,479.12	$36,282.02	$ 0	$18,518.49	$17,763.53
7	$17,763.53	$ 754.96	$18,518.49	$ 0	$18,518.49	$ 0

For beginning-of-year withdrawals we see:

Year	Beginning Balance	Minus Withdrawal	Balance After Withdrawal	Plus 4.25% Interest	Plus Deposit	Ending Balance
1	\$ 0	\$ 0	\$ 0	\$ 0	\$12,000.00	\$12,000.00
2	\$12,000.00	\$ 0	\$12,000.00	\$ 510.00	\$12,000.00	\$24,510.00
3	\$24,510.00	\$ 0	\$24,510.00	\$1,041.68	\$12,000.00	\$37,551.68
4	\$37,551.68	\$ 0	\$37,551.68	\$1,595.94	\$12,000.00	\$51,147.62
5	\$51,147.62	\$17,763.54	\$33,384.08	\$1,418.82	\$ 0	\$34,802.90
6	\$34,802.90	\$17,763.54	\$17,039.36	\$ 724.17	\$ 0	\$17,763.53
7	\$17,763.53	\$17,763.54	\$ 0	\$ 0	\$ 0	\$ 0

b. If Edith can earn a 4.25% average annual rate of return on any money that remains in her savings account from year to year, and if she saves \$12,000 at the beginning of each of her 4 working years, how much can she withdraw at the end of each of her 3 volunteer years? What if instead she makes her withdrawal at the beginning of each year?

The amount she will have saved after 4 years of level beginning-of-year deposits will be

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$12{,}000\left[\left(\frac{(1.0425)^4-1}{.0425}\right)(1.0425)\right] = \text{TOT}$$

$$\$12{,}000 \times 4.443450 = \$\underline{53{,}321.40}$$

We initially think of the \$53,321.40 as the future total to which the stream of 4 deposits grows. But then we have to move ourselves 4 years forward in time, and think of it as a present total on that day from which the stream of 3 level withdrawals can be taken. At the end of each of the next 3 years she can withdraw

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.0425}\right)^3}{.0425}\right) = \$53{,}321.40$$

$$\text{PMT} \times 2.761976 = \$53{,}321.40$$

$$\$53{,}321.40 \div 2.761976 = \text{PMT} = \$\underline{19{,}305.53}$$

(a larger amount than with year-end withdrawals in part a above, because beginning-of-year deposits have provided a larger total to draw from). At the beginning of each of the next 3 years she could withdraw a smaller

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1}{1.0425}\right)^3}{.0425}\right)(1.0425)\right] = \$53{,}321.40$$

$$\text{PMT} \times 2.879360 = \$53{,}321.40$$

$$\$53{,}321.40 \div 2.879360 = \text{PMT} = \$\underline{18{,}518.49}$$

Notice what happens year-to-year with end-of-year withdrawals:

Year	Beginning Balance	Plus Deposit	Total Accumulated	Plus 4.25% Interest	Minus Withdrawal	Ending Balance
1	\$0	\$12,000.00	\$12,000.00	\$ 510.00	\$ 0	\$12,510.00
2	\$12,510.00	\$12,000.00	\$24,510.00	\$1,041.68	\$ 0	\$25,551.68
3	\$25,551.68	\$12,000.00	\$37,551.68	\$1,595.95	\$ 0	\$39,147.63
4	\$39,147.63	\$12,000.00	\$51,147.63	\$2,173.77	\$ 0	\$53,321.40
5	\$53,321.40	\$ 0	\$53,321.40	\$2,266.16	\$19,305.53	\$36,282.03
6	\$36,282.03	\$ 0	\$36,282.03	\$1,541.99	\$19,305.53	\$18,518.49
7	\$18,518.49	\$ 0	\$18,518.49	\$ 787.04	\$19,305.53	\$ 0

With beginning-of-year withdrawals we see:

Year	Beginning Balance	Plus Deposit	Minus Withdrawal	Total Available	Plus 4.25% Interest	Ending Balance
1	$0	$12,000.00	$ 0	$12,000.00	$ 510.00	$12,510.00
2	$12,510.00	$12,000.00	$ 0	$24,510.00	$1,041.68	$25,551.68
3	$25,551.68	$12,000.00	$ 0	$37,551.68	$1,595.95	$39,147.63
4	$39,147.63	$12,000.00	$ 0	$51,147.63	$2,173.77	$53,321.40
5	$53,321.40	$ 0	$18,518.49	$34,802.91	$1,479.12	$36,282.03
6	$36,282.03	$ 0	$18,518.49	$17,763.54	$ 754.95	$18,518.49
7	$18,518.49	$ 0	$18,518.49	$ 0	$ 0	$ 0

We get the same withdrawal, $18,518.49, with end-of-year deposits and end-of-year withdrawals as with beginning-of-year deposits and beginning-of-year withdrawals. Why? Because the same amount of interest is earned before withdrawals are made in either case.

c. If Edith expects to earn a 4.25% average annual rate of return on any money that remains in her account from year to year, and if she wants to withdraw $20,000 at the end of each of her 3 volunteer years, how much must she save at the end of each of her 4 working years? What if instead she makes deposits at the beginning of each year?
Now we have the same arrangement as above, but with the level deposits as the unknown to solve for. Here we have to complete the steps in the order opposite that shown in parts a and b above, finding first the "present" value she must have on hand at the beginning of her volunteer period:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$20{,}000\left(\frac{1-\left(\frac{1}{1.0425}\right)^3}{.0425}\right) = \text{TOT}$$
$$\$20{,}000 \times 2.761976 = \$\underline{55{,}239.52}$$

Then for the case of year-end deposits we compute as

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\text{PMT}\left(\frac{(1.0425)^4-1}{.0425}\right) = \$55{,}239.52$$
$$\text{PMT} \times 4.262302 = \$55{,}239.52$$
$$\$55{,}239.52 \div 4.262302 = \text{PMT} = \$\underline{\underline{12{,}960.02}}$$

(it stands to reason that she would need to deposit a little more than $12,000 per year since $12,000 deposits, as shown above, provide for somewhat less than $20,000 in yearly withdrawals). If she made beginning-of-year deposits, then each year she could deposit a smaller amount:

$$\text{PMT}\left[\left(\frac{(1.0425)^4-1}{.0425}\right)(1.0425)\right] = \$55{,}239.52$$
$$\text{PMT} \times 4.443450 = \$55{,}239.52$$
$$\$55{,}239.52 \div 4.443450 = \text{PMT} = \$\underline{\underline{12{,}431.67}}$$

In parts a and b above we first compute the FV to which the series of deposits will grow and then solve for the withdrawals that can be taken; here in part c we first compute the lump sum PV needed for making the desired withdrawals and then solve for the deposits that will grow to that lump sum. The key to knowing which to do first is: which do we know the payments for? In parts a and b we know the deposits that will grow to a large future value, so we do the FV of annuity step first. In part c we know the withdrawals we want to take from a large present value, so we do the PV of annuity step first.

d. If Edith expects to earn a 4.25% average annual rate of return on any money that remains in her savings account from year to year, and if she wants to withdraw $20,000 at the beginning of each of her 3 volunteer years, how much must she save at the end of each of her 4 working years? What if instead she makes her savings deposit at the beginning of each year?

