This exercise is designed as a "back to basics" approach to time value of money computations, in the context of mortgage loan payment situations. Financial calculator steps are included in many of the accompanying solutions, but in the output you submit for credit you are to work with the appropriate annuity factors or other formulas, not your financial calculator (other than in problem 7, where it is fine to use your calculator to compute the lender's monthly periodic rate of return before converting to APR and EAR values). Too many students know how to compute simple answers with the calculator's function keys, without really knowing what those numbers mean, as evidenced by failure to understand more complicated situations like hybrid loans and the NPV of refinancing. ***[And remember that you can not use graphing calculators on our exams because of security concerns; work these practice problems using a calculator that you can use when you are tested on the mortgage loan computation material.]*** *Show steps for computing each relevant value, and be sure to answer all parts of each question. You are earning credit for working the problems carefully to be better prepared for the exam; points will be taken off for cutting corners. Detailed solutions, that are also intended as a tutorial on mortgage loan mechanics, are provided for each question.*

Please submit a scanned or photographed copy of your carefully hand-written (not typed) answers. (If you take photographs of individual pages with your phone or other device, put only one photo on each page, and please paste all of the pictures into a single Word file in the correct order before submitting. DO NOT submit multiple individual photographs; that kind of submission is very difficult for the instructor to grade.) Upload your submission to Canvas if you can; attach it to an e-mail to the instructor if you can not. Due date and time are shown with the assignment information on Canvas.

1. A borrower can afford to make a monthly principal-plus-interest payment of $1,395. If a local lending institution is willing to provide a fixed-interest rate, fixed-payment mortgage (FRM) loan at a 6.24% annual percentage rate (APR) of interest with equal end-of-month payments over 25 years, how much can the individual afford to borrow?

The amount a borrower can afford to borrow = amount a lender is willing to invest is just the sum of the present values of the individual payments to be made. If a lender charged a 7% annual interest rate, a borrower able to make three subsequent annual payments of $400, $600, and $900 could justify borrowing $400/(1.07)^1 + $600/(1.07)^2 + $900/(1.07)^3, or

$$\$400\left(\frac{1}{1.07}\right)^1 + \$600\left(\frac{1}{1.07}\right)^2 + \$900\left(\frac{1}{1.07}\right)^3 = \text{Total Borrowed}$$

$400 (.934579) + $600 (.873439) + $900 (.816298) = $373.83 + $524.06 + $734.67 = $1,632.56

For equal expected annual payments of $600 per year the answer would be

$$\$600\left(\frac{1}{1.07}\right)^1 + \$600\left(\frac{1}{1.07}\right)^2 + \$600\left(\frac{1}{1.07}\right)^3 = \text{Total Borrowed}$$

$600 (.934579) + $600 (.873439) + $600 (.816298) = $560.75 + $524.06 + $489.78 = $1,574.59

But with equal expected payments we can streamline the computing process with the distributive property:

$$\$600\left[\left(\frac{1}{1.07}\right)^1 + \left(\frac{1}{1.07}\right)^2 + \left(\frac{1}{1.07}\right)^3\right] = \text{Total Borrowed}$$

$600 [(.934579) + (.873439) + (.816298)] = $600 (2.624316) = $1,574.59

And knowing that $\left[\left(\frac{1}{1.07}\right)^1 + \left(\frac{1}{1.07}\right)^2 + \left(\frac{1}{1.07}\right)^3\right] = \left(\frac{1-\left(\frac{1}{1.07}\right)^3}{.07}\right)$ lets us streamline even more:

$$\$600 \left(\frac{1-\left(\frac{1}{1.07}\right)^3}{.07}\right) = \$600\ (2.624316) = \$1{,}574.59$$

The PV of an annuity factor is just the sum of what textbooks call the "PV of \$1" factors for the same periodic rate and number of periods; $\left(\frac{1-\left(\frac{1}{1.07}\right)^3}{.07}\right) = \left[\left(\frac{1}{1.07}\right)^1 + \left(\frac{1}{1.07}\right)^2 + \left(\frac{1}{1.07}\right)^3\right] = 2.624316$ is the present value of a level ordinary annuity factor for three time periods and a 7% periodic discount rate. A PV of annuity factor always has a value smaller than the number of payments; here the principal that can be lent is only 2.624316 (not a full 3) times the size of each payment, because the borrower must repay interest along with principal within the three payments made.

