Topic 12: Mortgage Loan Mechanics

In this discussion, we look at the computational steps used in solving for the payments, and some other relevant figures, for some common types of residential mortgage loans. ***Remember: you can not use graphing calculators on our exams because of concerns over the ability to retrieve stored alphanumeric information; as you study you should work with a calculator that can be used when you are tested on the mortgage loan computation material.***

I. Ways to classify mortgage loans

A. *By repayment method*

Think about how loan payments could be arranged.

1. *Negative amortization* – initial payments are so low that they *do not even cover interest owed* for the respective time periods. The shortfalls are added to the principal balance owed: *negative amortization* results (the amount the borrower owes after making the first payment is more than the amount initially borrowed). Think of the extreme case of a *pure discount* loan (no payment is made in periods 1 to $n - 1$, as with zero-coupon bonds); if the principal borrowed is $100,000 and the annual interest rate is 6%, the annual payment breakdown for a four-year term is

Year	(A) Principal Owed at Start of Year	(B) 6.0% Interest on That Principal	(C) Total Owed by End of Year (A + B)	(D) End of Year Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Year (C – D = A – E)
1	$100,000.00	$6,000.00	$106,000.00	$0.00	($6,000.00)	$106,000.00
2	$106,000.00	$6,360.00	$112,360.00	$0.00	($6,360.00)	$112,360.00
3	$112,360.00	$6,741.60	$119,101.60	$0.00	($6,741.60)	$119,101.60
4	$119,101.60	$7,146.10	$126,247.70	$126,247.70	$119,101.60	$0.00

A less extreme case might have something paid every period, but each period 1 through $n - 1$ amount still is less than applicable interest, with negative amortization again occurring (year 1's $2,000 should be easy to understand):

Year	(A) Principal Owed at Start of Year	(B) 6.0% Interest on That Principal	(C) Total Owed by End of Year (A + B)	(D) End of Year Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Year (C – D = A – E)
1	$100,000.00	$6,000.00	$106,000.00	$4,000.00	($2,000.00)	$102,000.00
2	$102,000.00	$6,120.00	$108,120.00	$4,000.00	($2,120.00)	$104,120.00
3	$104,120.00	$6,247.20	$110,367.20	$4,000.00	($2,247.20)	$106,367.20
4	$106,367.20	$6,382.03	$112,749.23	$112,749.23	$106,367.20	$0.00

Each year more principal is owed, but ultimately there must be a day of reckoning, with a large $126,247.70 or $112,749.23 *balloon payment* (a payment, usually at the end of a loan's term, that is considerably larger than other payments in the stream) made at the end of year 4. The *graduated payment mortgage* (GPM) loan is a type of negative amortization loan that sometimes might be well-suited to younger people with little money but much income growth potential. But the *option-adjustable rate mortgage* loan that gained popularity in the runup to the 2000s housing and mortgage lending crisis was a poorly conceived effort to keep payments low for cash-challenged borrowers, with the expectation (at least the hope) that interest rates would stay low, borrower incomes (and credit scores) would rise, and home prices would continue rising – none of which ended up coming true in many cases.

2. *No amortization* – a *straight term* (or "interest only") loan, with only interest paid during the loan's life and a balloon payment that includes the full principal amount at maturity (similar to the arrangement seen with coupon bonds issued by corporations). This structure has not traditionally been used much in practice (especially for loans on homes), but the concept should be easy to understand. Payments over a four-year term, if the principal borrowed is $100,000 and the interest rate charged is 6% per year, are

Year	(A) Principal Owed at Start of Year	(B) 6.0% Interest on That Principal	(C) Total Owed by End of Year (A + B)	(D) End of Year Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Year (C – D = A – E)
1	$100,000.00	$6,000.00	$106,000.00	$6,000.00	$0.00	$100,000.00
2	$100,000.00	$6,000.00	$106,000.00	$6,000.00	$0.00	$100,000.00
3	$100,000.00	$6,000.00	$106,000.00	$6,000.00	$0.00	$100,000.00
4	$100,000.00	$6,000.00	$106,000.00	$106,000.00	$100,000.00	$0.00

Here the balloon payment is only $106,000, because the borrower pays full interest each period (running in place with respect to principal owed, rather than actually falling behind as in the negative amortization examples). Stock brokerage firms have at times promoted interest-only home mortgage loans, touting the tax deductibility of the full payment (since the payments are solely interest and thus fully deductible – although the federal income tax benefits of mortgage loan interest paid were reduced with late 2017 tax law changes) and the accompanying ability to apply more money each month to other investments. A possibility put forth was to pair the interest-only loan with a whole life insurance policy, whose cash value would be expected to grow tax-free and ultimately equal the loan's principal.

3. *Partial amortization* – all interest owed is paid each period, and *some principal is repaid* during the loan's life, but there still is a sizable balloon payment owed at maturity. No current, commonly offered loan product has this feature, but the lack of prepayment penalties on home mortgage loans would allow a borrower to synthetically create this arrangement, by getting a straight term loan if available and then paying enough extra each month to reduce, but not fully repay, principal over the loan's life. Staying with our four-year/$100,000/6% example, if the borrower paid an amount somewhat larger than the interest owed in each of periods 1 – 3, let's say $9,000, payments would look as follows:

Year	(A) Principal Owed at Start of Year	(B) 6.0% Interest on That Principal	(C) Total Owed by End of Year (A + B)	(D) End of Year Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Year (C – D = A – E)
1	$100,000.00	$6,000.00	$106,000.00	$9,000.00	$3,000.00	$97,000.00
2	$97,000.00	$5,820.00	$102,820.00	$9,000.00	$3,180.00	$93,820.00
3	$93,820.00	$5,629.20	$99,449.20	$9,000.00	$3,370.80	$90,449.20
4	$90,449.20	$5,426.95	$95,876.15	$95,876.15	$90,449.20	$0.00

The amount of principal repaid in a particular period is the total payment minus the amount owed as interest for the period. In the straight term loan example above the payment in each of periods 1 – 3 is exactly the .06 x $100,000 = $6,000 due as interest, so nothing remains to reduce the remaining principal owed. But here even in year 1, when the greatest amount of interest is owed because the full $100,000 principal balance remains to be repaid, interest is just .06 x $100,000 = $6,000, so a $9,000 payment covers interest and chips away at the principal. Yet principal is not fully met through the regular payment stream, so there still is a balloon payment at the end of year 4 – but at $95,876.15 it is not as big as we saw in the two earlier cases when no principal was repaid ($106,000 balloon) or principal owed actually grew because too little was paid to cover each period's interest (*e.g.*, $126,247.70 balloon).

4. *Full amortization* – a fully amortizing loan provides for payment of periodic interest plus repayment of all lent principal over the stated term, with no need for a balloon payment (one considerably larger than the other payments in the stream) at the end. In the partial amortization case above involving a four-year/$100,000/6% loan, the period 1 – 3 payments of $9,000 each covered more than just interest, with modest amounts of principal retired. But if the period 1 – 3 yearly payments are not $0 or $6,000 or $9,000, but something more substantial, then four payments of reasonably similar magnitude can provide for full amortization, with no ending balloon. Consider the *constant amortization* mortgage (CAM) loan, characterized by payments that include equal amounts of principal every period. Because interest can be charged only on principal that remains unpaid, and more principal has been repaid with each passing period, the amount of interest owed gets smaller in each successive period, so the payments that consist of unchanging principal portions and declining interest portions get smaller in total as the periods pass. (This type loan is not used much in practice, although some agricultural loans seem to follow its general structure.) Each payment on a four-year/$100,000/6% constant amortization loan consists of $25,000 in principal and 6% interest on whatever principal is owed at the period's beginning; the repayment schedule shows as

Year	(A) Principal Owed at Start of Year	(B) 6.0% Interest on That Principal	(C) Total Owed by End of Year (A + B)	(D) End of Year Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Year (C – D = A – E)
1	$100,000.00	$6,000.00	$106,000.00	$31,000.00	$25,000.00	$75,000.00
2	$75,000.00	$4,500.00	$79,500.00	$29,500.00	$25,000.00	$50,000.00
3	$50,000.00	$3,000.00	$53,000.00	$28,000.00	$25,000.00	$25,000.00
4	$25,000.00	$1,500.00	$26,500.00	$26,500.00	$25,000.00	$0.00

Or consider the old standby: the fixed-payment, fixed interest rate mortgage loan (FRM). As with the CAM, each payment is large enough to cover interest due for the respective period and also repay enough principal that the last payment just covers the remaining principal owed – there is no balloon. But FRM payments stay exactly the same throughout the loan term: $28,859.15 for the four-year/$100,000/6% loan. The final $28,859.15 outlay is precisely what is needed to cover $1,633.54 due as interest on the $27,225.61 in principal owed coming into year 4, plus that $27,225.61 in principal. (With each passing period more principal has been repaid, so less remains owed, so in the next period less interest is charged, such that each successive period's unchanging payment consists less of interest/ more of principal.) The way we got that unusual $28,859.15 value is explained after amortization case 5, just below.

Year	(A) Principal Owed at Start of Year	(B) 6.0% Interest on That Principal	(C) Total Owed by End of Year (A + B)	(D) End of Year Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Year (C – D = A – E)
1	$100,000.00	$6,000.00	$106,000.00	$28,859.15	$22,859.15	$77,140.85
2	$77,140.85	$4,628.45	$81,769.30	$28,859.15	$24,230.70	$52,910.15
3	$52,910.15	$3,174.61	$56,084.76	$28,859.15	$25,684.54	$27,225.61
4	$27,225.61	$1,633.54	$28,859.15	$28,859.15	$27,225.61	$0.00

5. *More than full amortization* – each period the borrower pays more than the amount that would be paid under a fully amortizing CAM or FRM arrangement. The added money paid directly reduces principal owed, and thereby shortens the period over which payments must be made (or at least makes the final payment smaller than earlier periods' outlays – the opposite of a balloon payment). Consider a four-year case with $30,000 constant amortization each year for the first three years, leaving only $10,000 owed coming into year 4.

Year	(A) Principal Owed at Start of Year	(B) 6.0% Interest on That Principal	(C) Total Owed by End of Year (A + B)	(D) End of Year Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Year (C – D = A – E)
1	$100,000.00	$6,000.00	$106,000.00	$36,000.00	$30,000.00	$70,000.00
2	$70,000.00	$4,200.00	$74,200.00	$34,200.00	$30,000.00	$40,000.00
3	$40,000.00	$2,400.00	$42,400.00	$32,400.00	$30,000.00	$10,000.00
4	$10,000.00	$600.00	$10,600.00	$10,600.00	$10,000.00	$0.00

With home mortgage lending, a plan with accelerated principal repayment sometimes is called a growing equity mortgage (GEM) loan. The extra payment can be an official part of the loan agreement, or can be done informally by the borrower because no prepayment penalty typically can be charged on home loans. The *biweekly mortgage* loan (typically managed by a third-party service provider for a fee, rather than being an official part of the plan) might be viewed as an example; instead of one payment at the end of each month, the borrower pays about half that amount every two weeks. By paying off a little bit of principal sooner the borrower cuts the total interest owed over the loan's life (and speeds up repayment by getting in 26 "half"-payments per year instead of 12 "full" payments).

So examples 1 through 5 show that a loan can have less than none of the principal, none of the principal, some of the principal, all of the principal, or more than all of the required principal (accelerated repayment that shortens the payment stream or reduces the final payment) repaid over the loan's stated maturity period.