The level deposits that will grow to a large FV again are the unknown to solve for, while we know the $20,000 desired annual withdrawal, so we do the PV of annuity step first. But now the $20,000 level withdrawals are to come at the start of each year. Again we first compute the "present" value she must have on hand at the start of her volunteer period:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$20{,}000\left[\left(\frac{1-\left(\frac{1}{1.0425}\right)^3}{.0425}\right)(1.0425)\right] = \text{TOT}$$
$$\$20{,}000 \times 2.879360 = \$\underline{57{,}587.20}$$

(she must start with more in the first of her three withdrawal years if she wants to immediately take some money out). Then for the case of year-end deposits we compute as

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\text{PMT}\left(\frac{(1.0425)^4-1}{.0425}\right) = \$57{,}587.20$$
$$\text{PMT} \times 4.262302 = \$57{,}587.20$$
$$\$57{,}587.20 \div 4.262302 = \text{PMT} = \$\underline{\underline{13{,}510.82}}$$

while the deposit she would have to make at the beginning of each year is computed as

$$\text{PMT}\left[\left(\frac{(1.0425)^4-1}{.0425}\right)(1.0425)\right] = \$57{,}587.20$$
$$\text{PMT} \times 4.443450 = \$57{,}587.20$$
$$\$57{,}587.20 \div 4.443450 = \text{PMT} = \$\underline{\underline{12{,}960.02}}$$

e. If Edith saves only $12,000 at the end of each of her 4 working years, what annual rate of return will she have to earn if she wants to be able to withdraw $20,000 at the end of each of her 3 volunteer years?

Here we must solve for the rate of return r, a challenging computational task since we have r and r to a higher power in each of the two annuity factors. Note that we have to combine the two steps

$$\$12{,}000\left(\frac{(1+r)^4-1}{r}\right) = \text{TOT}$$

and

$$\$20{,}000\left(\frac{1-\left(\frac{1}{1+r}\right)^3}{r}\right) = \text{TOT}$$

(recall that the total "future value" TOT accumulated during the savings years becomes the total "present value" TOT to draw from in the withdrawal years) to form

$$\$12{,}000\left(\frac{(1+r)^4-1}{r}\right) = \$20{,}000\left(\frac{1-\left(\frac{1}{1+r}\right)^3}{r}\right)$$

Solving with trial and error iterations (which we would never ask you to do on an exam), we find r = 6.5475% (it has to be greater than 4.25% since part d above shows that \$20,000 withdrawals accompanied by a 4.25% annual return necessitate deposits of \$12,960.02). Let's double check:

$$\$12{,}000\left(\frac{(1.065475)^4-1}{.065475}\right) = \$20{,}000\left(\frac{1-\left(\frac{1}{1.065475}\right)^3}{.065475}\right)$$

$$\$12{,}000\ (4.410279) = \$20{,}000\ (2.646164)$$

$$\$52{,}923.30 = \$52{,}923.30 \ \checkmark$$

f. Assume that Edith can earn a 4.25% average annual rate of return on any money that remains in her savings account from year to year, and that she saves \$12,000 at the end of each of her 4 working years. She then joins Doctors Without Borders, and immediately decides that she would like to stay with the organization for 3 additional years. If she reduces her annual withdrawal to \$9,600, will her savings sustain her for 6 years of volunteer service?

Now our unknown becomes the number of years n in which she can withdraw \$9,600. As in part a above, we see that by the end of year 4 she expects to have

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$12{,}000\left(\frac{(1.0425)^4-1}{.0425}\right) = \text{TOT}$$

$$\$12{,}000 \times 4.262302 = \$\underline{51{,}147.62}$$

Recall that this \$51,147.62 is the present total from which the stream of level withdrawals can be taken. We find the number of \$9,600 year-end withdrawals she can take by solving for n below:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$9{,}600\left(\frac{1-\left(\frac{1}{1.0425}\right)^n}{.0425}\right) = \$51{,}147.62$$

$$\left(\frac{1-\left(\frac{1}{1.0425}\right)^n}{.0425}\right) = 5.327877$$

$$1-\left(\frac{1}{1.0425}\right)^n = .226435$$

$$1 - (.959233)^n = .226435$$

$$-(.959233)^n = -.773565$$

$$(.959233)^n = .773565$$

$$\ln[(.959233)^n] = \ln .773565$$

$$n \times \ln .959233 = \ln .773565$$

$$n\ (-.041622) = -.256746$$

n = $\underline{6.168555}$, or just over 6 years; <u>yes her plan will work</u>.

20. Doug, who is 47, is doing some planning toward meeting important financial goals.

a. He plans to deposit $750 at the end of every three-month period into a mutual fund account that holds his Roth IRA retirement savings plan. If the average rate of return he expects to earn is a 10% annual percentage rate (APR), but with quarterly compounding, how much will he have in his IRA after 13 years, when he turns 60? What if he makes beginning-of-quarter deposits? What if instead he deposits $700 per quarter for the first 3 years, $750 per quarter for the next 6 years, and $800 per quarter for the last 4 years? What Effective Annual Rate (EAR) of return will he be earning?

Type: FV of Annuity with Non-Annual Payments. Saving for retirement is a future value of an annuity application: a stream of equal or related payments corresponds in time-value adjusted terms to a large chunk of money that will not exist until a future date. Here the deposits are to be made quarterly rather than annually, so we must "shift mental gears" to quarterly mode. The quarterly periodic interest rate is 10% ÷ 4 = 2.5%, and we are dealing with 4 quarterly deposit periods each year for 13 years, for a total of 13 x 4 = 52 quarters. For end-of-quarter level deposits we find

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$750 \left(\frac{(1.025)^{52}-1}{.025}\right) = \text{TOT}$$
$$\$750 \times 104.444494 = \$\underline{\underline{78,333.37}}$$

If deposits instead were made at the beginning of each quarter he would accumulate

$$\$750 \left[\left(\frac{(1.025)^{52}-1}{.025}\right)(1.025)\right] = \text{TOT}$$
$$\$750 \times 107.055606 = \$\underline{\underline{80,291.70}}$$

(with beginning-of-period deposits he will earn interest for one additional quarter and thus have more by the time he retires). If his deposits instead were to rise sequentially over time, he would have a sequential or multi-stage annuity: $700 per quarter for 12 quarters (3 years), with the total reinvested for the remaining 40 quarters (10 years) of the investment period; $750 per quarter for the next 24 quarters, with the total reinvested for the remaining 16 quarters of the investment period; and $800 quarterly for the final 16 quarters until the end of the target period. The FV of this sequential annuity (the amount our saver is expected to have at retirement) is computed as

$$\$700 \left[\left(\frac{(1.025)^{12}-1}{.025}\right)(1.025)^{40}\right] + \$750\left[\left(\frac{(1.025)^{24}-1}{.025}\right)(1.025)^{16}\right] + \$800 \left[\left(\frac{(1.025)^{16}-1}{.025}\right)(1.025)^{0}\right]$$

$$= \$700\ [(13.795553)\ (2.685064)] + \$750\ [(32.349038)\ (1.484506)] + \$800\ (19.380225)$$
$$= \$700\ (37.041940) + \$750\ (48.022329) + \$800\ (19.380225)$$
$$= \$25,929.36 + \$36,016.75 + \$15,504.18 = \$\underline{\underline{77,450.29}} \text{ with end-of-quarter deposits and}$$

$$\$700 \left[\left(\frac{(1.025)^{12}-1}{.025}\right)(1.025)\ (1.025)^{40}\right] + \$750 \left[\left(\frac{(1.025)^{24}-1}{.025}\right)(1.025)\ (1.025)^{16}\right]$$
$$+ \$800 \left[\left(\frac{(1.025)^{16}-1}{.025}\right)(1.025)\ (1.025)^{0}\right]$$
$$= \$700\ (37.967989) + \$750\ (49.222887) + \$800\ (19.864730)$$
$$= \$26,577.59 + \$36,917.17 + \$15,891.78 = \$\underline{\underline{79,386.54}} \text{ with beginning-of-quarter deposits.}$$

With an APR of 10%, he will earn a quarterly periodic rate of 10% ÷ 4 = 2.5%; the corresponding EAR will be $(1.025)^4 - 1 = 10.3813\%$. We instead could have talked about the expected annual rate of return as a 10.3813% EAR, and found the quarterly periodic r to work with in the computations by undoing the compounding included in the EAR: $\sqrt[4]{1.103813}$ – 1 = .025, or 2.5%.

b. Assume that Doug will retire at age 60. He will then use the IRA balance to supplement his primary sources of retirement income (company pension and Social Security). He will want to make level withdrawals at the end of every six-month period from the account, and to be confident that he will not outlive his money (he projects that he will live to age 94 ½). If he expects to earn a 10.2% average annual percentage rate of return on any money that remains in the account from year to year, how much should he be able to withdraw every half-year beginning six months after his 60th birthday, and ending on his 94th birthday?