Repaying a fixed-rate, fixed payment mortgage loan is the classic example of a present value of a level, ordinary annuity situation: a series of equal, end-of-period cash flows that are equally spaced apart in time relate to the principal amount borrowed today, in the present. (An "ordinary" annuity has end-of-period payments, while payments on an "annuity due" occur at the beginning of each period. If loan payments were made at the start of each period the borrower would have to make a payment immediately after receiving the principal, thereby walking away from the transaction with less than the amount that was borrowed and presumably needed to pay for an asset, like a house. With most consumer or business loans the borrower receives the entire principal balance to use productively for one period before the first payment is due, and the lender charges interest for the use of that money – such that by the time the first payment is due the borrower owes more than was borrowed; see the period-by-period breakdown of payments in problems 3 and 4 below.) We can set up any annuity problem with the formula Payment x Factor = Total (abbreviated PMT x FAC = TOT); for an FRM loan problem we can be more specific and say

FRM Payment x PV of Annuity Factor = Total Borrowed

This structure lets us plug in the given "knowns" and solve for any "unknown." Here the lump sum total "TOT" principal that can be borrowed is the unknown we are solving for. With a 6.24% stated annual percentage rate (APR) of interest and monthly payments, our monthly periodic interest rate is .0624 ÷ 12 = .0052. With 25 years of monthly payments, we have 25 x 12 = 300 payment periods. Thus for a \$1,395 unchanging monthly payment we compute (recall we can discount each expected cash flow to a present value individually and add the resulting PV's together, or can streamline by using the distributive property and multiplying the steady payment amount by the annuity factor):

$$\$1{,}395\left(\frac{1}{1.0052}\right)^1 + \$1{,}395\left(\frac{1}{1.0052}\right)^2 + \cdots + \$1{,}395\left(\frac{1}{1.0052}\right)^{299} + \$1{,}395\left(\frac{1}{1.0052}\right)^{300} = \text{TOT}$$

$$\$1{,}395\left[\left(\frac{1}{1.0052}\right)^1 + \left(\frac{1}{1.0052}\right)^2 + \left(\frac{1}{1.0052}\right)^3 + \cdots + \left(\frac{1}{1.0052}\right)^{299} + \left(\frac{1}{1.0052}\right)^{300}\right] = \text{TOT}$$

Or, because $\left[\left(\frac{1}{1.0052}\right)^1 + \left(\frac{1}{1.0052}\right)^2 + \cdots + \left(\frac{1}{1.0052}\right)^{299} + \left(\frac{1}{1.0052}\right)^{300}\right] = \left(\frac{1-\left(\frac{1}{1.0052}\right)^{300}}{.0052}\right)$:

PMT x FAC = TOT

$$\$1{,}395\left(\frac{1-\left(\frac{1}{1.0052}\right)^{300}}{.0052}\right) = \text{TOT}$$

\$1,395 x 151.733239 = \$211,667.87

Let's say a home buyer borrows this amount and makes a 20% down-payment. How expensive a house can he afford to buy? If 80% of the purchase price is the $211,667.87 borrowed, the total he can pay is $211,667.87 ÷ .80 = $264,584.84, with 20% ($52,916.97) out-of-pocket and 80% ($211,667.87) borrowed. [These amounts would probably be rounded to something less precise.] On a Texas Instruments BA II Plus financial calculator (other brands/models may follow slightly different key sequences), set payments per period to 1 instead of the default setting of 12. Type in $1395 PMT, $0 FV, 25 x 12 = N (there are 300 payment periods), 6.24 ÷ 12 = I/Y (the periodic interest rate is 6.24% ÷ 12 = .52%, or .0052, shown on the calculator screen as a full percentage amount of .52); CPT PV. It should show $-211,667.87, just as we computed manually above (shown as negative on the calculator because that amount is what comes out of the lender's pocket).

2. What is the monthly payment on a $175,000 fixed-rate, fixed-payment (FRM) home mortgage loan with a 5.46% stated annual percentage rate (APR) of interest, if payments are to be made at the end of each month for 30 years? Based on this loan's interest rate and number of payment periods, show that a loan payment factor is the sum of the interest rate and sinking fund factor.