Now returning to the important fixed-payment, fixed interest rate FRM loan: where did the $28,859.15 payment come from? Let's first work backwards: think of a borrower who tells a mortgage loan officer that she can budget to pay $28,859.15 at the end of each year for four years; if the bank charges a 6% annual interest rate, how much can this individual afford to borrow? From a time value of money standpoint, the amount that can be borrowed or lent is the sum of the present values (PVs) of the payments that will be made or received. So if we know the payments the borrower can afford, we can solve for the principal that can be lent:

- Amount borrowed today that would correspond to a payment of $28,859.15 in 1 year is

  $PV\ (1.06)^1 = \$28,859.15 \Rightarrow \$28,859.15 \div (1.06)^1$ or $\$28,859.15\left(\frac{1}{1.06}\right)^1 = \$27,225.61$ PV
- Amount borrowed today that would correspond to a payment of $28,859.15 in 2 years is

  $PV\ (1.06)^2 = \$28,859.15 \Rightarrow \$28,859.15 \div (1.06)^2$ or $\$28,859.15\left(\frac{1}{1.06}\right)^2 = \$25,684.54$ PV
- Amount borrowed today that would correspond to a payment of $28,859.15 in 3 years is

  $PV\ (1.06)^3 = \$28,859.15 \Rightarrow \$28,859.15 \div (1.06)^3$ or $\$28,859.15\left(\frac{1}{1.06}\right)^3 = \$24,230.70$ PV
- Amount borrowed today that would correspond to a payment of $28,859.15 in 4 years is

  $PV\ (1.06)^4 = \$28,859.15 \Rightarrow \$28,859.15 \div (1.06)^4$ or $\$28,859.15\left(\frac{1}{1.06}\right)^4 = \$22,859.15$ PV

A lender can justify lending the total of the PVs of all the payments it expects to receive from the borrower. The fact that *the amount of principal owed, on a given date, is the sum of the PVs of the remaining payments*, discounted at a rate corresponding to the interest rate specified in the borrower/lender contract, is perhaps the most important thing to understand about loans. Here the PVs of the expected equal payments sum to $27,225.61 + $25,684.54 + $24,230.70 + $22,859.15 = $100,000. So $100,000 is the principal the lender is willing to extend (what the borrower can afford to repay), if the interest rate is 6% per year and the loan is to be "serviced" with four year-end payments of $28,859.15 each. The steps shown above can be summarized as

$$\$28{,}859.15\left(\frac{1}{1.06}\right)^1 + \$28{,}859.15\left(\frac{1}{1.06}\right)^2 + \$28{,}859.15\left(\frac{1}{1.06}\right)^3 + \$28{,}859.15\left(\frac{1}{1.06}\right)^4$$

and, through the distributive property, shown even more succinctly as

$$\$28{,}859.15\left[\left(\frac{1}{1.06}\right)^1 + \left(\frac{1}{1.06}\right)^2 + \left(\frac{1}{1.06}\right)^3 + \left(\frac{1}{1.06}\right)^4\right] = \$28{,}859.15\ (.943396 + .889996 + .839619 + .792094)$$

$$= \$28{,}859.15\left(\frac{1-\left(\frac{1}{1.06}\right)^4}{.06}\right) = \$28{,}859.15\ (3.465106) = \$100{,}000$$

Now reverse the order; a bank that lends $100,000 at a 6% annual interest rate, if the borrower is to make four equal year-end payments, can divide $100,000 by 3.465106 to compute the $28,859.15 annual payment. This 3.465106 is the *present value of a level ordinary annuity* factor for a 6% periodic interest rate and four time periods. The PV of annuity situation equates a series of equal or related payments to a large dollar sum that exists intact in the present but will decline over time; repaying a loan is the classic example. The factor for the PV of a level ordinary annuity, with "ordinary" indicating end-of-period payments and "level" meaning that all payments are equal, is simply the sum of the present values of the single dollar amount factors for the same periodic discount rate and same number of time periods:

$$\left[\left(\frac{1}{1.06}\right)^1 + \left(\frac{1}{1.06}\right)^2 + \left(\frac{1}{1.06}\right)^3 + \left(\frac{1}{1.06}\right)^4\right] = \left(\frac{1-\left(\frac{1}{1.06}\right)^4}{.06}\right)$$

$$(.943396 + .889996 + .839619 + .792094) = 3.465106$$

$$\text{so FRM Payment} = \$100{,}000 \div \left(\frac{1-\left(\frac{1}{1.06}\right)^4}{.06}\right) = \$100{,}000 \div 3.465106 = \$28{,}859.15$$

(The present value of an annuity factor's magnitude is smaller than the number of payments. A borrower could not submit a mere $100,000 ÷ 4 = $25,000 at the end of each year to repay this $100,000 loan by the end of year 4; four payments of $100,000 ÷ 3.465106 = $28,859.15 each, which together account for interest along with principal repayment, would be needed.) What more typically is actually used in computing the steady periodic payment on an FRM loan, however, is the PV of a level ordinary annuity factor's *reciprocal* (here 1 ÷ 3.465106 = .288591); we multiply borrowed principal by this *loan payment factor*, also called the *mortgage loan constant*, to compute the periodic payment:

$$\text{FRM Payment} = \$100{,}000\left(\frac{.06}{1-\left(\frac{1}{1.06}\right)^4}\right) = \$100{,}000\ (.288591) = \$28{,}859.15$$

Or what if a $100,000 loan with a 6% annual interest rate were to be repaid with equal year-end payments over fifteen years rather than four? Then the unchanging annual payment would be computed as

$$\text{FRM Payment} = \$100{,}000 \div \left(\frac{1-\left(\frac{1}{1.06}\right)^{15}}{.06}\right) = \$100{,}000\left(\frac{.06}{1-\left(\frac{1}{1.06}\right)^{15}}\right)$$

$$= \$100{,}000\ (.1029628) = \$10{,}296.28$$

(The payment is actually $10,296.27640; it is shown as $10,296.28 in column D of the amortization schedule below, but Excel keeps track of the added decimal places, and that is why the total repaid comes out to exactly $100,000.00 and the final amount owed comes out to exactly $0.00.) Notice in the following grid how the principal portions of the fifteen payments made add up to be the $100,000 borrowed, and the $154,444.15 in total payments made is the sum of the $100,000 in principal repaid plus the $54,444.15 interest paid over the loan's life. Compare this grid to the similar breakdown just below it for the $100,000 loan with a 6% annual interest rate and equal payments made over four years. Its shorter period for repaying necessitates a higher annual payment of $28,859.15, but since debt is outstanding for fewer years a much lower $15,436.60 in total interest is paid over the loan's life. Repaying over a longer period keeps the individual payments lower/more affordable, but at a cost of more in total interest paid.

$100,000, 6% interest rate loan with equal payments over fifteen years (lower payments, more total interest paid):

Year	(A) Principal Owed at Start of Year	(B) 6.0% Interest on That Principal	(C) Total Owed by End of Year (A + B)	(D) End of Year Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Year (C – D = A – E)
1	$100,000.00	$6,000.00	$106,000.00	$10,296.28	$4,296.28	$95,703.72
2	$95,703.72	$5,742.22	$101,445.95	$10,296.28	$4,554.05	$91,149.67
3	$91,149.67	$5,468.98	$96,618.65	$10,296.28	$4,827.30	$86,322.37
4	$86,322.37	$5,179.34	$91,501.72	$10,296.28	$5,116.93	$81,205.44
5	$81,205.44	$4,872.33	$86,077.77	$10,296.28	$5,423.95	$75,781.49
6	$75,781.49	$4,546.89	$80,328.38	$10,296.28	$5,749.39	$70,032.10
7	$70,032.10	$4,201.93	$74,234.03	$10,296.28	$6,094.35	$63,937.75
8	$63,937.75	$3,836.27	$67,774.02	$10,296.28	$6,460.01	$57,477.74
9	$57,477.74	$3,448.66	$60,926.41	$10,296.28	$6,847.61	$50,630.13
10	$50,630.13	$3,037.81	$53,667.94	$10,296.28	$7,258.47	$43,371.66
11	$43,371.66	$2,602.30	$45,973.96	$10,296.28	$7,693.98	$35,677.69
12	$35,677.69	$2,140.66	$37,818.35	$10,296.28	$8,155.62	$27,522.07
13	$27,522.07	$1,651.32	$29,173.39	$10,296.28	$8,644.95	$18,877.12
14	$18,877.12	$1,132.63	$20,009.74	$10,296.28	$9,163.65	$9,713.47
15	$9,713.47	$582.81	$10,296.28	$10,296.28	$9,713.47	$0.00
15-Year Totals →		$54,444.15		$154,444.15	$100,000.00	

$100,000, 6% interest rate loan with equal payments over four years (higher payments, less total interest paid):

Year	(A) Principal Owed at Start of Year	(B) 6.0% Interest on That Principal	(C) Total Owed by End of Year (A + B)	(D) End of Year Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Year (C – D = A – E)
1	$100,000.00	$6,000.00	$106,000.00	$28,859.15	$22,859.15	$77,140.85
2	$77,140.85	$4,628.45	$81,769.30	$28,859.15	$24,230.70	$52,910.15
3	$52,910.15	$3,174.61	$56,084.76	$28,859.15	$25,684.54	$27,225.61
4	$27,225.61	$1,633.54	$28,859.15	$28,859.15	$27,225.61	$0.00
4-Year Totals →		$15,436.60		$115,436.60	$100,000.00	

[Loan payments almost always are made at the end of each period; a loan with beginning-of-period payments would be quite awkward – the lender hands the borrower $100,000 and then immediately asks for a payment, after which the borrower no longer has the full $100,000 needed. Standard practice is for the borrower to receive principal at the beginning of period 1, and then during period 1 the lender adds interest to what the borrower owes, such that by the end of period 1 the borrower owes more than was borrowed (see year 1 in the grid for either loan above). But because the borrower has been able to use the money productively, she is able to make a payment at the end of the period, and if that payment exceeds the interest that was applied then some principal is repaid as well.]

The 30-year FRM loan emerged in the 1930s with the creation of FHA. Earlier loans had been shorter term, such as five years, interest-only with full balloon payments. 30-year or other long-term FRMs were seen as being good for:

- borrowers, because of cash flow benefits (ability to spread payments over a long period).
- lenders, because the amortization feature reduced the likelihood that the borrower would default.

But the long-term FRM can be bad for:

- lenders because of interest rate risk, especially in light of the short-term nature of the deposit base of banks and other depository institutions that have been traditional home mortgage lenders.
- borrowers because the unchanging interest rate charged must include an inflation premium that is the average of annual inflation rates expected over the loan's entire term. A result is the *"tilt" problem: while FRM payments are fixed in nominal dollar terms, if expected inflation occurs they will decline for the borrower in real dollar terms over time – but that means they are expensive with respect to the borrower's initial purchasing power. This need to pay toward expected future inflation now, before higher living costs would bring pay raises, could price struggling borrowers out of the housing market.*

Recall that the premium for expected inflation is an important component of interest rates. A lender making a 30-year FRM loan is likely to expect inflation to occur at some point over that period, but even if the inflation is not expected for 10 years the compensation for that future inflation must be built into the fixed interest rate quoted at origination – of course, the borrower's income probably will not rise until year 10, when the inflation happens. Greater relative affordability of the payments over time certainly can have benefits for the borrower. But the need to pay from the outset for problems anticipated in later years makes the early-year payments expensive in real dollar terms (the FRM forces young families to subsidize their later, higher-earning years).

Computing Monthly Payments for the Fixed Rate, Fixed Payment Mortgage Loan
Up to this point we have dealt with fairly short-term loans with annual payments, toward being able to follow the numbers carefully through the repayment structures. But our main concern in Topic 12 is residential mortgage loans, which almost always are longer-term, with payments made at the end of each *month* rather than each year. A 30-year (360 month) repayment plan is most typical, although 15 years also is common; and 10, 20, and 25-year amortization periods sometimes are available (40-year loans have been offered in some cases, even 50-year terms have been discussed), and lenders now generally can create amortizations for any periods borrower desire. [Some lenders extended payment periods from 30 to 40 years to forestall defaults in the 2000s mortgage meltdown; other "loan modification" tools included reduced interest rates and deferred, or some forgiven, principal repayment.]