A plan of saving systematically during the working years, and then withdrawing systematically during retirement years, combines FV of an annuity with PV of an annuity computations. The savings phase, as shown in part a above, is a future value of an annuity situation. Then the individual has a large sum intact at the retirement date; this amount is a present value at that time, so the withdrawal phase is a present value of an annuity situation. Here Doug's expected half-year periodic rate of return is 10.2% ÷ 2 = 5.1%, and he wants to be prepared to make withdrawals over 34 years x 2 = 68 half-year periods. If he has saved the smaller amount associated with end-of-quarter deposits, then the amount he can withdraw at the end of each half-year of expected retirement is

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.051}\right)^{68}}{.051}\right) = \$78{,}333.37$$
$$\text{PMT} \times 18.941888 = \$78{,}333.37$$
$$\$78{,}333.37 \div 18.941888 = \text{PMT} = \$\underline{4{,}135.46}\ ,$$

or 2 x $4,135.46 = $8,270.92 in total each year. Beginning-of-quarter deposits would allow withdrawals at the end of each half-year of retirement to be

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.051}\right)^{68}}{.051}\right) = \$80{,}291.70$$
$$\text{PMT} \times 18.941888 = \$80{,}291.70$$
$$\$80{,}291.70 \div 18.941888 = \text{PMT} = \$\underline{4{,}238.84}\ ,$$

or 2 x $4,238.84 = $8,477.68 in total each year. How can Doug start with only about $80,000 and then withdraw a total of about $4,000 x 68 = $272,000 over time? Note that 10.2% is a fairly high expected annual return; if he withdrew only "interest" and left "principal" intact he could withdraw about $8,000 each year. It should stand to reason that he can take amounts slightly higher than that each year if he is to take some principal along with the interest and systematically deplete the account to a zero balance over the 34 years.

21. Cierra obtains a student loan that is to be repaid in equal monthly installments.

a. If she borrows $8,000 at a 6.72% Annual Percentage Rate (APR) of interest for a five-year term, and the contract calls for her to pay at the end of each month, how much will each payment be? What if the payments instead are to be made at the beginning of each month (even though loan payments usually are made at the end of each period)?

Type: PV of Annuity with Non-Annual Payments. Loan amortization (making equal periodic payments to retire a loan) is a present value of an annuity application: a stream of equal or related payments corresponds in time-value adjusted terms to a large chunk of money that exists intact in the present (the lender hands the loan principal to the borrower today). The twist here is that the payments are to be made monthly rather than annually, so we must "shift mental gears" to monthly mode. Our monthly periodic interest rate is 6.72% ÷ 12 = .56%, and we are dealing with 12 monthly payment periods over five years, for a total of 5 x 12 = 60 months. (We instead could have talked about the annual interest rate as a 6.9309% EAR, and found the monthly periodic r to work with by undoing the compounding included in the EAR: $\sqrt[12]{1.069309}$ – 1 = .0056, or .56%.) The level periodic payment, if made at the end of each month, is

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.0056}\right)^{60}}{.0056}\right) = \$8{,}000$$

$$\text{PMT} \times 50.840493 = \$8{,}000$$

$$\$8{,}000 \div 50.840493 = \text{PMT} = \$\underline{\underline{157.35}}$$

In the unlikely case of level beginning-of-period payments, the monthly outlay would be

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1}{1.0056}\right)^{60}}{.0056}\right)(1.0056)\right] = \$8{,}000$$

$$\text{PMT} \times 51.125200 = \$8{,}000$$

$$\$8{,}000 \div 51.125200 = \text{PMT} = \$\underline{\underline{156.48}}$$

(if she starts repaying sooner, she can pay less each time and still meet her obligations).

b. If she can afford to make a $185 monthly payment over a five-year term, and the contract calls for her to pay at the end of each month, how much principal can Cierra afford to borrow? What if the payments instead are to be made at the beginning of each month?

The structure of the problem is the same as in part a, but now the total amount she can borrow is the unknown to solve for. For level end-of-month payments we find

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$185\left(\frac{1-\left(\frac{1}{1.0056}\right)^{60}}{.0056}\right) = \text{TOT}$$

$$\$185 \times 50.840493 = \$\underline{\underline{9{,}405.49}}$$

In the more unlikely case of level beginning-of-month payments, she could afford to borrow

$$\$185\left[\left(\frac{1-\left(\frac{1}{1.0056}\right)^{60}}{.0056}\right)(1.0056)\right] = \text{TOT}$$

$$\$185 \times 51.125200 = \$\underline{\underline{9{,}458.16}}$$

(she can service a larger loan if she is willing to start making payments sooner).

c. If Cierra borrows $8,000 at a 6.72% APR and can afford to pay back only $107.96 at the end of each month, how long will it take for her to repay all principal and the applicable interest? What if instead she makes the $107.96 payment at the beginning of each month?

Again the structure of the problem is the same, but now we must solve for n (and since we are in monthly mode, n is a number of months, not years). For level end-of-month payments we find

PMT x FAC = TOT

$$\$107.96\left(\frac{1-\left(\frac{1}{1.0056}\right)^{n}}{.0056}\right) = \$8{,}000$$

$$\frac{1-\left(\frac{1}{1.0056}\right)^{n}}{.0056} = 74.101519$$

$$1-\left(\frac{1}{1.0056}\right)^{n} = .414969$$

$$-\left(\frac{1}{1.0056}\right)^{n} = -.585031$$

$$-(.994431)^{n} = -.585031$$

$$(.994431)^{n} = .585031$$

$$\ln[(.994431)^{n}] = \ln .585031$$

$$n(\ln .994431) = \ln .585031$$

$$n(-.005584) = -.536090$$

$$n = \underline{\underline{96}} \text{ months,}$$

or 8 years, to repay the loan with such small payments. In the unlikely case of beginning-of-month level payments, repayment of all principal and accompanying interest would require a slightly lower

PMT x FAC = TOT

$$\$107.96\left[\left(\frac{1-\left(\frac{1}{1.0056}\right)^{n}}{.0056}\right)(1.0056)\right] = \$8{,}000$$

$$\left(\frac{1-\left(\frac{1}{1.0056}\right)^{n}}{.0056}\right)(1.0056) = 74.101519$$

$$\frac{1-\left(\frac{1}{1.0056}\right)^{n}}{.0056} = 73.688861$$

$$1-\left(\frac{1}{1.0056}\right)^{n} = .412658$$

$$-\left(\frac{1}{1.0056}\right)^{n} = -.587342$$

$$-(.994431)^{n} = -.587342$$

$$(.994431)^{n} = .587342$$

$$\ln[(.994431)^{n}] = \ln .587342$$

$$n(\ln .994431) = \ln .587342$$

$$n(-.005584) = -.532147$$

$$n = \underline{\underline{95.29}} \text{ months, or just under 8 years}$$

d. If Cierra borrows $8,000 and pays back $163.75 at the end of each month over a five-year period, what monthly periodic rate of interest is she paying? What corresponding annual percentage rate (APR) of interest would the lender be charging? What would be the corresponding Effective Annual Rate (EAR) of interest?