The $175,000 lump sum "TOT" is lent today, in the present, so of course we have a present value of annuity problem – as noted, repaying a loan with regular, equal periodic payments is a very common example of a PV of annuity situation (and with those equal payments made at the end of each period it is a level "ordinary" annuity). Here we use the same skeleton as in problem 1 above, and as always we plug in the "knowns" and solve for the "unknown." But now it is the regular periodic loan payment "PMT" that we solve for. With a 5.46% stated annual percentage rate (APR) of interest and monthly payments, our monthly periodic interest rate is .0546 ÷ 12 = .00455, or .455%. With 30 years of monthly payments, we have 30 x 12 = 360 payment periods. Therefore, with $175,000 borrowed we divide the known total by the PV of annuity factor to compute the unknown regular payment:

FRM Payment x PV of Annuity Factor = Total Borrowed, or PMT x FAC = TOT

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.00455}\right)^{360}}{.00455}\right) = \$175{,}000$$

PMT x 176.902891 = $175,000

So PMT = $175,000 ÷ 176.902891 = $989.24

Actually however, when computing loan payments we typically replace the PV of an annuity factor (here, 176.902891) with its reciprocal, the *loan payment factor*, so we can *multiply* the amount borrowed by this factor to compute the periodic payment. 1 ÷ 176.902891 = .005653, so

PMT = $175,000 x .005653 = $989.24

The loan payment factor (for monthly payments) is the sum of the monthly interest rate and the monthly *sinking fund factor* (reciprocal of future value of a level ordinary annuity factor for the same periodic interest rate and number of time periods); here the sinking fund factor is computed as

$$\left[1/\left(\frac{(1.00455)^{360}-1}{.00455}\right)\right] = \left(\frac{.00455}{(1.00455)^{360}-1}\right) = .001103$$

Adding the .00455 monthly interest rate and the .001103 monthly sinking fund factor gives us the loan payment factor: .00455 + .001103 = .005653. So computing the payment with the loan payment factor rather than the PV of annuity factor provides a nice visual; recall that every FRM payment must cover the period's interest and also provide for repayment of some principal. The sinking fund factor – the reciprocal of the FV of annuity factor – accounts for that repayment of principal.

On the TI BA II Plus type in $175,000 +/- PV, $0 FV, 30 x 12 = N, 5.46 ÷ 12 = I/Y (this periodic interest rate shows as .455); CPT PMT. It should show $989.24, like we computed manually above.

3. Show the full amortization schedule for a $275,000 loan with a 4% stated annual interest rate, interest-only payments for the first five years, and a balloon payment at the end of year 6 that includes the year's applicable interest plus all principal owed. Then show the full amortization schedule for a $275,000 loan with a 4% stated annual interest rate and six years of equal year-end payments (the loans described in this problem would not likely be home mortgage loans). Indicate the amount of interest the borrower pays during the second year on the amortization schedule for the second loan, and then compute that year 2 interest amount with the formulas.

The periodic payment on an interest-only loan is just the periodic interest rate multiplied by the amount of principal borrowed, here .04 x $275,000 = $11,000 (with no principal being paid the sinking fund factor is 0, so the loan payment factor is the periodic interest rate + 0, meaning just the interest rate). And because interest must be paid on all principal that remains owed (and no principal is being repaid) that $11,000 annual interest payment persists throughout the loan's term, until the final period when a balloon payment that includes the entire $275,000 principal borrowed must be paid, in addition to the $11,000 in interest owed for the year. The year-by-year breakdown of payments is as follows:

Year	(A) Principal Owed at Start of Year	(B) 4.0% Interest on That Principal	(C) Total Owed by End of Year (A + B)	(D) End of Year Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Year (C – D = A – E)
1	$275,000.00	$11,000.00	$286,000.00	$11,000.00	$0.00	$275,000.00
2	$275,000.00	$11,000.00	$286,000.00	$11,000.00	$0.00	$275,000.00
3	$275,000.00	$11,000.00	$286,000.00	$11,000.00	$0.00	$275,000.00
4	$275,000.00	$11,000.00	$286,000.00	$11,000.00	$0.00	$275,000.00
5	$275,000.00	$11,000.00	$286,000.00	$11,000.00	$0.00	$275,000.00
6	$275,000.00	$11,000.00	$286,000.00	$286,000.00	$275,000.00	$0.00

[The amortization schedule for a loan with yearly payments and a one-year maturity would look just like the final year of the amortization for an interest-only loan with a multiple-year maturity:

Year	(A) Principal Owed at Start of Year	(B) 4.0% Interest on That Principal	(C) Total Owed by End of Year (A + B)	(D) End of Year Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Year (C – D = A – E)
1	$275,000.00	$11,000.00	$286,000.00	$286,000.00	$275,000.00	$0.00]

Then for the fully-amortizing six-year fixed-payment FRM loan, we compute the annual payment as

PMT x FAC = TOT

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.04}\right)^{6}}{.04}\right) = \$275{,}000$$