Recall that computing the equal periodic payment for an FRM is a *present value of an annuity* exercise (the large principal amount lent today corresponds to a series of equal payments spaced equally through time). Assume again that we have a $100,000 loan with a 6% annual percentage rate (APR) of interest, but now amortization occurs with equal end-of-*month* payments over 30 years. The relevant number of (monthly) time periods is 30 x 12 = 360, and the relevant monthly interest rate is .06 ÷ 12 = .005. Our computation formula is

$$\text{FRM Payment}\left(\frac{1-\left(\frac{1}{1+r}\right)^{n}}{r}\right) = \text{Total Principal Borrowed}$$

$$\text{FRM Payment}\left(\frac{1-\left(\frac{1}{1.005}\right)^{360}}{.005}\right) = \text{FRM Payment } (166.791614) = \$100{,}000$$

$$\text{FRM Payment} = \$100{,}000 \div (166.791614) = \$599.55$$

or, equivalently, we multiply the loan amount by a payment factor (reciprocal of the PV of annuity factor) to compute the payment:

$$\$100{,}000\left(\frac{.005}{1-\left(\frac{1}{1.005}\right)^{360}}\right) = \$100{,}000\ (.005996) = \$\underline{\underline{599.55}}$$

[Excel computes the payment very expediently as =PMT(6%/12,30*12,100000), but that shortcut shows us nothing about the logic behind loan payments and amortizations.] For an FRM loan, if the life is the typical 30 years and the annual interest rate is in the historically typical range of 7 – 9%, the payment factor is somewhere in the neighborhood of .007 (think of a famous spy). Note also that the monthly payment factor is a number slightly greater than the monthly interest rate (in the above example, that monthly rate is 6% ÷ 12 = .005, and the monthly payment factor is the slightly larger .005996). Why? Because each monthly payment contains .005 times the remaining principal balance to cover the month's interest, plus a little bit more to chip away further at the remaining principal. [The loan payment factor actually consists of the interest rate for each period plus the *sinking fund factor* for accumulating a lump sum – the principal borrowed – for the loan's amortization term and periodic interest rate. The sinking fund factor is the reciprocal of the *future* value of an annuity factor for the same discount rate and same number of time periods. With the numbers in this example the future value of a level ordinary annuity factor (which is the sum of the future values of the single dollar amount factors for the same periodic compounding rate and same number of time periods, with end-of-period payments) is

$$[(1.005)^{359} + (1.005)^{358} + \cdots + (1.005)^{1} + (1.005)^{0}] = \left(\frac{(1.005)^{360}-1}{.005}\right) = 1{,}004.515042$$

and its reciprocal, the sinking fund factor, is $\left(\frac{.005}{(1.005)^{360}-1}\right) = 1 \div 1{,}004.515042 = .0009955$

(The FV of an annuity factor's magnitude is larger than the number of payments. A saver making 360 deposits would end up with 1,004.515 times each deposit, because of the substantial interest buildup over all those periods.) So the interest rate plus the sinking fund factor total to .005 + .000996 = .005996, which is the loan payment factor.

Someone wanting to borrow \$100,000 with a 6% APR interest rate and end-of-month payments for 30 years could pay .005996 x \$100,000 = \$599.55 each month under a standard FRM arrangement. But alternatively, that borrower could get an interest-only loan, directly paying interest of .005 x \$100,000 = \$500 each month, while simultaneously putting .000996 x \$100,000 = \$99.55 into a "sinking fund" savings account each month toward saving \$100,000 that would be paid as a balloon payment (along with the final interest payment) at the end of month 360. Note that an interest-only loan has no regular amortization of principal, so the sinking fund factor is 0 and the payment factor is the periodic interest rate + 0 = just the periodic interest rate. If an interest-only loan's principal borrowed is \$100,000 and the APR is 6% the monthly *r* would be .5% or .005, and the monthly payment every month would be just \$100,000 (.005) = \$500, with a balloon payment of \$100,000 to be made at the end of the loan's term.]

The amortization plan for this loan's first and last years is as shown in the grid below. Note how, with each passing month, the portion of the unchanging \$599.55 payment devoted to interest gets smaller and the remainder that goes to retiring principal gets bigger. During year 1 most of each payment goes to paying interest and a small portion reduces principal, while in year 30 most of the \$100,000 original principal already has been repaid, so only a small piece of each monthly payment is needed for interest and the much larger remainder goes to paying back principal.

Month	(A) Principal Owed at Start of Month	(B) 0.5% Interest on That Principal	(C) Total Owed by End of Month (A + B)	(D) End of Month Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Month (C – D = A – E)
1	\$100,000.00	\$500.00	\$100,500.00	\$599.55	\$99.55	\$99,900.45
2	\$99,900.45	\$499.50	\$100,399.95	\$599.55	\$100.05	\$99,800.40
3	\$99,800.40	\$499.00	\$100,299.40	\$599.55	\$100.55	\$99,699.85
4	\$99,699.85	\$498.50	\$100,198.35	\$599.55	\$101.05	\$99,598.80
5	\$99,598.80	\$497.99	\$100,096.80	\$599.55	\$101.56	\$99,497.24
6	\$99,497.24	\$497.49	\$99,994.73	\$599.55	\$102.06	\$99,395.18
7	\$99,395.18	\$496.98	\$99,892.16	\$599.55	\$102.57	\$99,292.61
8	\$99,292.61	\$496.46	\$99,789.07	\$599.55	\$103.09	\$99,189.52
9	\$99,189.52	\$495.95	\$99,685.47	\$599.55	\$103.60	\$99,085.92
10	\$99,085.92	\$495.43	\$99,581.35	\$599.55	\$104.12	\$98,981.79
11	\$98,981.79	\$494.91	\$99,476.70	\$599.55	\$104.64	\$98,877.15
12	\$98,877.15	\$494.39	\$99,371.54	\$599.55	\$105.16	\$98,771.99
Year 1 Totals →		\$5,966.59		\$7,194.61	\$1,228.01	
349	\$6,966.14	\$34.83	\$7,000.97	\$599.55	\$564.72	\$6,401.42
350	\$6,401.42	\$32.01	\$6,433.42	\$599.55	\$567.54	\$5,833.87
351	\$5,833.87	\$29.17	\$5,863.04	\$599.55	\$570.38	\$5,263.49
352	\$5,263.49	\$26.32	\$5,289.81	\$599.55	\$573.23	\$4,690.26
353	\$4,690.26	\$23.45	\$4,713.71	\$599.55	\$576.10	\$4,114.16
354	\$4,114.16	\$20.57	\$4,134.73	\$599.55	\$578.98	\$3,535.18
355	\$3,535.18	\$17.68	\$3,552.86	\$599.55	\$581.87	\$2,953.31
365	\$2,953.31	\$14.77	\$2,968.07	\$599.55	\$584.78	\$2,368.52
357	\$2,368.52	\$11.84	\$2,380.36	\$599.55	\$587.71	\$1,780.81
358	\$1,780.81	\$8.90	\$1,789.72	\$599.55	\$590.65	\$1,190.17
359	\$1,190.17	\$5.95	\$1,196.12	\$599.55	\$593.60	\$596.57
360	\$596.57	\$2.98	\$599.55	\$599.55	\$596.57	\$0.00
Year 30 Totals →		\$228.47		\$7,194.61	\$6,966.14	
30-Year Totals →		\$115,838.19		\$215,838.19	\$100,000.00	

Note that *the monthly payment for a 30-year/\$100,000/6% APR loan is not just ¹⁄₁₂ of what would be paid on a loan with similar terms and annual payments*. The unchanging payment on a \$100,000 loan with a 6% annual interest rate and 30 annual year-end payment periods would be

$$\text{FRM Payment} = \$100{,}000 \div \left(\frac{1-\left(\frac{1}{1.06}\right)^{30}}{.06}\right) = \$100{,}000 \div (13.764831) = \$100{,}000\ (.072649) = \$7{,}264.89,$$

and ¹⁄₁₂ of that amount is \$7,264.89 ÷ 12 = \$605.41 – which is more than the \$599.55 true monthly payment. If the first payment is not made until the end of *year* 1 then interest is owed on the full \$100,000 principal for the entire first year rather than for just one month, indeed with annual payments there is more principal remaining unpaid, on average, to pay interest on over the entire life of the loan. So more money in total must be paid to the lender with annual payments than with monthly payments based on the same principal, annual interest rate, and maturity.

One final point: the interest rate in the example above, with 30 years of monthly payments, is stated as a 6% annual percentage rate (APR), which we converted to a monthly working rate r of .06 ÷ 12 = .005 or .5% per month. But we could have presented the interest rate in the initial discussion phase instead as an effective annual rate (EAR) of 6.1678%, which we would convert to a monthly working rate r of $\sqrt[12]{1.061678} - 1$ = .005 or .5% per month.

We talk about interest rates in annual terms, but in time value of money exercises we must work with a rate that corresponds to the timing of the payments and compounding. The APR is an annual interest rate measure we can talk about that includes the periodic rate and number of periods in a year, but not the impact of compounding that occurs within the year (it is a *simple interest* measure); here the APR is .005 x 12 = .06 or 6% per year. But the EAR is an annual rate measure we can talk about that is adjusted for the impact of compounding that occurs within the year (a *compound interest* measure); it is computed as $(1.005)^{12} - 1$ = .061678, or 6.1678% per year. Someone who keeps money for a full year in a plan that earns .5% per month, and that interest compounds, ends the year 6.1678% ahead of where she started, not just 6%. We might say the APR is a convenient measure that does not accurately reflect someone's true gain in wealth (or opportunity cost), while the EAR is an accurate measure that is not convenient to use (dividing .06 by 12 is much easier than taking the 12th root of 1.061678).

6. *Fully amortizing mortgage loans without fixed interest rates*. The long-term FRM's unchanging interest rate causes a tilt problem for the borrower, and also imposes considerable interest rate risk on the lender. Once the lender buys the note from the borrower the lender is stuck for up to 30 years; if interest rates in the market rise far above what the lender had anticipated the lender will continue to earn 6% annually for many remaining years while newly negotiated loans similar in risk earn, *e.g.*, 9%. But if interest rates in the market fall the lender does not get to enjoy earning 6% per year overall for many remaining years while new loans would generate, *e.g.*, 4%, because home mortgage loan borrowers usually can prepay some or all principal owed, with no direct penalty. As a result, when market interest rates decline many borrowers *refinance* (pay back existing high-interest-rate loans with money obtained through new loans carrying the new, lower interest rate), and then the lender must re-lend that money at the lower prevailing interest rate. Heads the borrower wins, tails the lender loses. [This issue destroyed the savings and loan industry. In the late 1970s S&Ls made 30-year FRM loans with 8% annual interest rates, funded with savings accounts on which they paid savers 5 or 6% in annual interest. But the savings accounts were short-term, and as rates across the market rose abruptly *circa* 1980 – largely over inflation concerns – the savers demanded higher returns on their money that the S&Ls had tied up in 30-year FRM loans. Ultimately the institutions were paying 10% annual interest for the continued use of money they were, in turn, earning only 8% on – a fatal condition.]

The two problems created by the FRM's fixed interest rate can be addressed with a plan that lets the lender change the borrower's interest rate on some scheduled basis to align the lender's returns with current market conditions. Then instead of worrying about possible inflation or other matters over 30 coming years, the lender is concerned primarily with the impact those issues could have before the interest rate can be reset – and can quote a lower initial interest rate accordingly (fewer unforeseen crises are likely to arise over five years than over 30). A loan that allows for changing interest rates typically is called an *adjustable rate* or *variable rate* mortgage loan (ARM or VRM) if the first rate reset date is just a year into the loan's term, or a *hybrid* or *two-step* loan if the interest rate can not change until a longer period (often five or seven years) has passed. The interest rate quoted to the borrower consists of an *index*, such as the interest rate being paid on U.S. government bonds of various maturities, plus a *margin* of 200 – 300 basis points since a mortgagor should expect to pay a higher interest rate than the federal government pays when it borrows. (The lender can not change the borrower's interest rate on a whim; rate changes must reflect changes in the verifiable index – and if the index rate falls the lender must, in turn, reduce the borrower's interest rate.) A common feature on these loans is to restrict, or "cap," the interest rate increase to a specified percentage per year and a specified percentage over the life of the loan so the borrower does not have to worry about enormous increases in payments; thus the lender's interest rate risk is reduced, but not eliminated. Of course, if possible rate increases are capped then the lender in a competitive market must charge a higher rate initially than if possible increases were unlimited. (Loans with changing interest rates were not much sought by borrowers during the years when 30-year FRM annual interest rates hovered in the 3-4% range or lower, but by fall of 2022, when the 30-year fixed rate hovered in the 7% or higher annual range, adjustable-rate and hybrid loans became popular again.)

The key to understanding loans with changeable interest rates is the important point, as noted earlier, that the principal still owed on a loan at any date is the present value of the stream of remaining payments, with the loan's contract interest rate used as the discount rate. Consider a $100,000 loan with end-of-month payments to be made over 30 years (360 months). The initial quoted interest rate is a 4.8% APR (perhaps a below-market "teaser" rate, which is less than the index plus the margin, designed to attract borrowers to this type of loan, toward reducing the lender's interest rate risk over the longer term).