Once again the structure of the problem is the same, but now we must solve for the monthly periodic rate r (and since we are in monthly mode, n is a number of months, not years). For year-end payments we find

PMT x FAC = TOT

$$\$163.75\left(\frac{1-\left(\frac{1}{1+r}\right)^{60}}{r}\right) = \$8{,}000$$

With both r and r^{60} in the factor, we can not isolate r on one side of the = sign and have only non-r terms on the other side. With trial and error we ultimately find the answer to be .007, or $\underline{\underline{.7\%}}$:

$$\$163.75\left(\frac{1-\left(\frac{1}{1.007}\right)^{60}}{.007}\right) = \$8{,}000 \ \checkmark$$

With a .7% periodic rate, the APR (which does not incorporate the impact of intra-year compounding) would be .7% x 12 = $\underline{8.4\%}$, and the EAR (which is adjusted for intra-year compounding) would be $(1.007)^{12} - 1 = \underline{8.7311\%}$.

e. If Cierra's monthly-payment loan carries an EAR of 7.1864%, what is the monthly periodic rate? What is the corresponding APR that the lender would be likely to quote as the interest rate on her loan?

EAR reflects the impact of intra-year compounding, so we "un-compound" to find the accompanying periodic rate. If payments and compounding are monthly the EAR is $(1 + \text{monthly periodic rate})^{12} - 1$, so here the monthly periodic rate is $\sqrt[12]{1+\text{EAR}} - 1 = \sqrt[12]{1.071864} - 1 = 1.071864^{.083333} - 1 = .0058$, or $\underline{.58\%}$. The attendant APR is simply .58% x 12 = $\underline{6.96\%}$. Recall that in time value analysis we must work with a rate that corresponds to the timing of the payments and compounding (here, monthly). But we talk about rates of return or cost in annual terms. So in this situation the annual rate could be talked about as a 6.96% APR (.0696 ÷12 = .0058) or as a 7.1864% EAR ($\sqrt[12]{1.071864} - 1 = .0058$).

22. [FIL 404 only.] Pete, who is 38, decides to start a long-overdue retirement savings plan. He will make a series of annual deposits, but they will not be equal in amount. Instead, each deposit will exceed the previous one by 2.25%, which is the average annual inflation rate that he expects to observe over future decades.

a. If he expects to earn a 5.75% compounded average annual rate of return on his account's growing balance, and if he makes year-end deposits starting with $3,000 in the current year, how much money will Pete have in 27 years when he turns 65? What if instead he makes 27 beginning-of-year deposits starting with $3,000?

Type: Future Value of a Changing Annuity. This situation is a future value of an annuity application; a series of related cash flows corresponds to a lump sum of money that will not exist intact until a future date. But whereas in most of our earlier annuity examples the cash flows remained constant from period to period, here they change by a constant percentage with each passing period. We compute for the case of end-of-year deposits with our future value of a changing (rather than level) ordinary annuity formula:

$$\text{PMT x FAC = TOT}$$

$$\$3{,}000\left(\frac{(1.0575)^{27}-(1.0225)^{27}}{.0575-.0225}\right) = \text{TOT}$$

$$\$3{,}000 \times 77.170844 = \$\underline{\underline{231{,}512.53}}$$

With beginning-of-year deposits (the future value of a changing annuity due), he should accumulate a larger total:

$$\$3{,}000\left[\left(\frac{(1.0575)^{27}-(1.0225)^{27}}{.0575-.0225}\right)(1.0575)\right] = \text{TOT}$$

$$\$3{,}000 \times 81.608168 = \$\underline{\underline{244{,}824.50}}$$

b. If Pete expects to earn a 5.75% compounded average annual rate of return on his account's growing balance, and he wants to have $350,000 when he turns 65 in 27 years, how large should the first of his growing deposits be if he makes his deposits at the end of each year? How large should the first deposit be if instead he makes beginning-of-year deposits that grow by 2.25% annually?

Again a series of related cash flows corresponds to a lump sum of money that will not exist intact until a future date, so the structure of the problem is the same; we still have a future value of a changing annuity situation. But now the amount of each payment in the series is the unknown. For end-of-year deposits we find that the first deposit's magnitude should be:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\text{PMT}\left(\frac{(1.0575)^{27}-(1.0225)^{27}}{.0575-.0225}\right) = \$350{,}000$$
$$\text{PMT} \times 77.170844 = \$350{,}000$$
$$\text{PMT} = \$350{,}000 \div 77.170844 = \$\underline{\underline{4{,}535.39}}$$

If deposits were made at the start of each year, so that more interest would be earned over the plan's life, Pete could reach the $350,000 target with a smaller initial deposit of:

$$\text{PMT}\left[\left(\frac{(1.0575)^{27}-(1.0225)^{27}}{.0575-.0225}\right)(1.0575)\right] = \$350{,}000$$
$$\text{PMT} \times 81.608168 = \$350{,}000$$
$$\text{PMT} = \$350{,}000 \div 81.608168 = \$\underline{\underline{4{,}288.79}}$$

Then each subsequent deposit would be higher than its predecessor by 2.25%, and the last deposit in the stream would be $\$4{,}535.39 \times (1.0225)^{26} = \$8{,}088.37$ with year-end deposits and $\$4{,}288.79 \times (1.0225)^{26} = \$7{,}648.58$ if there were beginning-of-year deposits.

c. If Pete makes year-end deposits starting at $3,000 and growing by 2.25% annually, how high an average annual rate of return must he earn for his account balance to reach $500,000 by the time he turns 65 in 27 years? What if instead he makes beginning-of-year deposits starting at $3,000 and growing by 2.25% annually?

The structure again is the same, but here the annual rate of return indicated by these cash flows is the unknown to solve for. In the end-of-year deposit case, we solve as:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$3{,}000\left(\frac{(1+r)^{27}-(1.0225)^{27}}{r-.0225}\right) = \$500{,}000$$

With both r and r^{27} in the factor, we can not isolate r on one side of the = sign and have only non-r terms on the other. With trial and error we find an answer of approximately .108581, or <u>10.8581%</u>:

$$\$3{,}000\left(\frac{(1.108581)^{27}-(1.0225)^{27}}{.108581-.0225}\right) = \text{TOT}$$
$$\$3{,}000 \times 166.667201 = \$500{,}000 \checkmark$$

For beginning-of-year deposits we find:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\$3{,}000\left[\left(\frac{(1+r)^{27}-(1.0225)^{27}}{r-.0225}\right)(1+r)\right] = \$500{,}000$$

Having r and r^{27} in the factor, we ultimately find the answer with trial and error to be approximately .102393, or <u>10.2393%</u>:

$$\$3{,}000\left[\left(\frac{(1.102393)^{27}-(1.0225)^{27}}{.102393-.0225}\right)(1.102393)\right] = \text{TOT}$$
$$\$3{,}000 \times 166.666362 = \$500{,}000 \checkmark$$

d. If Pete expects to earn a 5.75% compounded average annual rate of return on his growing balance, and if he makes a series of year-end deposits starting with $3,000 and growing by 2.25% annually, how many years will it take for his account to grow to $200,000? What if instead he makes beginning-of-year deposits starting at $3,000 and growing by 2.25% annually?