PMT x 5.242137 = $275,000

So PMT = $275,000 ÷ 5.242137 = $52,459.52 OR

PMT = $275,000 x .190762 = $52,459.52

(here we have no specific need to find the annual sinking fund factor, but we know that it has to be the .190762 annual loan payment factor minus the .04 annual interest rate, or .150762). The accompanying amortization schedule is as follows; note that year 1's interest owed is the same $11,000 that was seen in every year of the interest-only loan with the same principal/interest rate/maturity, while a smaller $9,341.62 in interest is paid during year 2:

Year	(A) Principal Owed at Start of Year	(B) 4.0% Interest on That Principal	(C) Total Owed by End of Year (A + B)	(D) End of Year Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Year (C – D = A – E)
1	$275,000.00	$11,000.00	$286,000.00	$52,459.52	$41,459.52	$233,540.48
2	$233,540.48	$9,341.62	$242,882.10	$52,459.52	$43,117.90	$190,422.57
3	$190,422.57	$7,616.90	$198,039.48	$52,459.52	$44,842.62	$145,579.95
4	$145,579.95	$5,823.20	$151,403.15	$52,459.52	$46,636.33	$98,943.63
5	$98,943.63	$3,957.75	$102,901.37	$52,459.52	$48,501.78	$50,441.85
6	$50,441.85	$2,017.67	$52,459.52	$52,459.52	$50,441.85	$0.00

We can also compute the interest amount for year 2 here (or for any time period, for any loan) by knowing that a loan's remaining principal balance, any time during its amortization period, is the present value of the stream of remaining scheduled payments, discounted at the loan's periodic interest rate. Here at the end of year 1, when five years of payments are still scheduled to be made, the PV of that remaining payment stream is

PMT x FAC = TOT

$$\$52{,}459.52\left(\frac{1-\left(\frac{1}{1.04}\right)^{5}}{.04}\right) = \$52{,}459.52 \times 4.451822 = \$233{,}540.48$$

(as we see at the end of year 1 in the year-by-year breakdown above). Then at the end of year 2, with 4 years of remaining payments, the remaining payment stream's PV is

$$\$52{,}459.52\left(\frac{1-\left(\frac{1}{1.04}\right)^{4}}{.04}\right) = \$52{,}459.52 \times 3.629895 = \$190{,}422.57$$

(again seen at the end of year 2 in the "brute force" breakdown above). If the remaining principal owed declines from $233,540.48 at the end of year 1 = beginning of year 2 to $190,422.57 at the end of year 2, then principal repaid during year 2 was $233,540.48 – $190,422,57 = $43,117.90 (also seen in the year-by-year breakdown above); the remainder of the payment was interest. So we can note that

Total year 2 payment	$52,459.52
Minus principal portion	43,117.90
Equals interest paid during year 2	$ 9,341.62

And yes, we could more quickly have just computed the $233,540.48 owed at the end of year 1 = beginning of year 2 and multiplied it by .04, but the process shown above is needed for computing interest owed over a series of payment periods, as we will see in part c of problem 4 that follows.

4. A borrower obtains a $160,000 fixed-rate, fixed-payment mortgage loan (FRM) with a 4.5% stated annual percentage rate (APR) of interest and end-of-month payments to be made over 20 years.

Question a: How much is each monthly payment?

With a 4.5% stated annual percentage rate (APR) of interest and monthly payments, our monthly periodic interest rate is .045 ÷ 12 = .00375. With 20 years of monthly payments, we have 20 x 12 = 240 payment periods. With $160,000 borrowed we compute

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.00375}\right)^{240}}{.00375}\right) = \$160{,}000$$

PMT x 158.065437 = $160,000

So PMT = $160,000 ÷ 158.065437 = $1,012.24 OR

PMT = $160,000 x .006326 = $1,012.24

On a financial calculator, type in $160,000 +/- PV, $0 FV, 240 N, 4.5 ÷ 12 = I/Y (the periodic interest rate is 4.5% ÷ 12 = .375%, or .00375); CPT PMT. It should show $1,012.24, just as we computed manually above.

b. How much interest does the borrower pay during the first *month* of the loan's life?