Case 1: Interest rate is reset after one year
Step 1: the payment for every month of the first year is based on a 30 x 12 = 360-month term n and a monthly periodic interest rate r of .048 ÷ 12 = .004.

$$\text{Payment}\left(\frac{1-\left(\frac{1}{1.004}\right)^{360}}{.004}\right) = \text{Payment } (190.597681) = \$100{,}000$$

Payment = \$100,000 ÷ 190.597681 = \$524.67 OR
Payment = \$100,000 (.005247) = \$524.67

Step 2: the remaining principal balance at end of year 1 is the PV of the 348-month remaining payment stream, with discounting based on the loan's initial contract interest rate (4.8% APR or .004 monthly *r*):

$$\$524.67\left(\frac{1-\left(\frac{1}{1.004}\right)^{348}}{.004}\right) = \$524.67\ (187.682797) = \$98{,}470.66 \text{ ("goodbye, old loan")}$$

So coming into year 2 we have a brand new loan with a \$98,470.66 initial principal, payments for 29 years = 348 months, and a monthly periodic interest rate *r* (let's say the reset date margin-plus-index APR is 6%) of .06 ÷ 12 = .005, for a year-2 monthly payment of

$$\text{Payment}\left(\frac{1-\left(\frac{1}{1.005}\right)^{348}}{.005}\right) = \text{Payment } (164.743394) = \$98{,}470.66 \text{ ("hello, new loan")}$$

Payment = \$98,470.66 ÷ 164.743394 = \$597.72 OR
Payment = \$98,470.66 (.006070) = \$597.72

[It should be obvious that if there is no change in the year 2 interest rate, because the index has not changed, then the payment should remain the same. Let's demonstrate this result by computing the payment on a 348-month loan with a 4.8% stated APR (.004 monthly rate) and a \$98,470.66 remaining balance:

$$\text{Payment}\left(\frac{1-\left(\frac{1}{1.004}\right)^{348}}{.004}\right) = \text{Payment } (187.682797) = \$98{,}470.66 \text{ ("hello, hypothetical new loan")}$$

Payment = \$98,470.66 ÷ 187.682797 = \$524.67 OR
Payment = \$98,470.66 (.005328) = \$524.67]

Then with this loan structure we would see a similar process play out in each successive year; if the market interest rate (index plus margin) coming into year 3 is 7.2% APR we would have a brand new loan based on the principal that remains owed at the start of year 3, 28 years = 336 monthly payments, and a .072 ÷ 12 = .006 monthly periodic interest rate. The amount still owed coming into year 3 would be computed as

$$\$597.72\left(\frac{1-\left(\frac{1}{1.005}\right)^{336}}{.005}\right) = \$597.72\ (162.568844) = \$97{,}170.88 \text{ ("goodbye, old loan")}$$

and year 3's monthly payment would be

$$\text{Payment}\left(\frac{1-\left(\frac{1}{1.006}\right)^{336}}{.006}\right) = \text{Payment } (144.334687) = \$97{,}170.88 \text{ ("hello, new loan")}$$

Payment = \$97,170.88 ÷ 144.334687 = \$673.23 OR
Payment = \$97,170.88 (.006928) = \$673.23

Case 2: Interest rate is reset after multiple years
Step 1: the payment that applies to every month in the first seven years of this "7/23" hybrid or "two-step" loan is based on a 30 x 12 = 360-month amortization term and .054 ÷ 12 = .0045 monthly periodic interest rate (an annual rate slightly higher than in case 1 above, because the payment will remain in place for 7 x 12 = 84 months rather than just year 1's 12 months before the rate can change, so the lender faces seven years of interest rate risk).

$$\text{Payment}\left(\frac{1-\left(\frac{1}{1.0045}\right)^{360}}{.0045}\right) = \text{Payment } (178.084624) = \$100{,}000$$

Payment = $100,000 ÷ 178.084624 = $561.53 OR
Payment = $100,000 (.005615) = $561.53

Step 2: the remaining principal balance at end of year 7 (month 84) is the PV of the 360 – 84 = 276-month remaining payment stream, based on the 5.4% APR = .0045 monthly *r* original contract interest rate:

$$\$561.53\left(\frac{1-\left(\frac{1}{1.0045}\right)^{276}}{.0045}\right) = \$561.53\ (157.864043) = \$\underline{88{,}645.52}\ \text{(“goodbye, old loan)}$$

So coming into year 8 we have a brand new loan with an $88,645.52 initial principal, payments for 23 years = 276 months, and a monthly periodic interest rate *r* (now we will assume that the index-plus-margin market interest rate at the reset date is a 6.6% APR, for a monthly *r* of .066 ÷ 12 = .0055), giving a monthly payment during year 8 of

$$\text{Payment}\left(\frac{1-\left(\frac{1}{1.0055}\right)^{276}}{.0055}\right) = \text{Payment } (141.806701) = \$88{,}645.52\ \text{(“hello, new loan”)}$$

Payment = $88,645.52 ÷ 141.806701 = $625.12 OR
Payment = $88,645.52 (.007052) = $625.12

Of course this payment will not remain the same for the amortization term's remaining 23 years; once the first reset date arrives the subsequent rate resets are likely to occur yearly (or perhaps after six months), regardless of whether the initial rate period was one year, or five or seven years, or some other interval. Continuing with our two-step loan example, if the interest rate is reset yearly and the index gives a 7.8% APR reset rate (.078 ÷ 12 = .0065 monthly *r*) coming into year 9, when 8 years have passed so 22 years or 264 months of initially scheduled payments remain, the year 9 payment is computed as:

$$\$625.12\left(\frac{1-\left(\frac{1}{1.0055}\right)^{264}}{.0055}\right) = \$625.12\ (139.084577) = \$\underline{86{,}943.88}\ \text{(“goodbye, old loan”)}$$

$$\text{Payment}\left(\frac{1-\left(\frac{1}{1.0065}\right)^{264}}{.0065}\right) = \text{Payment } (126.033084) = \$86{,}943.88\ \text{(“hello, new loan”)}$$

Payment = $86,943.88 ÷ 126.033084 = $86,943.88 (.007934) = $689.85

(The first interest rate reset date is not restricted by law or even industry practice; between mid 2022 and early 2023 we saw hybrid loans with 30-year amortizations and interest rates that stayed fixed for 10 or 15 years). In fact, sometimes the hybrid loan is described as, *e.g.*, "7/1" or "7/6" rather than "7/23" to signify that after the first interest rate reset the following resets will occur yearly or after six months (5/1 or 5/6 for the hybrid whose interest rate remains fixed for five years, a popular choice as fixed-rate loan interest rates started rising in fall 2022). The lender trying to reduce interest rate risk certainly would not want to be bound to a new rate for a 23- or 25-year period.[1]

The two-step loan (initial rate locked in for multiple years) is a very clever and useful, somewhat-recent twist on the longer-established variable rate loan (initial rate locked in for only one year). Even though 30-year amortization plans are typical (to keep payments affordable), most mortgage borrowers do not keep their loans outstanding for 30 years; loans tend to be repaid more quickly (studies have shown average periods of 7 – 10 years), for reasons such as people moving (recall the alienation or "due-on-sale" clause), inheriting money, or refinancing. With a 30-year FRM the lender must charge a high interest rate because it faces 30 years of interest rate risk, but then the borrower knows she need not fear a rate increase for 30 years. With the original ARM adjustable rate loans the lender could quote a low year-1 interest rate because it could increase the rate, if market conditions (the index plus margin) so indicated, after just a year, while the borrower had to worry about possible rate (and payment) increases year after year from the outset. (Rates on some adjustable-rate loans on commercial real estate can change every month.) But a 5/25 or 7/23 hybrid allows the lender to charge an initial interest rate higher than an ARM's low figure but lower than the FRM's high figure, knowing it is stuck with that initial rate only for a few years – while the borrower need not be concerned about a rate increase until a date by which many borrowers already have repaid their loans. This arrangement could work nicely for a buyer who does not plan to stay in the home for more than a few years.

Some interesting variations on the hybrid mortgage loan have been tried. One seen several years ago was the "20/20" loan, with an interest rate adjustment 20 years into the loan's 40-year amortization. Another had a 45 (even 50) year amortization, but a balloon payment after 30 years and a rate that remained fixed for only five or ten years (after which it adjusted annually to market levels). Another interesting, and potentially financially hazardous, twist on the hybrid is to have the first few years of payments be interest-only.

7. *Shared appreciation mortgage* (SAM), also called shared equity mortgage (SEM) loan – a lending institution or private investor provides favorable financing in return for a promised share of the possible increase in the underlying property's value. A recent twist on this theme is the "100% financing" loan, in which a third-party investor puts up part or all of the borrower's down-payment, although this lender's reward may be a high interest rate rather than a share of the property's appreciation in value. Or the third party on a 100% loan may be a relative who pledges securities or other valuable property as collateral that the lender can claim in the event of default.

8. *Reverse mortgage* (sometimes called reverse annuity mortgage, RAM) loan – a home owner 62 years of age or older with a lot of equity borrows against the value of the home; repayment is made from the future sale proceeds when the borrower permanently vacates the premises – frequently after the borrower becomes incapacitated, or dies. (Repayment also is triggered if the borrower fails to pay property taxes or homeowner's insurance premiums, or to adequately maintain the property.) The borrower can take the money either in a large lump sum or in monthly installments (the "annuity" aspect). The loan's structure is similar to that of a zero-coupon bond; the lender gives the borrower cash, and then the amount owed grows as interest is applied to that principal – so the initial loan-to-value ratio has to be low enough that the growing amount owed will not be expected ever to exceed the home's value. The borrower gets more money, lump sum or installments, if the loan is taken out at an older age (with less time left until the borrower would be expected to die or otherwise repay). Income tax is not paid on the money the borrower receives, because that money represents loan principal that must be repaid. The reverse mortgage has become sufficiently popular that there is now a National Reverse Mortgage Lenders Association trade group.

With a reverse mortgage loan, the borrower's income and credit history are not an issue, since the borrower makes no direct repayments (reverse mortgage loans are nonrecourse; if the home's ultimate sale price is not enough to repay the loan the lender is made whole through a government guarantee rather than through value from other assets held by the borrower or family members). But closing costs and other fees often are high; a news account stated that a couple aged 65 getting a $250,000 reverse mortgage loan on a $500,000 house could face costs of up to $18,000.[2] And as suggested above, borrowers must budget to pay property taxes, insurance premiums, and maintenance costs. Controversy has arisen over reverse mortgage loans having been heavily promoted to low-income older people, who in some cases failed to meet the required expenses and ended up losing their homes to foreclosure as fees mounted. So a federal law requires seniors who obtain reverse mortgage loans to get financial counseling first.

Most reverse mortgage loans come in the form of FHA-backed home equity conversion mortgages (HECMs), for which there are no borrower income limits (the amount that can be borrowed follows the standard FHA limit) and no restrictions on how the borrowed money can be used. These loans typically carry variable interest rates. The borrower pays 2% of the principal borrowed as an origination fee, and annual insurance premiums of a percentage of the principal owed. Because these premiums, along with the interest owed each month, normally are added to the loan principal, HECMs typically have negative amortization: the outstanding principal increases over time, and thus so do the interest dollars applied and the loan insurance costs. It seems unlikely that much value would remain for a typical reverse mortgage loan borrower's heirs when the home ultimately is sold (especially since a non-borrower surviving spouse generally is allowed to keep living in the home if the borrower dies).

Finally, while getting a reverse mortgage loan was long seen largely as a last resort for cash-strapped seniors to get needed money, some financial advisors now encourage clients to tap their homes' equity through reverse mortgage loans at the earliest possible date – to diversify their investments with borrowed money that never has to be repaid.

9. *Price level adjusted mortgage* (PLAM) – a recent development in loan repayment theory, not yet used in the U.S. (has been used in some countries experiencing high inflation; its structure is similar to that of U.S. government Treasury Inflation Protected Securities). The idea is to *hold the annual interest rate constant* at about *3%,* and *increase the principal owed* on a fully-amortizing loan to directly reflect inflation after-the-fact, instead of the ARM idea of indirectly adjusting for inflation by raising the interest rate to reflect anticipated inflation before-the-fact.