The structure again is the same, but now the number of time periods n needed for the stream of deposits to reach $200,000 is the unknown to solve for. For year-end deposits, we solve as:

$$\text{PMT x FAC = TOT}$$
$$\$3{,}000\left(\frac{(1.0575)^n-(1.0225)^n}{.0575-.0225}\right)=\$200{,}000$$
$$\left(\frac{(1.0575)^n-(1.0225)^n}{.0575-.0225}\right)=66.666667$$
$$(1.0575)^n-(1.0225)^n=2.333333$$

In earlier examples in which n was the unknown, we noted that trial and error would be a legitimate, though cumbersome, way to solve from this stage, and we used logarithms instead. In this example, unfortunately, there is no convenient way to use logarithms. We have no easy means of simplifying $\ln[(1.0575)^n - (1.0225)^n]$; it is not equal to $\ln(1.0575)^n - \ln(1.0225)^n$. Thus we are stuck with trial and error, and find the answer to be n = 25.16795, or just over 25 years. Let's double-check:

$$\$3{,}000\left(\frac{(1.0575)^{25.16795}-(1.0225)^{25.16795}}{.0575-.0225}\right)=\$200{,}000 \checkmark$$

For beginning-of-year deposits we find:

$$\text{PMT x FAC = TOT}$$
$$\$3{,}000\left[\left(\frac{(1.0575)^n-(1.0225)^n}{.0575-.0225}\right)(1.0575)\right]=\$200{,}000$$
$$\left[\left(\frac{(1.0575)^n-(1.0225)^n}{.0575-.0225}\right)(1.0575)\right]=66.666667$$
$$\frac{(1.0575)^n-(1.0225)^n}{.0575-.0225}=63.041765$$
$$(1.0575)^n-(1.0225)^n=2.206462$$

Again logarithms do not help us; through trial and error we find n = 24.48345, or about 24½ years. Let's double-check to be sure:

$$\$3{,}000\left[\left(\frac{(1.0575)^{24.48345}-(1.0225)^{24.48345}}{.0575-.0225}\right)(1.0575)\right]=\$200{,}000 \checkmark$$

e. Now assume that Pete wants to have $350,000 when he turns 65 in 27 years, but instead of making deposits for all 27 years he plans to make equal annual deposits for only 20 years, and then just let the accumulated balance earn an average of 5.75% annually until he turns 65. How much must he deposit at the end of each of the 20 years to reach his goal? What if instead he made his twenty deposits at the beginning of each year? If he wanted his annual deposits to grow each year by 2.25%, how much should his first deposit be (compute for both end-of-year and beginning-of-year deposit cases)?

Now we are dealing with the future value of a truncated annuity (a situation with similarities to the present value of a deferred annuity). We can think of the situation either as the sum of the future values of individual deposits made in years 1 – 20 and compounded until the end of year 27, or else as the future value of a 20-year annuity that is then compounded for 7 more years. The level amount that he should put aside each year is

PMT x FAC = TOT

$$\text{PMT}\ \left[\left(\frac{(1.0575)^{27}-1}{.0575}\right)-\left(\frac{(1.0575)^{7}-1}{.0575}\right)\right] = \$350{,}000$$

PMT (61.295573 – 8.330107) = $350,000

PMT (52.965466) = $350,000

PMT = $350,000 ÷ 52.965466 = $6,608.09 or

$$\text{PMT}\left[\left(\frac{(1.0575)^{20}-1}{.0575}\right)(1.0575)^{7}\right] = \$350{,}000$$

PMT [(35.812131)(1.478981)] = $350,000

PMT (52.965466) = $350,000

PMT = $350,000 ÷ 52.965466 = $6,608.09 with year-end deposits and a smaller

PMT x FAC = TOT

$$\text{PMT}\left(\left[\left(\frac{(1.0575)^{27}-1}{.0575}\right)-\left(\frac{(1.0575)^{7}-1}{.0575}\right)\right](1.0575)\right) = \$350{,}000$$

PMT = $350,000 ÷ 56.010980 = $6,248.77 or

$$\text{PMT}\ \left[\left(\frac{(1.0575)^{20}-1}{.0575}\right)(1.0575)\ (1.0575)^{7}\right] = \$350{,}000$$

PMT = $350,000 ÷ 56.010980 = $6,248.77 with beginning-of-year deposits.

If the contributions are to increase by 2.25% per year (a changing truncated annuity), then the year-1 deposit should be

PMT x FAC = TOT

$$\text{PMT}\ \left[\left(\frac{(1.0575)^{20}-(1.0225)^{20}}{.0575-.0225}\right)(1.0575)^{7}\right] = \$350{,}000$$

PMT [(42.819667)(1.478981)] = $350,000

PMT (63.329479) = $350,000

PMT = $350,000 ÷ 63.329479 = $5,526.65 with year-end deposits and a smaller

$$\text{PMT}\left[\left(\frac{(1.0575)^{20}-(1.0225)^{20}}{.0575-.0225}\right)(1.0575)(1.0575)^{7}\right] = \$350{,}000$$

PMT [(42.819667)(1.478981)(1.0575)] = $350,000

PMT (66.970924) = $350,000

PMT = $350,000 ÷ 66.970924 = $5,226.15 with beginning-of-year deposits.

In dealing with future value of changing truncated (or present value of changing deferred) annuity situations, what we have labeled the "compound (or discount) the annuity" technique is easier to use than what we are calling the "compute a factor" technique.

f. Now assume that Pete wants to have $350,000 when he turns 65 in 27 years, but instead of making annual deposits he decides to put money into the account semiannually (every 6 months). If he can earn a 5.75% Annual Percentage Rate (APR) of return on his growing balance, and plans to increase his deposits by 1.125% (half of 2.25%) every 6 months, how much should he plan to deposit at the end of each semiannual period? What if he instead makes the deposits at the beginning of each semiannual period? What Effective Annual Rate (EAR) of interest would he be earning on his account's growing balance?

Here the structure of the problem is the same; we still have a future value of a changing annuity situation. But now the initial payment (the unknown to solve for) is a semiannual value. So here the number of semiannual periods is 27 x 2 = 54 and the periodic rate of return is half of the 5.75% APR: .0575 ÷2 = .02875. For deposits made at the end of each semiannual period we solve as:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left(\frac{(1.02875)^{54}-(1.01125)^{54}}{.02875-.01125}\right) = \$350{,}000$$

$$\text{PMT} \times 159.504904 = \$350{,}000$$

$$\text{PMT} = \$350{,}000 \div 159.504904 = \$\underline{\underline{2{,}194.29}}$$

For deposits made at the start of each six-month period we find:

$$\text{PMT}\left[\left(\frac{(1.02875)^{54}-(1.01125)^{54}}{.02875-.01125}\right)(1.02875)\right] = \$350{,}000$$

$$\text{PMT} \times 164.090670 = \$350{,}000$$

$$\text{PMT} = \$350{,}000 \div 164.090670 = \$\underline{\underline{2{,}132.97}}$$

Then each subsequent deposit would be higher than its predecessor by 1.125%, and the last deposit in the stream would be $2,194.29 × $(1.01125)^{53}$ = $3,970.05 with deposits made at the end of each half-year and $2,132.97 × $(1.01125)^{53}$ = $3,859.10 if deposits were at the start of each half-year. The EAR earned on the account's growing balance would be $(1.02875)^2$ – 1 = .058327, or 5.8327%.

23. [FIL 404 only] Andrea just received a large inheritance. She wants to give most of this money to charity, but because she also wants to quit her job and do full-time volunteer work for the next 9 years she must use part of the inheritance to fund her living expenses. Specifically, she will keep some of the inherited money to open an account, and then will withdraw yearly amounts that increase by 1.6% per year (her estimate of the average annual inflation rate during her 9-year expected period of volunteer work).

a. If she expects to earn a 4.8% average annual rate of return on any money remaining in her account from year to year, how large an account must she open today to fund a series of year-end withdrawals that start at $25,000? What if instead she wants to make beginning-of-year withdrawals?