For example, you borrow $101,948.66 for 30 years at 3% APR interest (monthly payment of $429.82). At the end of year 1 you still owe $100,000 under the original plan, but there has been 6% inflation. The balance owed is increased to $106,000, and year 2's payments will be based on a 29-year amortization of $106,000 at 3% interest (the new $456.43 payment is about 6% higher than the old payment, and should be equally affordable – in real dollar terms – if borrower income has matched inflation). Later year payments are computed in a manner similar to that of the ARM. PLAM amortization is *negative* over much of the loan's life in *nominal* terms if there is inflation, but *positive* in *real* terms (dollars owed keep rising even with payments made as scheduled, but the buying power of the money still owed keeps falling, and any remaining principal is repaid as part of the last scheduled payment).

10. *Buydown* – a builder, seller, or buyer (or even an employer that transfers someone who wants to buy a home) pays a lump sum at the time a loan is originated, in return for lower payments during part or all of the loan's life. The "3 – 2 – 1 buydown" plan, mentioned in a 2022 news account, has a rate that rises by 1% (100 basis points) each year for three years before leveling off for the remainder of the amortization period.[3] The National Association of Home Builders reported that 62% of builders were offering buydowns and other incentives to buyers amid the low inventory/high interest rate environment of early 2024.[4]

B. A second way to classify mortgage loans is by the *property included as security*

1. *Package Mortgage* – some personal property may be included (though we know that including the value of personal property in a loan otherwise secured by real estate can be problematic).

2. *Blanket or "Cross-Collateral" Mortgage* – more than one property serves as security for the same loan. One account states that a lender offering this type of loan generally will lend up to 70% of the properties' total appraised values. So someone with little cash but owing no money on a $300,000 home, and wanting to buy a new $600,000 residence, could borrow 70% of ($300,000 + $600,000) = $630,000 – and pay all cash for the new home, without having to sell the old one first.[5] Of course the buyer would have to pay interest on a large amount of debt for some period, but having access to cash can sometimes provide valuable flexibility and bargaining power in a purchase.

3. *Open-end Mortgage* – the borrower can borrow added funds, up to the original total, as the remaining principal balance is reduced through the payment stream (this plan exhibits some features of a line of credit).

4. *Purchase Money Mortgage* – the home seller lends the buyer the money to buy (by not requiring full payment at the closing, and instead collecting payments over time), either because the buyer can not borrow elsewhere or because the seller is in the real estate business and wants favorable "installment sale" treatment. (The term purchase money mortgage sometimes is used more broadly to mean a loan used for buying a primary residence, rather than one that refinances a previous loan, often done for the purpose of getting a lower interest rate.)

5. *Senior (first) vs. Junior (second) Mortgage* – more than one loan can be extended based on a particular property's value. But if the borrower is unable to make payments on all such loans, the first lender's claim must be satisfied in full before the lender holding a second, third, *etc*. lien is paid anything. In a recent innovation called the *piggyback* loan, a lender provides both a first mortgage loan (which is sold in the secondary market) for 80% of the home's value and a second mortgage loan (which the lender keeps in its own portfolio) for 15% of the home's value, so the borrower needs only a 5% down-payment (a December 11, 2022 *Chicago Tribune* article stated that the average down-payment for first-time home buyers had fallen to 6%). This arrangement is said to offer the borrower a sometimes-attractive alternative to buying private mortgage insurance.

A second mortgage may also be in the form of a "home equity loan," with the borrower getting additional money based on the difference between the home's value and the amount still owed on the first mortgage loan. A home equity loan could involve a lump sum lent at a fixed interest rate, or could come in the form of a home equity line of credit (HELOC), with the borrower possibly getting no money at the loan's closing, but getting the right to borrow later, perhaps under a credit card arrangement, up to a pre-determined amount (again, based on the difference between the home's value and the amount still owed on existing mortgage loans) and with a variable interest rate paid. During the early 2020 pandemic, lenders cut back considerably on making home equity loans because of concerns over the stability of home values and borrowers' incomes, particularly in light of the subordinated position of second mortgage loans.

6. *Wrap-around Mortgage* – the borrower obtains a second mortgage loan, then makes one monthly payment to the second mortgage lender, who in turn makes the needed payment on the first mortgage loan. (This arrangement assures the second lender that payments on the senior loan are being received.)

7. *Construction loan* – this type of loan finances a new building during construction. The growing value of the improvements serves as collateral (on loans for constructing single-family homes the lender typically makes payments to the builder in three stages: when construction is approximately ⅓, ⅔, and 100% completed). The long-term financing that begins after the construction is finished is a "takeout" (or permanent) loan.

8. *Pledged Asset Loan* (PAL): A home buyer can borrow 100% of the home's value if the borrower or a friendly party pledges a CD (if a bank arranges the loan), or stocks/bonds/mutual fund shares (if a securities firm arranges the loan), as additional collateral. Typically, securities pledged as assets must be worth 30% or so of the home's purchase price (to allow for some value decline without bringing the loan-to-value [L/V] ratio above 80%), and the

agreement may call for added securities to be pledged if those already pledged fall substantially in value. PALs are often structured as 5/1 two-step loans. They allow the borrower to keep financial assets without having to pay the private mortgage insurance (PMI) premiums typically required on high L/V loans. In one variation, the borrower pays only interest for the first 5 – 7 years, and then the payments rise high enough to amortize all principal over the loan's remaining life. The lender may require a higher down payment and charge higher origination fees.

C. A third way to classify mortgage loans is by *government guarantee or insurance*

1. FHA-insured
2. VA-guaranteed (some other federal agencies mentioned in our Topic 11 outline also offer guarantees)
3. Conventional (there is no government involvement)

II. Computing FRM Loan Interest Paid Over a Specified Time Period

To determine the total amount of interest contained in an FRM's payments over a specified time interval, we need to remember two key ideas discussed earlier. First is that the principal still owed on a loan at any date is the PV of the stream of remaining payments (discounted at the contract interest rate). Second is that the portion of a payment that is not owed as interest reduces principal, and conversely the portion that does not reduce principal is interest. Think of the loan discussed earlier, with $100,000 in principal to be repaid with equal monthly payments for 360 months and a 6% APR contract interest rate = monthly periodic rate r of .06 ÷ 12 = .005. The first step is to compute the monthly payment – but here we already know the payment, from repeatedly computing it earlier. How much of that first month's $599.55 payment consists of interest? The interest component of a first period's loan payment is always easy to compute; here in month 1 the borrower owes .5% interest on the full $100,000 original balance for interest of .005 x $100,000 = $500. Because the total payment is $599.55 the difference of $599.55 – $500 interest = $99.55 in principal repaid during month 1 (so $99,900.45 in principal remains owed at the end of month 1 = start of month 2). These values all appear on the month 1 line in the grid shown earlier on page 7, and reproduced below. That grid also shows, in a month-by-month "brute force" breakdown, that total interest paid in year 1 is a sizable $5,966.59 while total interest paid during final year 30, when little principal remains owed for the borrower to have to pay interest on, is a much smaller $228.47. Let's compute those values with formulas rather than brute force.

Total interest paid over the first year (or any time period that starts on the loan's origination date) for a loan with monthly payments is not too cumbersome to compute (but it is not just 12 times the month-1 interest!!), because the amount owed at the start of the interval is simply the amount of principal that was borrowed. For this loan the entire $100,000 in principal is owed at the start of year 1, while at the end of year 1, when 29 years = 348 months of payments remain to be made, the amount of principal still owed (the PV of the stream of remaining payments) is

$$\$599.55\left(\frac{1-\left(\frac{1}{1.005}\right)^{348}}{.005}\right) = \$599.55\ (164.743394) = \$98{,}771.99$$

(as shown at the end of month 12 in the grid below). If that portion of the $100,000 borrowed remains owed, then the difference of $100,000 – $98,771.99 = $1,228.01 is principal that has been retired, and the rest of what was paid during year 1 is interest:

Total payments during year 1: 12 x $599.55	$ 7,194.60
Minus principal portion	1,228.01
Equals interest paid during year 1	$ 5,966.59

(as also seen in the grid). Computing total interest paid over the final year (or any interval that ends with the last payment date) for a loan with monthly payments is similarly facilitated by the fact that principal owed at the end of the amortization period already is known (it is $0). At the start of year 30, when just 12 months of payments remain, principal still owed is

$$\$599.55\left(\frac{1-\left(\frac{1}{1.005}\right)^{12}}{.005}\right) = \$599.55\ (11.618932) = \$6{,}966.14$$

(as shown at the start of month 349 in the grid). If principal still owed is $6,966.14 when the year opens and $0 when the year ends, then the difference (the entire $6,966.14) constitutes principal paid during year 30. The rest of what was paid during year 30 is interest (as seen in the grid below, which as noted was shown initially on page 7):

Total payments during year 30: 12 x $599.55	$ 7,194.60
Minus principal portion	6,966.14
Equals interest paid during year 30	$ 228.47

It is important to genuinely understand how interest is paid, and remaining principal owed declines, on a loan over a specific time interval. Hitting the AMORT key and ↓ ↓ ↓ on the Texas Instruments BA II Plus financial calculator, even though we walk through it in our homework solutions, shows no understanding of that process (any more than entering numbers into a loan payment app on some web site demonstrates knowledge of how payments are computed). ***No credit will be awarded on an exam question relating to interest or principal paid over some part of a loan amortization period unless enough steps are shown for the grader to be confident that the student understands the process; using the AMORT ↓ ↓ ↓ keys earns no points.***

Month	(A) Principal Owed at Start of Month	(B) 0.5% Interest on That Principal	(C) Total Owed by End of Month (A + B)	(D) End of Month Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Month (C – D = A – E)
1	$100,000.00	$500.00	$100,500.00	$599.55	$99.55	$99,900.45
2	$99,900.45	$499.50	$100,399.95	$599.55	$100.05	$99,800.40
3	$99,800.40	$499.00	$100,299.40	$599.55	$100.55	$99,699.85
4	$99,699.85	$498.50	$100,198.35	$599.55	$101.05	$99,598.80
5	$99,598.80	$497.99	$100,096.80	$599.55	$101.56	$99,497.24
6	$99,497.24	$497.49	$99,994.73	$599.55	$102.06	$99,395.18
7	$99,395.18	$496.98	$99,892.16	$599.55	$102.57	$99,292.61
8	$99,292.61	$496.46	$99,789.07	$599.55	$103.09	$99,189.52
9	$99,189.52	$495.95	$99,685.47	$599.55	$103.60	$99,085.92
10	$99,085.92	$495.43	$99,581.35	$599.55	$104.12	$98,981.79
11	$98,981.79	$494.91	$99,476.70	$599.55	$104.64	$98,877.15
12	$98,877.15	$494.39	$99,371.54	$599.55	$105.16	$98,771.99
Year 1 Totals →		$5,966.59		$7,194.61	$1,228.01	
349	$6,966.14	$34.83	$7,000.97	$599.55	$564.72	$6,401.42
350	$6,401.42	$32.01	$6,433.42	$599.55	$567.54	$5,833.87
351	$5,833.87	$29.17	$5,863.04	$599.55	$570.38	$5,263.49
352	$5,263.49	$26.32	$5,289.81	$599.55	$573.23	$4,690.26
353	$4,690.26	$23.45	$4,713.71	$599.55	$576.10	$4,114.16
354	$4,114.16	$20.57	$4,134.73	$599.55	$578.98	$3,535.18
355	$3,535.18	$17.68	$3,552.86	$599.55	$581.87	$2,953.31
365	$2,953.31	$14.77	$2,968.07	$599.55	$584.78	$2,368.52
357	$2,368.52	$11.84	$2,380.36	$599.55	$587.71	$1,780.81
358	$1,780.81	$8.90	$1,789.72	$599.55	$590.65	$1,190.17
359	$1,190.17	$5.95	$1,196.12	$599.55	$593.60	$596.57
360	$596.57	$2.98	$599.55	$599.55	$596.57	$0.00
Year 30 Totals →		$228.47		$7,194.61	$6,966.14	
30-Year Totals →		$115,838.19		$215,838.19	$100,000.00	

Among reasons for wanting to know the amount of interest paid over an interval is that interest on a home mortgage loan during a given year is a potential deduction for reducing the income on which a household must pay federal income tax (though 2017 income tax law changes reduced this benefit relative to what it had previously been). Now let's compute the interest portion of the total payments made over years 9 to 14 for the loan described above. Again we can coast a bit by not having to first compute the monthly payment, but now must compute remaining principal owed at two separate dates. At the start of year 9 (after eight years = 96 months have passed), with 22 years or 360 – 96 = 264 months of payments to go, principal still outstanding is the PV of the 264 remaining payments:

$$\$599.55\left(\frac{1-\left(\frac{1}{1.005}\right)^{264}}{.005}\right) = \$599.55\ (146.396927) = \$87{,}772.35$$

At the end of year 14, with 16 years (192 months) of payments remaining, principal still owed is the PV of the 192 remaining payments:

$$\$599.55\left(\frac{1-\left(\frac{1}{1.005}\right)^{192}}{.005}\right) = \$599.55\ (123.238025) = \$73{,}887.42$$

The difference in the principal balances owed at the beginning and end of the six-year interval is $87,772.35 – $73,887.42 = $13,884.93. The rest of what was paid during those six years = 72 months is interest:

Total payments during years 9 – 14: 72 x $599.55	$43,167.64
Minus principal portion	13,884.93
Equals interest paid during years 9 – 14	$29,282.71

III. Time Needed to Repay a Loan (solving for *n* in the FRM equation)

Recall that residential mortgage loan borrowers typically are permitted (by state laws and secondary mortgage market practices) to pay back principal early, in small or large amounts, with no direct penalty (call premium) charged. How long would it take to fully repay our \$100,000 mortgage loan, with the 360-month amortization plan and 6% APR = .005 monthly *r* interest rate, if instead of paying \$599.55 each month the borrower paid a higher \$725? Or a lower \$500? The structure of our analysis is just as it has been above, except now we have a PV of annuity situation with *n* as the unknown to solve for. The \$725 monthly payment possibility yields

$$\$725\left(\frac{1-\left(\frac{1}{1.005}\right)^n}{.005}\right) = \$100{,}000$$

$$\left(\frac{1-\left(\frac{1}{1.005}\right)^n}{.005}\right) = 137.931034$$

$$1-\left(\frac{1}{1.005}\right)^n = .689655$$

$$\left(\frac{1}{1.005}\right)^n = .310345 \;\Rightarrow\; (.995025)^n = .310345$$

$$\ln\,[(.995025)^n] = \ln .310345 \;\Rightarrow\; n \ln(.995025) = \ln .310345$$

$$n\,(-.004988) = -1.170071$$

n = <u>234.5988</u>, not quite 235 months (just under 20 years)

$$\text{Double check as } \$725\left(\frac{1-\left(\frac{1}{1.005}\right)^{234.5988}}{.005}\right) = \$100{,}000 \;\checkmark$$

So paying about an extra \$125 per month reduces the repayment period by approximately ten years. For the \$500 monthly payment scenario we solve for *n* as

$$\$500\left(\frac{1-\left(\frac{1}{1.005}\right)^n}{.005}\right) = \$100{,}000$$

$$\left(\frac{1-\left(\frac{1}{1.005}\right)^n}{.005}\right) = 200$$

$$1-\left(\frac{1}{1.005}\right)^n = 1 \;\Rightarrow\; \left(\frac{1}{1.005}\right)^n = 0$$

This result is not plausible; a quotient can be 0 only if the numerator is 0 (and here it is not) – something is wrong. The problem is that the month 1 payment interest portion should be .005 x \$100,000 = \$500. So if exactly \$500 is paid at the end of month 1, the entire \$100,000 principal is still owed going into month 2. And this situation repeats itself every month; each payment covers just interest owed, so no principal is ever repaid. This example involves an interest-only loan that would go on perpetually, or else require a 100% balloon payment at some future date.

IV. Computing the Effective Cost of Borrowing (solving for *r* in the FRM equation)

A percentage return (cost) for a lender (borrower) is determined by what is received (given up) in one or more subsequent periods relative to what is given up (received) initially. The percentage cost or return on a mortgage loan is not merely the stated annual interest rate. First, rates generally are quoted without recognition of intra-year compounding (APR's, not EAR's). Second, the lender might quote a low interest rate but then require the borrower to pay fees in order to get approval; such an arrangement might not represent a very good deal for the borrower. Philosophically it could be like an offer of "free" meals for those who pay to stay at a terribly overpriced resort.

If no extra fees are charged, then the stated interest rate does indeed express the borrower's actual cost of borrowing and lender's rate of return in APR (but not EAR) terms. Recall that the APR is the periodic interest cost multiplied by the number of periods in a year, while the EAR is $[(1 + \text{periodic interest cost})^{\text{number of periods in a year}} - 1]$. Consider a \$100,000 loan with a 6% stated annual interest rate, end-of-month payments over a 30-year amortization period, and no extra fees charged. The monthly payment, computed as in so many earlier examples, is \$599.55. If every payment is made as scheduled, what is the effective borrowing cost? (And unless a middle party is paid for bringing the borrower and lender together, the lender's effective annual rate of return is equal to the borrower's effective annual percentage borrowing cost.) We use trial and error to solve for *r* in the equation

$$\$599.55\left(\frac{1-\left(\frac{1}{1+r}\right)^{360}}{r}\right) = \$100{,}000$$

The answer is .005 or .5%, which is a monthly rate since \$599.55 is a monthly payment and 360 is the number of monthly periods. Of course while we work with rate measures that correspond to the timing of the payments, we talk about rate measures in annual terms, APRs and EARs. The APR corresponding to this monthly borrowing cost is .005 x 12 = .06, the 6% stated interest rate (there are no added fees in this example). But even here the true effective opportunity cost, taking intra-year compounding into account, is a higher $(1.005)^{12} - 1 = 6.1678\%$ EAR.

Now consider a case with other required outlays. Extra mortgage loan fees often are applied in the form of "points." Do not confuse this "discount" point (1% of the loan amount, charged to increase the lender's return) with a basis point or "bp" or "bip" (1⁄100 of 1%, a measure that helps us clarify interest rate discussions and specify rate changes in the small units in which they usually occur). Discount points a lender charges reduce the net amount of money the borrower receives at closing, but do not reduce the payments relative to a no-point loan example with the same principal and stated interest rate. Think in terms of this discussion. Lender: "You are borrowing \$100,000 at a 6% stated annual interest rate, and thus will pay us \$599.55 at the end of each month for 30 years, agreed?" Borrower: "Yes." Lender: "Great, now we charge 2 points, so here is your \$100,000 minus 2% of \$100,000, or \$98,000." (Points might be compared to a cover charge levied at a night club – you hand over a fee to get in, but if you still have to pay the full menu prices for your food and drinks the night out costs more than it might initially appear.)

Looking at what the borrower gives up relative to what is received, we compute

$$\$599.55\left(\frac{1-\left(\frac{1}{1+r}\right)^{360}}{r}\right) = \$98{,}000;$$

trial and error yields a monthly rate of $r = .0051579$. The lender quotes a non-compounded percentage as the annual rate, but with the points paid the borrower's yearly cost, even in non-compounded APR terms, is .001579 x 12 = 6.1895%, and with compounding we find an even higher EAR cost of $(1.0051579)^{12} - 1 = 6.3681\%$.

Charging points may look on the surface to be a bit sneaky, but it is not sinister. First, the financial impact of the points is made clear to the borrower ("Regulation Z" under the TILA/RESPA reporting discussed in Topic 8 with closings) requires the lender to report the 6.1895% APR, though not the 6.3681% EAR, to the borrower). Second, lenders sometimes can offer a range of choices based on rate/point combinations, through which the borrower "buys down" a rate (as with buydown arrangements discussed earlier) by paying a chunk of money – discount points – up front, with each point paid reducing the annual interest rate by perhaps .25%. Because these points are a one-time charge for the benefit of a lower rate, a choice with higher points and a lower rate might be advantageous to a borrower who intends to keep the loan intact for its entire 25- or 30-year amortization period, while a borrower planning to keep the loan for only a short time typically would be better off eating a higher rate for a few years and avoiding the large up-front payment of discount points.

V. Computing Payment Without Knowing Principal Initially Borrowed (or Even Original Amortization Period)

Knowing that principal owed on a loan at any time is just the PV of the stream of remaining payments also lets us compute the regular monthly payment on an FRM loan at any date during the scheduled repayment period without knowing what originally was borrowed – as long as we know the interest rate, what still is owed, and how many scheduled payments remain. On the day the borrower gets a \$100,000 loan with a 6% APR interest rate and 360-month amortization, the amount owed is \$100,000 and the number of remaining payments is 360, so the monthly payment is

$$\$100{,}000 \div \left(\frac{1-\left(\frac{1}{1.005}\right)^{360}}{.005}\right) = \$100{,}000\left(\frac{.005}{1-\left(\frac{1}{1.005}\right)^{360}}\right) = \$100{,}000\ (.005996) = \$599.55,$$

as repeatedly seen earlier. But what if instead we were told that the borrower owes \$98,877.15 on this 6.0% APR interest rate loan when 360 – 11 = 349 months of scheduled monthly payments remain (after 11 months have passed/going into month 12, as shown below in part of the grid that was first seen earlier on page 7)?

Month	(A) Principal Owed at Start of Month	(B) 0.5% Interest on That Principal	(C) Total Owed by End of Month (A + B)	(D) End of Month Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Month (C – D = A – E)
1	$100,000.00	$500.00	$100,500.00	$599.55	$99.55	$99,900.45
2	$99,900.45	$499.50	$100,399.95	$599.55	$100.05	$99,800.40
3	$99,800.40	$499.00	$100,299.40	$599.55	$100.55	$99,699.85
4	$99,699.85	$498.50	$100,198.35	$599.55	$101.05	$99,598.80
5	$99,598.80	$497.99	$100,096.80	$599.55	$101.56	$99,497.24
6	$99,497.24	$497.49	$99,994.73	$599.55	$102.06	$99,395.18
7	$99,395.18	$496.98	$99,892.16	$599.55	$102.57	$99,292.61
8	$99,292.61	$496.46	$99,789.07	$599.55	$103.09	$99,189.52
9	$99,189.52	$495.95	$99,685.47	$599.55	$103.60	$99,085.92
10	$99,085.92	$495.43	$99,581.35	$599.55	$104.12	$98,981.79
11	$98,981.79	$494.91	$99,476.70	$599.55	$104.64	$98,877.15
12	$98,877.15	$494.39	$99,371.54	$599.55	$105.16	$98,771.99
Year 1 Totals →		$5,966.59		$7,194.61	$1,228.01	

The exact same unchanging monthly payment could be computed as

$$\$98{,}877.15 \div \left(\frac{1-\left(\frac{1}{1.005}\right)^{349}}{.005}\right) = \$98{,}877.15\left(\frac{.005}{1-\left(\frac{1}{1.005}\right)^{349}}\right) = \$98{,}877.15\,(.006064) = \$599.55;$$

the $100,000 originally borrowed and even the 360 originally scheduled payments would not have to be known. Or use a totally different set of numbers: $124,422.39 remains owed on a loan with a 4.8% APR = .4% monthly interest rate and 9 years of scheduled monthly payments remaining (if we know that 9 x 12 = 108 payments remain it does not matter whether the loan is 21 years into a 30-year payment plan or 6 years into a 15-year plan, *etc*., as long as we know that 108 payments remain). The monthly payment can be computed as

$$\$124{,}422.39 \div \left(\frac{1-\left(\frac{1}{1.004}\right)^{108}}{.004}\right) = \$124{,}422.39\left(\frac{.004}{1-\left(\frac{1}{1.004}\right)^{108}}\right) = \$124{,}422.39\,(.011421) = \$1{,}421.03.$$

(Let's say that it started out as a $248,000 loan to be repaid over 25 years = 300 months:

$$\$1{,}421.03\left(\frac{1-\left(\frac{1}{1.004}\right)^{300}}{.004}\right) = \$1{,}421.03\,(174.520995) = \$248{,}000,$$

but we do not have to know those figures to compute the monthly payment if we know the periodic interest rate, the amount of principal still owed, and the number of initially scheduled payments that remain.)