Type: Present Value of a Changing Annuity. This situation is a present value of an annuity application; a series of related cash flows corresponds to a lump sum of money that exists intact in the present. As in the previous question, the cash flows here are expected to change by a constant percentage from period to period, rather than remaining level over time. We compute for the end-of-year case with our present value of a changing ordinary annuity formula:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$25{,}000\left(\frac{1-\left(\frac{1.016}{1.048}\right)^9}{.048-.016}\right) = \text{TOT}$$

$$\$25{,}000 \times 7.610306 = \$\underline{\underline{190{,}257.65}}$$

That amount must be on deposit today to fund a series of nine $25,000 year-end withdrawals; for beginning-of-year withdrawals she must start out with a larger:

$$\$25{,}000\left[\left(\frac{1-\left(\frac{1.016}{1.048}\right)^9}{.048-.016}\right)(1.048)\right] = \text{TOT}$$

$$\$25{,}000 \times 7.975601 = \$\underline{\underline{199{,}390.02}}$$

b. If Andrea opens a $200,000 account, if she expects to earn a 4.8% average annual rate of return on any money remaining in her account from year to year, and if she makes year-end withdrawals, how much should the first of her stream of growing withdrawals be? What if instead she makes beginning-of-year withdrawals?

Again a series of related cash flows corresponds to a lump sum of money that exists intact in the present, so we still have a present value of a changing annuity situation. But now the amount of each payment in the series is the unknown. For end-of-year withdrawals we solve as:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left(\frac{1-\left(\frac{1.016}{1.048}\right)^9}{.048-.016}\right) = \$200{,}000$$

$$\text{PMT} \times 7.610306 = \$200{,}000$$

$$\text{PMT} = \$200{,}000 \div 7.610306 = \$\underline{\underline{26{,}280.15}}$$

Her first in a series of beginning-of-year withdrawals would be a smaller:

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1.016}{1.048}\right)^9}{.048-.016}\right)(1.048)\right] = \$200{,}000$$

$$\text{PMT} \times 7.975601 = \$200{,}000$$

$$\text{PMT} = \$200{,}000 \div 7.975601 = \$\underline{\underline{25{,}076.48}}$$

Then each subsequent withdrawal would exceed its predecessor by 1.6%, and the last withdrawal in the stream would be $26,280.15 $(1.016)^8$ = $30,315.95 for the year-end case and $25,076.48 $(1.016)^9$ = $28,927.44 if there were beginning-of-year withdrawals. Note that if she planned to take a series of nine unchanging year-end withdrawals, the amount of each would be something in between $26,280.15 and $30,315.95:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.048}\right)^9}{.048}\right) = \$200{,}000$$

$$\text{PMT} \times 7.171555 = \$200{,}000$$

$$\text{PMT} = \$200{,}000 \div 7.171555 = \$\underline{\underline{27{,}887.95}}$$

c. If Andrea opens a $200,000 account, and she wants to make nine withdrawals that start at $30,000 and then increase by 1.6% per year, what average annual rate of return does she have to earn on her account's declining balance if she makes her withdrawal at the end of each year? What if she makes beginning-of-year withdrawals?

The structure again is the same, but here the average annual rate of return suggested by these cash flows is the unknown to solve for. For year-end withdrawals, we solve as:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$30{,}000\left(\frac{1-\left(\frac{1.016}{1+r}\right)^9}{r-.016}\right) = \$200{,}000$$

With both r and r^9 in the factor, we can not isolate r on one side of the = sign and have only non-r terms on the other. With trial and error we find an answer of approximately .077935, or <u>7.7935%</u>:

$$\$30{,}000\left(\frac{1-\left(\frac{1.016}{1.077935}\right)^9}{.077935-.016}\right) = \$200{,}000$$

$$\$30{,}000 \times 6.666707 = \$200{,}000 \ \checkmark$$

For beginning-of-year withdrawals we find:

PMT x FAC = TOT

$$\$30{,}000\left[\left(\frac{1-\left(\frac{1.016}{1+r}\right)^{9}}{r-.016}\right)(1+r)\right] = \$200{,}000$$

Having both r and r^9 in the factor we must solve with trial and error, ultimately finding the answer to be approximately .101046, or 10.1046%:

$$\$30{,}000\left[\left(\frac{1-\left(\frac{1.016}{1.101046}\right)^{9}}{.101046-.016}\right)(1.101046)\right] = \$200{,}000$$

$$\$30{,}000 \times 6.666677 = \$200{,}000 \checkmark$$

d. Assume that Andrea opens a $200,000 account, but then realizes that she may come to enjoy her volunteer work so much that she will want to stay with it for more than nine years. Of course, doing so will require her to live on less money each year. For how many years could she continue to make year-end withdrawals, starting with $20,000 and growing with inflation by 1.6% annually, if she can earn a 4.8% average annual rate of return on any money remaining in her account from year to year? What if instead she makes beginning-of-year withdrawals?

The structure is again the same, but now the unknown to solve for is the number of time periods n over which the $200,000 account will allow for a stream of withdrawals starting with $20,000 and growing by 1.6% annually. For year-end withdrawals, we solve as:

PMT x FAC = TOT

$$\$20{,}000\left(\frac{1-\left(\frac{1.016}{1.048}\right)^{n}}{.048-.016}\right) = \$200{,}000$$

$$\frac{1-\left(\frac{1.016}{1.048}\right)^{n}}{.048-.016} = 10$$

$$1-\left(\frac{1.016}{1.048}\right)^{n} = .320000$$

$$\left(\frac{1.016}{1.048}\right)^{n} = .680000$$

$$(.969466)^{n} = .680000$$

$$\ln[(.969466)^{n}] = \ln .680000$$

$$n \ln .969466 = \ln .680000$$

$$n(-.031010) = -.385662$$

$$n = \underline{\underline{12.436618}},$$

or just under $12\frac{1}{2}$ years. Double-check:

$$\$20{,}000\left(\frac{1-\left(\frac{1.016}{1.048}\right)^{12.436618}}{.048-.016}\right) = \$200{,}000 \checkmark$$

She could keep the described series of beginning-of-year withdrawals going for:

PMT x FAC = TOT

$$\$20{,}000\left[\left(\frac{1-\left(\frac{1.016}{1.048}\right)^{n}}{.048-.016}\right)(1.048)\right] = \$200{,}000$$

$$\left[\left(\frac{1-\left(\frac{1.016}{1.048}\right)^{n}}{.048-.016}\right)(1.048)\right] = 10$$

$$\left(\frac{1-\left(\frac{1.016}{1.048}\right)^{n}}{.048-.016}\right) = 9.541985$$

$$1-\left(\frac{1.016}{1.048}\right)^{n} = .305344$$

$$\left(\frac{1.016}{1.048}\right)^{n} = .694656$$

$$(.969466)^{n} = .694656$$

$$\ln[(.969466)^{n}] = \ln .694656$$

$$n \ln .969466 = \ln .694656$$

$$n\,(-.031010) = -.364338$$

$$n = \underline{\underline{11.748953}},$$

or about $11\frac{3}{4}$ years. Let's again double-check to be sure:

$$\$20{,}000\left[\left(\frac{1-\left(\frac{1.016}{1.048}\right)^{11.748953}}{.048-.016}\right)(1.048)\right] = \$200{,}000 \ \checkmark$$

e. Now assume that Andrea opens a $200,000 account, and then starts making preparations to quit her job. But her boss asks if she would not reconsider and stay for three more years, to oversee completion of the Forbes project to which she has devoted so much of her career. After briefly thinking it over, she agrees; she will simply wait three years before starting her nine-year volunteer program. If she can earn a 4.8% average annual rate of return on any money remaining in her account from year to year, and if she will make a series of nine year-end withdrawals that grow by 1.6% per year after three years have passed (so withdrawals will come in years 4 through 12), how much should the first of her stream of growing withdrawals be? What if withdrawals instead are at the start of each year?