VI. Loan with Lower Interest Rate Amortizes More Rapidly than Otherwise Similar Loan with Higher Rate

Think of two individuals who, on the same day, borrow the same amount of principal, with equal end-of-month payments to be made over identical terms to maturity (30 years is common, but it could be any number of years). But one borrower is seen as less credit-worthy, perhaps because of a late credit card payment in recent months, and thus is charged a higher interest rate. It might seem surprising, but even though the higher interest rate loan carries bigger monthly payments, the high-rate borrower owes more in remaining principal at any date during the payment term than does the borrower charged the lower interest rate – the bigger payments do not cause principal to amortize more rapidly, or even as rapidly, relative to the loan with the lower interest rate. To see why this result holds, we might first recall that the amount owed on a loan at any time is the PV of the stream of remaining payments:

$$\text{Principal originally borrowed}\left(\frac{r}{1-\left(\frac{1}{1+r}\right)^{n}}\right)\left(\frac{1-\left(\frac{1}{1+r}\right)^{n-m}}{r}\right)$$

Compute monthly payment — *Discount payment stream to PV*

(here n is the longer original amortization period, m the shorter number of periods that has passed already: 120 months into a 360-month amortization period there are $n – m$ = 360 – 120 = 240 payments remaining).

Note that when we multiply $\left(\frac{r}{1-\left(\frac{1}{1+r}\right)^{n}}\right)$ by $\left(\frac{1-\left(\frac{1}{1+r}\right)^{n-m}}{r}\right)$ the r's cancel, and we can restate the product as $\left(\frac{1-\left(\frac{1}{1+r}\right)^{n-m}}{1-\left(\frac{1}{1+r}\right)^{n}}\right)$. Multiply it by $\frac{(1+r)^n}{(1+r)^n}$ (multiplying by 1, thus not changing the value) gives us $\left(\frac{(1+r)^n-(1+r)^m}{(1+r)^n-1}\right)$.

If we subtract 1 and add 1 (subtract –1) in the numerator we do not change the value but can restate the relationship as $\left(\frac{((1+r)^n-1)-((1+r)^m-1)}{(1+r)^n-1}\right)$. Then distribute the denominator over the numerator's terms: $\left[\left(\frac{(1+r)^n-1}{(1+r)^n-1}\right)-\left(\frac{(1+r)^m-1}{(1+r)^n-1}\right)\right]$, which equals $\left[1-\left(\frac{(1+r)^m-1}{(1+r)^n-1}\right)\right]$.[6] If $\left[1-\left(\frac{(1+r)^m-1}{(1+r)^n-1}\right)\right]$ is the proportion of a loan that *has not* been repaid and thus still is owed, then $\left(\frac{(1+r)^m-1}{(1+r)^n-1}\right)$ is the proportion that *has* been repaid. Let's refer to *m* as the short period and *n* as the long period:

$$\left(\frac{(1+r)^{short\ period}-1}{(1+r)^{long\ period}-1}\right)$$

In this fraction representing the proportion of a loan that has been repaid, the numerator gets bigger as the interest rate rises. But the denominator grows by an even larger proportion (a bigger value is taken to an even higher power). So for any date in the repayment period, as the interest rate increases the denominator increases by a higher multiple than the numerator does, indicating that a smaller proportion of the original principal has been repaid.

For example, 211 months into a 300-month amortization period, *A* paying interest at a rate of .32% per month has repaid $\left(\frac{(1.0032)^{211}-1}{(1.0032)^{300}-1}\right)=\left(\frac{.962309}{1.607697}\right)$ = 59.86% of the borrowed principal, while *B* paying .42% per month in interest has repaid only $\left(\frac{(1.0042)^{211}-1}{(1.0042)^{300}-1}\right)=\left(\frac{1.421396}{2.516132}\right)$ = 56.49%. The higher-rate loan's numerator is 1.421396 ÷ .962309 = 1.477 times the lower-rate loan's, but the higher-rate loan's denominator is an even larger 2.516132 ÷ 1.607697 = 1.565 multiple of its lower-rate counterpart. (Check to see that 17 months into the 300-month amortization period *A* has repaid 3.47% of principal, while *B* has repaid only 2.94%.)

The impact of the denominator's higher exponent causes the higher-rate loan always to lag behind in the proportion of principal that has been repaid. If two loans have equal initially borrowed principal and equal amortization schedules, but different interest rates, only at the time of origination when each borrower owes the amount originally borrowed, and after the final payment when each owes $0, are the two loans' owed principal balances the same.

VII. Net Present Value of Refinancing an FRM Loan

Let's say that three years ago someone got a $100,000 FRM loan with a 6% APR interest rate (.005 monthly *r*) and 360-month amortization period, as described in earlier examples. Today she learns that the interest rate quoted on new long-term FRM loans is a lower 5.1% APR (.051 ÷ 12 = .00425 monthly *r*). With 27 years of payments remaining on her current loan, does it make economic sense for her to refinance to a new 27-year loan? (While loan maturities traditionally were 30 or 25 years or other "round" numbers, Quicken and some other lenders in recent years have offered loans with flexible maturities to meet borrower preferences – which we in FIL 260 appreciate, because our computations in an example like this one are much simpler if the replacement loan's term is equal to the remaining life on the original loan.) Note that the reduction in payments from having a lower interest rate would create a savings stream for the borrower, equivalent to receiving a series of cash inflows. And there is no direct penalty for repaying the original loan early with proceeds from a new loan – however, our borrower would incur closing and other transaction costs in connection with getting the new, lower-rate replacement loan.[7] She should consider refinancing if the Net Present Value of refinancing (PV of cash inflows – PV of cash outflows) is positive.

Step 1: compute the current monthly loan payment, based on a 360-month life and a .005 monthly periodic interest rate; as seen numerous times above:

$$\text{FRM Payment}\left(\frac{1-\left(\frac{1}{1.005}\right)^{360}}{.005}\right)=\text{FRM Payment }(166.791614)=\$100{,}000$$

FRM Payment = $100,000 ÷ (166.791614) = $599.55 OR
FRM Payment = $100,000 (.005996) = $599.55

Step 2: compute the monthly loan payment that would be made on a new loan, based on a 27-year = 324-month life and monthly periodic interest rate *r* of .051 ÷ 12 = .00425. Principal that remains unpaid/still owed 36 months into the original loan's 360-month life (*i.e.*, with 360 – 36 = 324 months remaining) is, as seen so many times before, just the present value of the stream of remaining scheduled payments based on the original agreed interest rate:

$$\$599.55\left(\frac{1-\left(\frac{1}{1.005}\right)^{324}}{.005}\right)=\$599.55\ (160.260172)=\$96{,}084.07 \text{ ("goodbye, old loan")}$$

Then the monthly payment on a new \$96,084.07, 324-month, 5.1% ÷ 12 = .00425 monthly periodic rate *r* loan is:

$$\text{FRM Payment}\left(\frac{1-\left(\frac{1}{1.00425}\right)^{324}}{.00425}\right) = \$96{,}084.07 \text{ ("hello, new loan")}$$

FRM Payment (175.747818) = \$96,084.07

So FRM Payment = \$96,084.07 ÷ 175.747818 = \$546.72 OR
FRM Payment = \$96,084.07 (.005690) = \$<u>546.72</u>

[Notice how *Steps 1 and 2 here are identical in structure to Steps 1 and 2 with the "hybrid" or "two-step" loan* discussed earlier: compute the initial level payment, then determine how much still is owed when the interest rate changes due to a pre-scheduled change or the borrower's decision to refinance, and then compute the new payment based on the principal still owed, number of payment periods that remain in the loan's initial term, and new interest rate. The difference is that the two-step loan analysis concludes with those two steps; we might simply take note of how the new payment differs from the initial level payment – whereas in refinancing analysis we must compute the difference in the two payments and then discount that expected savings stream to a present value, as follows.]

Step 3: refinancing after 3 years with a new 27-year loan creates a monthly payment reduction for the borrower of \$599.55 – \$546.72 = \$<u>52.83</u>, to be realized every month for 27 years, or 324 months. If we discount the savings stream to a PV using the current mortgage lending interest rate as the discount rate, we find the value to be

$$\$52.83\left(\frac{1-\left(\frac{1}{1.00425}\right)^{324}}{.00425}\right) = \$52.83\ (175.747818) = \$\underline{9{,}285.63}$$

Step 4: the borrower's cost of refinancing consists of the closing and other costs related to getting the new loan; let's say that total is \$5,000 (and since the outlay would be made when the loan is received it already is a present value). The NPV of refinancing is PV of savings stream minus PV of refinancing costs = \$9,285.63 – \$5,000 = \$<u>4,285.63</u>. NPV is positive, so refinancing creates wealth for the borrower, in an amount a bit over \$4,000.

[A student asked why our NPV computation differs from what was done in another Finance course. It turned out the only difference was that in the other course the cash flows were shown in chronological order (we subtract the CF_0 cost of getting the new loan as the last step rather than the first), and the cash flows expected in periods 1 – n were given (whereas we had to compute the \$52.83 expected monthly payment savings). Restate our computation as:

$$\text{NPV} = \sum_0^n \text{CF}_x\left(\frac{1}{1+r}\right)^x = \text{CF}_0\left(\frac{1}{1+r}\right)^0 + \text{CF}_1\left(\frac{1}{1+r}\right)^1 + \text{CF}_2\left(\frac{1}{1+r}\right)^2 + \cdots + \text{CF}_{n-1}\left(\frac{1}{1+r}\right)^{n-1} + \text{CF}_n\left(\frac{1}{1+r}\right)^n$$

$$= -\$5{,}000\left(\frac{1}{1.00425}\right)^0 + \$52.83\left(\frac{1}{1.00425}\right)^1 + \$52.83\left(\frac{1}{1.00425}\right)^2 + \cdots + \$52.83\left(\frac{1}{1.00425}\right)^{324}$$

$$= -\$5{,}000 + \$52.83\left(\frac{1-\left(\frac{1}{1.00425}\right)^{324}}{.00425}\right) = -\$5{,}000 + \$9{,}285.63 = \$4{,}285.63.]$$

While textbooks generally state that the rate used in discounting the savings stream to a PV in a loan refinancing analysis should be the mortgage lending interest rate at the refinancing date,[8] which is what we used above, we also could argue that the presumed discount or reinvestment rate should be lower than the mortgage lending interest rate, something closer to a risk-free rate, since getting a lower interest rate through refinancing guarantees the borrower a stream of reduced monthly payments.[9] [The situation is not entirely risk-free for the borrower, because if interest rates available on new loans decline even farther the borrower will have exercised the option to refinance at a suboptimal time; thus, we state above that refinancing should be *considered* if NPV is positive.] The bank balance today that would allow us to withdraw \$52.83 per month for 324 months, based on a low-risk discount rate (let's say 3% APR or .25% monthly *r*), is a higher

$$\$52.83\left(\frac{1-\left(\frac{1}{1.0025}\right)^{324}}{.0025}\right) = \$52.83\ (221.876815) = \$\underline{11{,}722.85}$$

(a fairly certain stream of $52.83 expected inflows is worth more than a less-certain stream of $52.83 expected inflows). Based on this low-risk discount rate the NPV of refinancing is a higher $11,722.85 – $5,000 = $6,722.85. Refinancing appears more attractive when we view the stream of monthly payment savings as a guaranteed benefit.

Now let's add a twist to the analysis. The initial loan's remaining term is 27 years, and if the borrower refinances it will be to a new loan with a 27-year amortization – but she plans to retire and sell the house (and make a balloon payment to repay any remaining principal owed, as required under the alienation clause) in 91 months, whether she keeps the current .5% monthly interest rate loan or refinances to the new available .425% rate. The PV of the stream of reduced monthly payments from refinancing (based on the low-risk .25% monthly discount rate) will be just

$$\$52.83\left(\frac{1-\left(\frac{1}{1.0025}\right)^{91}}{.0025}\right) = \text{TOT}$$

$$\$52.83 \times 81.300565 = \$4{,}295.51,$$

such that the PV of the savings stream seems, at first glance, to be too low to cover the $5,000 refinancing cost.