This situation involves the present value of a changing deferred annuity: we want to know how large an amount must be on deposit today to allow for a stream of changing withdrawals that will not begin until some number of periods has passed. So our computations must combine the changing annuity idea with the deferred annuity idea. The easiest approach is our "discount the annuity" method for handling deferred annuities: compute the present value of a nine-year changing annuity, and then discount it for three periods (the "compute a factor" method is not practical with changing annuities). For year-end withdrawals we find a first withdrawal of:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1.016}{1.048}\right)^{9}}{.048-.016}\right)\left(\frac{1}{1.048}\right)^{3}\right] = \$200{,}000$$

$$\text{PMT} \times 6.611778 = \$200{,}000$$

$$\text{PMT} = \$200{,}000 \div 6.611778 = \$\underline{\underline{30{,}249.05}}$$

Let's work through the cash flows for this changing deferred ordinary annuity year-by-year:

Year	Beginning Balance	Plus 4.8% Interest	Total Available	Minus Withdrawal	Ending Balance
1	$200,000.00	$ 9,600.00	$209,600.00	$ 0	$209,600.00
2	$209,600.00	$10,060.80	$219,660.80	$ 0	$219,660.80
3	$219,660.80	$10,543.72	$230,204.52	$ 0	$230,204.52
4	$230,204.52	$11,049.82	$241,254.34	$30,249.05	$211,005.29

5	$211,005.29	$10,128.25	$221,133.54	$30,733.03	$190,400.51
6	$190,400.51	$ 9,139.22	$199,539.73	$31,224.76	$168,314.97
7	$168,314.97	$ 8,079.12	$176,394.09	$31,724.36	$144,669.73
8	$144,669.73	$ 6,944.15	$151,613.88	$32,231.95	$119,381.93
9	$119,381.93	$ 5,730.33	$125,112.26	$32,747.66	$ 92,364.60
10	$ 92,364.60	$ 4,433.50	$ 96,798.10	$33,271.62	$ 63,526.48
11	$ 63,526.48	$ 3,049.27	$ 66,575.75	$33,803.97	$ 32,771.78
12	$ 32,771.78	$ 1,573.05	$ 34,344.83	$34,344.83	$ 0

The first in a deferred series of nine beginning-of-year withdrawals is computed as:

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1.016}{1.048}\right)^9}{.048-.016}\right)(1.048)\left(\frac{1}{1.048}\right)^3\right] = \$200{,}000$$

$$\text{PMT} \times 6.929144 = \$200{,}000$$

$$\text{PMT} = \$200{,}000 \div 6.929144 = \$\underline{\underline{28{,}863.60}}$$

Working through the cash flows for this changing deferred annuity due year-by-year shows:

Year	Beginning Balance	Minus Withdrawal	Total Available	Plus 4.8% Interest	Ending Balance
1	$200,000.00	$ 0	$200,000.00	$ 9,600.00	$209,600.00
2	$209,600.00	$ 0	$209,600.00	$10,060.80	$219,660.80
3	$219,660.80	$ 0	$219,660.80	$10,543.72	$230,204.52
4	$230,204.52	$28,863.60	$201,340.92	$ 9,664.36	$211,005.28
5	$211,005.28	$29,325.42	$181,679.86	$ 8,720.63	$190,400.49
6	$190,400.49	$29,794.62	$160,605.87	$ 7,709.08	$168,314.95
7	$168,314.95	$30,271.34	$138,043.61	$ 6,626.09	$144,669.70
8	$144,669.70	$30,755.68	$113,914.02	$ 5,467.87	$119,381.89
9	$119,381.89	$31,247.77	$ 88,134.12	$ 4,230.44	$ 92,364.56
10	$ 92,364.56	$31,747.73	$ 60,616.83	$ 2,909.61	$ 63,526.44
11	$ 63,526.44	$32,255.70	$ 31,270.74	$ 1,501.00	$ 32,771.74
12	$ 32,771.74	$32,771.79	$ 0	$ 0	$ 0

(a slight 5¢ rounding difference). She will have to be content taking out less in the first year (and in each subsequent year) if she wants to make her first withdrawal at the beginning, rather than at the end, of year 4. Note that, in the year-end case, her first withdrawal would be $28,863.60 and her last withdrawal, eight years later in year 12, would be $28,863.60 $(1.016)^8$ = $32,771.79. If she planned to take a series of nine deferred unchanging year-end withdrawals, the amount of each would be something in between $28,863.60 and $32,771.79:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1}{1.048}\right)^9}{.048}\right)\left(\frac{1}{1.048}\right)^3\right] = \$200{,}000$$

$$\text{PMT} \times 6.230595 = \$200{,}000$$

$$\text{PMT} = \$200{,}000 \div 6.230595 = \$\underline{\underline{32{,}099.66}}$$

f. Again assume that, after opening a $200,000 account, Andrea agrees to wait the three years before starting her volunteer period. But now assume she realizes that it is difficult to budget when your income arrives in one large annual chunk, so instead she wants to make quarterly withdrawals. If she can earn a 4.8% Annual Percentage Rate (APR) of interest on any money remaining in her account from year to year, and if she wants her withdrawals to increase by .4% per quarter, how much can she expect to withdraw at the end of each quarter in years 4 through 12? What if instead she makes the withdrawals at the beginning of each quarter? What Effective Annual Rate (EAR) of return would she be earning on her account's declining balance?

Again we have the PV of a changing deferred annuity, but now each payment (the unknown to solve for) is a quarterly value to be withdrawn over 36 quarters: quarters 13 (first quarter of year 4) through 48 (last quarter in year 12). An APR of 4.8% corresponds to a quarterly periodic rate of .048 ÷ 4 = .012. We solve for the first in a series of end-of-quarter withdrawals as:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1.004}{1.012}\right)^{36}}{.012-.004}\right)\left(\frac{1}{1.012}\right)^{12}\right] = \$200{,}000$$

$$\text{PMT} \times 26.922284 = \$200{,}000$$

$$\text{PMT} = \$200{,}000 \div 26.922284 = \$\underline{\underline{7{,}428.79}}$$

The first in a series of beginning-of-quarter withdrawals would be:

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left[\left(\frac{1-\left(\frac{1.004}{1.012}\right)^{36}}{.012-.004}\right)(1.012)\left(\frac{1}{1.012}\right)^{12}\right] = \$200{,}000$$

$$\text{PMT} \times 27.245352 = \$200{,}000$$

$$\text{PMT} = \$200{,}000 \div 27.245352 = \$\underline{\underline{7{,}340.70}}$$

Then each subsequent withdrawal would be higher than its predecessor by .4%, and the last withdrawal in the stream (35 quarters later) would be $\$7{,}428.79\ (1.004)^{35} = \$8{,}542.76$ with end-of-quarter withdrawals and $\$7{,}340.70\ (1.004)^{35} = \$8{,}441.46$ with start-of-quarter withdrawals. The EAR earned on the account's declining balance would be $(1.012)^4 - 1 = .048871$, or 4.8871%.

24. [FIL 404 only] ISU's wealthiest professor, Joe Solberg, donates 20 acres of land to the campus for a small park. He also wants to give enough money to provide for park maintenance (mowing, clean-up, snow removal) forever into the future. ISU finance director Regina McRedbird tells him that grounds-keeping costs go up by 2.4% per year, on average.

a. If Solberg feels that the university will be able to earn a 7.2% average annual rate of return on money it invests, and if maintaining 20 acres of open space currently costs about $17,000 per year, how much must he contribute today to fund the growing maintenance budget? Compute for both the case of end-of-year maintenance payments and that of beginning-of-year outlays.