But there is more to the story. Right now, the borrower owes $96,084.07, which she will repay through a loan with 27 years = 324 months of scheduled payments, either by continuing with the existing 30-year loan that is three years into its amortization period, or by refinancing to a new 27-year loan. If she keeps her existing loan with the .5% monthly r and 324 remaining $599.55 scheduled monthly payments (as computed above), then 91 months from today (36 + 91 = 127 months into the existing loan's amortization), when she retires and sells the house, she will have to pay her current lender a final regular monthly payment plus all principal that remains owed. That amount will be the PV of the stream of 360 – 127 (counting from the origination date) or 324 – 91 (measured from today) = 233 scheduled payments that will then remain on the existing loan:

$$\$559.55\left(\frac{1-\left(\frac{1}{1.005}\right)^{233}}{.005}\right) = \$559.55 \times 137.434113 = \$82{,}398.69$$

If instead she refinances the $96,084.07 currently owed with a new 27-year = 324-month loan that has a .425% monthly interest rate and $546.72 monthly payment (as computed above), then in 91 months, when she sells the house after making her month 91 payment, the amount of principal still owed will be the present value of the stream of 324 – 91 = 233 remaining scheduled payments that will not be made:

$$\$546.72\left(\frac{1-\left(\frac{1}{1.00425}\right)^{233}}{.00425}\right) = \$546.72 \times 147.702559 = \$80{,}751.29$$

So if she refinances today, then in 91 months when she sells the house she will owe the new lender $82,398.69 – $80,751.29 = $1,647.40 less than if she had not refinanced and would have to pay off the bigger amount that would be owed to the original lender. (As we can attest from the lesson of part VI above, a loan with 324 months of payments and a .425% monthly interest rate amortizes more rapidly than an equally sized loan with 324 months of payments and a .5% monthly rate.)

And the present value of a $1,1647.40 benefit that would be received in 91 months, if we use the low-risk discount rate since that smaller obligation would be assured if refinancing were to occur, is

$$\$1{,}647.40\left(\frac{1}{1.0025}\right)^{91} = \$1{,}647.40 \times .796749 = \$1{,}312.57$$

Thus the present value of the benefits of refinancing becomes $4,295.51 + $1,312.57 = $5,608.08. Based on these figures, the NPV of refinancing, when the cost of doing so (already in present value terms) is $5,000 and the PV of the benefits (paying $52.83 less per month for just 91 months, and then settling up in 91 months at a cost $1,647.40 less than with no refinancing) is slightly positive, and thus in theory refinancing creates wealth for the home owner of $5,608.08 – $5,000 = $608.08. (Another way to state the issue is that if the computed NPV is positive then the borrower's option to refinance is in-the-money.)

Refinancing need not be accompanied by a replacement loan with a life equal to the remaining maturity on the initial loan, and the replacement loan's principal need not be simply the amount still owed on the initial loan rolled over to a new lender. (The "new" lender can actually be the "old" lender; our borrower in the example above could approach her current loan's originator about refinancing when interest rates have declined. Perhaps that originator already has sold the note in the secondary mortgage market to Fannie Mae or Freddie Mac or another buyer, and thus does not care that the new loan will produce lower monthly payments. Or the originator may still hold the higher-rate note in its portfolio, and is not thrilled to see the monthly payments it collects decline, but the borrower could as easily refinance with a different lender, so the current loan's originator might as well at least earn the fees the borrower will pay when the new loan is created.)

Keeping the potential replacement loan's term and principal owed equal to the remaining term and remaining principal owed on the existing loan keeps things simple in this example, so we can focus on the main points that concern us in our FIL 260 introductory coverage. The analysis is more complicated if a borrower refinances to a loan with a maturity shorter or longer than the remaining life of the existing loan, or if the replacement loan's initial principal balance is higher (perhaps a "cash-out refinance," in which the mortgagor can borrow more because equity has built up through the repayment of principal or the home's having gone up in value) or lower than what is owed on the loan being replaced. We would tend to look at situations like those in the FIL 360 course.

VIII. A Few Final Interesting Points

A. The net present value of refinancing a mortgage loan to a lower interest rate might well be negative if the new loan's origination cost is high, the interest rate reduction is modest, and/or the borrower does not expect to remain in the house for a long period. Recall that in computing NPV we compare the present value of the savings stream (lower payments) to the project's cost (already in PV terms), to see whether making the change would increase the borrower's wealth (positive NPV). Deductibility of interest payments from adjusted gross income in computing the income on which federal income tax must be paid also can play a role (albeit to a lesser extent than was true before 2018); a loan with a lower interest rate has smaller deductible interest payments, so if the income tax deductibility of home mortgage loan interest has value to the borrower, then a lower-rate loan may not be as cheap as it seems.

B. *It is not always wise to prepay your mortgage loan* in order to reduce the number of interest dollars paid. Because of possible income tax deductibility (likely applying to fewer people now than was the case before late 2017 federal income tax law changes, which went into effect for tax year 2018) and the strong collateral, borrowing against your home's value typically is your cheapest form of available borrowing. You would not want to free up money to pay more on your mortgage loan each month by running up your credit card bills at 19% annual interest. (Do not try to make the decision based on the number of dollars of interest paid over the loan's life – remember that you always pay interest only on the loan's remaining outstanding principal balance. It is interesting to note that annual interest rates charged on 30-Year FRM loans have varied from about 7% in the late 1970s to 18% in the early 1980s to 10% in 1990 to 8% in 2000 to less than 3% in late 2021, to more than 7% again by fall of 2023.)

But do not take the argument too far. People sometimes say something like, "If you owe $100,000 on a 7% APR mortgage loan and you inherit $100,000, you are better off putting the money in a growth stock mutual fund with an expected return of 12% APR than paying off the loan; you can earn a 5% spread." The problem here is that the investor would be crossing risk classes; the mutual fund is a risky instrument, but the mortgage loan is risk-free – in that we know with certainty that the regular payment will be due next month.

Finally, it would not make sense for most people to pay added principal on their mortgage loans if they did not first maintain savings balances of liquid emergency funds, and if they did not make the maximum allowed yearly deposits into tax-favored Individual Retirement Accounts and employer-sponsored 401-k or 403-b plans. For example, someone who generally wants to both save for retirement and pay extra mortgage loan principal can pay back extra loan principal, in large or small amounts, at any time. But someone who misses the opportunity to make a given year's maximum IRA contribution ($7,000 for 2024, can be contributed until April 15 of 2025 but then the window closes forever) can not simply choose to contribute twice as much to the IRA the following year.

C. Your loan's remaining principal balance can be valuable information to have. Under the federal Homeowners Protection Act (HOPA) or "PMI Cancellation Act" passed in 1998, your lender must stop requiring premiums to be paid for private mortgage insurance when your equity reaches 22% (still owe only 78%), based on the original property value and original loan balance. Another possibility for terminating your need to pay for PMI is to get an appraisal showing that your equity has risen to 20% of the purchase price[10] (unless you have been late with payments, or the appraisal is thought to be defective).

D. The more general, observable interest rate that home mortgage loan interest rates tend to relate to is what the U.S. government pays on the 10-year U.S. Treasury Note – with rates paid on less risky T-Notes lower, on a given day, than those paid on riskier home mortgage loans (mortgage lenders face default and prepayment risk that T-Note buyers do not). In late 2022, after a year of substantial mortgage loan interest rate increases amid ongoing inflation concerns, the spread between T-Note and mortgage loan interest rates was the highest it had been in many years. Reasons were thought to be illiquidity risks in mortgage lending as the Federal Reserve stopped buying mortgage-backed securities, and as costs for bank deposits were expected to rise while expected terms on existing mortgage loans were lengthening, since higher interest rates would discourage early repayment through refinancing (the asset/liability maturity mismatch problem that brought down the savings and loan industry a generation earlier).

E. While *prepayment penalties* typically are non-existent with single-family home mortgage loans in the U.S., there can be some exceptions. One is "subprime" loans made to borrowers without strong credit histories. Some state laws also allow prepayment penalties as long as the contract interest rate is not excessively high. A penalty also can be negotiated in return for a lower contract interest rate, a situation that protects a lender from some reinvestment rate risk while benefiting a borrower who is sure that he/she will remain in the home for several years. When penalties exist, they typically apply for 5 – 7 years and are levied as a percentage of the loan's remaining principal balance. (Practices can differ around the globe; British lenders, for example, are more likely to levy penalties when borrowers prepay standard home mortgage loans by refinancing to obtain lower interest rates. There is no absolute rule, and lenders seem to have some case-by-case discretion; one approach some lenders in the U.K. follow is an "overpayment allowance" of up to 10%, but no more, of a loan's principal to be repaid without penalty each year).[11]

F. Also on the international front: according to a late 2022 news article, many countries offer hybrid loans with initial fixed interest rate periods for a few years, as short as two in the U.K., Canada, Australia, Spain, Portugal, and Italy (in New Zealand can be as short as 12 months, like the original 1980s variable rate loans in the U.S.). Many borrowers faced high payment increases when their short-term fixed interest rates reset to long-time highs in late 2022. Canadian borrowers also can obtain loans with payments that stay fixed while the interest rate varies; more of each month's unchanging payment goes to interest (and thus less to amortizing principal) when the market interest rate index moves higher.[12]

G. Leasing sometimes is a useful real estate financing tool for businesses. They can "borrow" 100% of the price of a property, thereby freeing funds for other uses, and sometimes can deduct all of the lease payments from income as business expenses for income tax purposes.

H. In recent periods, lenders (especially Internet-based) have tried some creative lending ideas, such as *portable* loans, in which the borrower pays a slightly higher interest rate and then has an option to apply the remaining balance to a new home purchase one time over the loan's original amortization period.[13] Or loan terms might reflect the borrower's geography (for example, if the borrower lives in a part of the country where fewer borrowers tend to prepay their loans, a lower interest rate might be offered since the lender perceives less in prepayment/reinvestment risk). •

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[1] The most prevalent type of home mortgage loan in England is a hybrid with a fixed interest rate for either two or five years, and then yearly changes. See Mitchell, Josh. "U.K. Market Turmoil Ripples Into Home Loans." *The Wall Street Journal*, September 27, 2022.
[2] Ioannou, Lori. "The New Math of Reverse Mortgages for Retirees." *The Wall Street Journal*, June 3, 2022.
[3] Heeb, Gina and Friedman, Nicole. "Mortgage Buydowns Are Making a Comeback." *The Wall Street Journal*, December 18, 2022.
[4] Friedman, Nicole. "Home Buyers Are Finding a Wider Inventory." *The Wall Street Journal*, February 6, 2024, A2.
[5] Friedman, Robyn A. "Three Ways to Pay for a Home if a Traditional Mortgage Isn't on the Table." *The Wall Street Journal*, October 13, 2023, M4.
[6] This algebra is shown in Trefzger, Joseph W. and Cannaday, Roger E. "Key Lessons in Fixed Rate Mortgage Loan Payment Mechanics." *Journal of Economics & Finance Education*, forthcoming.
[7] A late 2023 news story stated that some loans provide the right to refinance at a future date with no out-of-pocket costs. See Dagher, Veronica. "Buyers Still Have Some Room to Negotiate." *The Wall Street Journal*, October 10, 2023, A9. In a competitive market the borrower would have to pay for an option like this through higher outlays in another part of the transaction, such as a higher interest rate, or higher up-front fees, than a loan without such an option would carry.
[8] See, for example, Clauretie. Terrence M. and Sirmans, G. Stacy. *Real Estate Finance: Theory and Practice, 7th Edition*, p. 133. OnCourse Learning, 2014.
[9] See Trefzger, Joseph W. and Cannaday, Roger E. "NPV of Refinancing: Rethinking the Borrower's Discount Rate." *Journal of Real Estate Practice and Education*, Vol. 18 No. 1, 2015, 1 – 33.
[10] Ostrowski, Jeff. "How to Get Rid of Private Mortgage Insurance." *Bankrate*, August 10, 2023.
[11] Wallis, Virginia. "Should I Overpay My Mortgage Monthly or With a Lump sum at the End of Each Year?" *The Guardian*, August 7, 2023.
[12] Glynn, James. "Epic Housing Booms Meet Their Match in Australia, Canada, New Zealand." *The Wall Street Journal*, November 6, 2022.
[13] Masters, Terry. "What Is a Portable Mortgage?" *SmartCapitalMind*, August 18, 2023.