Type: Present Value of a Changing Perpetuity. Now we have the PV of an annuity whose changing cash flows are expected to continue forever. The present value of a changing perpetuity is much easier to handle computationally than is the PV of a changing finite annuity, just as other perpetuity situations generally offer easier computations than do their corresponding finite annuities. It is easy because the factor for the PV of a changing perpetual annuity with year-end cash flows is simply 1/(r – g). So for the case of year-end withdrawals in the problem at hand, we compute

PMT x FAC = TOT

$$\$17{,}000\left(\frac{1}{.072-.024}\right) = \text{TOT}$$

$17,000 x 20.833333 = $354,166.67

To allow for an infinite stream of beginning-of-year withdrawals, he must endow a larger fund:

$$\$17{,}000\left[\left(\frac{1}{.072-.024}\right)(1.072)\right] = \text{TOT}$$

$17,000 x 22.333333 = $379,666.67

b. Assume instead that Solberg simply contributes $350,000 along with the land, specifying that the money is to be used toward maintaining the park forever. If the university can earn a 7.2% average annual rate of return on money it invests, how much can it apply to the park's growing maintenance budget in the first year if it accounts for the money at the end of each year? What about the more realistic case of needing the money for park maintenance at the start of each year?

Now we have the same structure, but with the annual withdrawal as the unknown to solve for. In the case of year-end cash flows we find a first-year available maintenance budget of

PMT x FAC = TOT

$$\text{PMT}\left(\frac{1}{.072-.024}\right) = \$350{,}000$$

PMT x 20.833333 = $350,000

PMT = $350,000 ÷ 20.833333 = $16,800.00

If the withdrawal were made at the beginning of each year, the first in the series of withdrawals would have to be smaller because there would be no return earned before the first withdrawal:

$$\text{PMT}\left[\left(\frac{1}{.072-.024}\right)(1.072)\right] = \$350{,}000$$

PMT x 22.333333 = $350,000.00

PMT = $350,000 ÷ 22.333333 = $15,671.64

c. Assume again that Solberg contributes $350,000. If the university wants to spend amounts that start at $19,500 and then increase by 2.4% per year, what average annual rate of return must it earn on invested money? Compute for both the end-of-year payments case and the beginning-of-year payments case.

Now we have the same present value of a changing perpetuity situation, but with the rate of return as the unknown to solve for. We can solve directly, because we do not end up with both r and r-to-a-power in the same equation:

PMT x FAC = TOT

$$\$19{,}500\left(\frac{1}{r-.024}\right) = \$350{,}000$$

$$\left(\frac{1}{r-.024}\right) = 17.948718$$

1 = (r – .024)(17.948718)

1 = 17.948718 r – .430769

1.430769 = 17.948718 r

r = .079714 or 7.9714%

If withdrawals are made at the start of each year, the mathematics are slightly more complicated, but we still can solve directly. Note that the average annual rate of return earned would have to be higher because interest would be earned one less time toward providing for the maintenance costs:

$$\$19{,}500\left[\left(\frac{1}{r-.024}\right)(1+r)\right] = \$350{,}000$$
$$\left(\frac{1+r}{r-.024}\right) = 17.948718$$
$$1 + r = (r - .024)(17.948718)$$
$$1 + r = 17.948718\ r - .430769$$
$$1 = 17.948718\ r - .430769 - r$$
$$1 = 16.948718\ r - .430769$$
$$1.430769 = 16.948718\ r$$
$$r = .084418 \text{ or } \underline{\underline{8.4418\%}}$$

d. Now assume that Solberg gives the land and $350,000, but the university tells him that the land will be used by the ISU Agriculture program to grow experimental crops for six years before it is converted into a park. If the university can earn a 7.2% average annual rate of return on money it invests, how much can it spend on maintenance in year 7 (the first year in a perpetual stream of outlays that will grow by 2.4% annually) if it pays such expenses at the end of each year? What if it pays for park maintenance at the beginning of year 7 and each subsequent year?

This example deals with the present value of a deferred, changing perpetuity. In the case of year-end cash flows we find a year 7 value of

$$\text{PMT} \times \text{FAC} = \text{TOT}$$
$$\text{PMT}\left[\left(\frac{1}{.072-.024}\right)\left(\frac{1}{1.072}\right)^6\right] = \$350{,}000$$
$$\text{PMT} \times (20.833333)(.658918) = \$350{,}000$$
$$\text{PMT} = \$350{,}000 \div 13.727456 = \$\underline{\underline{25{,}496.35}}$$

If the withdrawal for paying maintenance costs instead were made at the beginning of each year, the first (year 7) in the series of withdrawals would be computed as

$$\text{PMT}\left[\left(\frac{1}{.072-.024}\right)(1.072)\left(\frac{1}{1.072}\right)^6\right] = \$350{,}000$$
$$\text{PMT} \times (20.833333)(.658918)(1.072) = \$350{,}000$$
$$\text{PMT} = \$350{,}000 \div 14.715832 = \$\underline{\underline{23{,}783.91}}$$

and then each subsequent year's withdrawal would be 2.4% higher; for example, in year 10 the university could expect to spend \$25,496.35 $(1.024)^3$ = \$27,376.50 if it paid for maintenance at the end of the year and \$23,783.91 $(1.024)^3$ = \$25,537.78 if it paid for maintenance at the start of the year. It stands to reason that if the university holds the money for 6 years before withdrawing anything, it can make larger withdrawals each time than in the non-deferred case described in part b above. However, again we see that the annual withdrawal must be smaller if taken at the beginning of each year, before the rate of return can be earned an initial time.

e. Assume again that Solberg gives the land and $350,000, and that the university will use the land for agricultural experiments for six years before converting it to a park. However, now the university wants to restate its budget to show monthly payments for park maintenance. If it expects to earn a 7.2% Annual Percentage Rate (APR) of return on money it invests, and if the amount it needs to spend is expected to rise by .2% per month, how much can it spend on maintenance at the end of the first month of year 7? What if it pays for park maintenance at the beginning of each month? What Effective Annual Rate (EAR) of return will ISU be earning on its money invested?

Now we have the present value of a deferred, changing perpetuity with monthly payments. Here our monthly periodic rate is 7.2% ÷ 12 = .006, or .6%. In the case of end-of-period cash flows we find that the amount that can be spent on maintenance in month 73 (the first month of year 7) is

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\text{PMT}\left[\left(\frac{1}{.006-.002}\right)\left(\frac{1}{1.006}\right)^{72}\right] = \$350{,}000$$

$$\text{PMT} \times (250)(.650048) = \$350{,}000$$

$$\text{PMT} = \$350{,}000 \div 162.511986 = \$\underline{\underline{2{,}153.69}}$$

If the withdrawal for paying maintenance costs instead were made at the start of each month, the first (month 73) in the series of withdrawals for park maintenance would be computed as a smaller

$$\text{PMT}\left[\left(\frac{1}{.006-.002}\right)(1.006)\left(\frac{1}{1.006}\right)^{72}\right] = \$350{,}000$$

$$\text{PMT} \times (250)(.650048)(1.006) = \$350{,}000$$

$$\text{PMT} = \$350{,}000 \div 163.487058 = \$\underline{\underline{2{,}140.84}}$$

The EAR earned on the account's declining balance would be $(1.006)^{12} - 1 = .074424$, or 7.4424%. Note that because perpetuities involve infinite cash flow streams, it makes no sense to solve for an unknown number of time periods in a perpetuity situation; the answer always would be "infinite."