This part is easy. Interest paid in any period is just the periodic (here monthly) rate times the amount of principal owed at the start of the period, just as we saw for years 1 and 2 in problem 3 above. In the first month the entire $160,000 principal borrowed is still owed, so the interest component of the first month's payment is simply .00375 x $160,000 = $600.00.

c. How much interest does the borrower pay during the first *year* of the loan's life? First, compute by hand a complete amortization for the first twelve months of this 20-year loan, and then compute the answer with formulas. [*Remember: on an exam question of this type you must show enough steps to assure the grader that you understand the ideas; hitting the BA II Plus AMORT key and ↓ ↓ ↓ shows no understanding of how interest and principal paid over a specified time period are computed.*]

This part is harder; the answer certainly is not as simple as $600 x 12 = $7,200 (it is less than that, because the interest component of each payment declines as time passes). A manually-computed month-by-month "brute force" approach shows:

Month	(A) Principal Owed at Start of Month	(B) .045 ÷ 12 = .00375 Int. on That Principal	(C) Total Owed by End of Month (A + B)	(D) End of Month Payment	(E) Principal Portion of Payment (D – B)	(F) Principal Owed at End of Month (C – D = A – E)
1	$160,000.00	$600.00	$160,600.00	$1,012.24	$412.24	$159,587.76
2	$159,587.76	$598.45	$160,186.22	$1,012.24	$413.78	$159,173.98
3	$159,173.98	$596.90	$159,770.88	$1,012.24	$415.34	$158,758.64
4	$158,758.64	$595.34	$159,353.98	$1,012.24	$416.89	$158,341.75
5	$158,341.75	$593.78	$158,935.53	$1,012.24	$418.46	$157,923.29
6	$157,923.29	$592.21	$158,515.50	$1,012.24	$420.03	$157,503.26
7	$157,503.26	$590.64	$158,093.90	$1,012.24	$421.60	$157,081.66
8	$157,081.66	$589.06	$157,670.72	$1,012.24	$423.18	$156,658.48
9	$156,658.48	$587.47	$157,245.95	$1,012.24	$424.77	$156,233.71
10	$156,233.71	$585.88	$156,819.58	$1,012.24	$426.36	$155,807.34
11	$155,807.34	$584.28	$156,391.62	$1,012.24	$427.96	$155,379.38
12	$155,379.38	$582.67	$155,962.06	$1,012.24	$429.57	$154,949.82
	Yr. 1 Totals→	$7,096.68		$12,146.87	$5,050.18	

It is worth computing an amortization schedule by hand for a short period, like 12 months, to build more complete understanding of loan repayment mechanics. This "brute force" approach is more practical to do with a spreadsheet, of course, especially if dealing with a longer period (think of doing this for 10 years, or 120 months, if we wanted to know total interest paid in year 10!). A more direct approach is to recall that the remaining principal balance, at any point during the life of a long-term loan, is the present value of the stream of remaining scheduled payments when discounted at the original loan contract's periodic interest rate. Here, at the end of year 1, with 19 years (228 months) of payments left to make, the PV of the remaining payment stream is

PMT x FAC = TOT

$$\$1{,}012.24\left(\frac{1-\left(\frac{1}{1.00375}\right)^{228}}{.00375}\right) = \text{TOT}$$

$1,012.24 x 153.076315 = $154,949.82

(as we see at the end of month 12 in the "brute force" breakdown above). If that amount of the original $160,000 principal is still unpaid, then the remaining $160,000 – $154,949.82 = $5,050.18 of principal has been repaid. Everything else paid during year 1 has been interest. So here we note that

Total payments during year 1: 12 x $1,012.24	$12,146.87
Minus principal portion	5,050.18
Equals interest paid during year 1	$ 7,096.68

Yes, a financial calculator is useful here as well, but your aging instructor fears that people who simply learn to hit buttons on a financial calculator may not truly understand the underlying ideas. But for you financial calculator jocks, you should already have entered $160,000 +/- PV, $0 FV, 240 N, 4.5 ÷ 12 = I/Y (the periodic interest rate is 4.5% ÷ 12 = .375%, or .00375); CPT PMT (and gotten $1,012.24). Now, to figure out what has happened during months 1 – 12 with the financial calculator's quick (but perhaps not very transparent) amortization tool, type 2nd AMORT 1 ENTER ↓ 12 ENTER ↓ (it should show $154,949.82 as the principal balance still owed 12 months into the loan's 240-month life), then ↓ again (it should show $5,050.18 as principal repaid over the first 12 months), and then ↓ again (it should show $7,096.68 as the interest paid over those 12 months). [*The BA II Plus AMORT and ↓ ↓ ↓ key sequence is handy for computing, but shows no understanding of how interest and principal paid over a given time period are computed; thus no credit is awarded on an exam question for finding an interval's interest/principal payments with AMORT ↓ ↓ ↓.*]

d. If the borrower makes regular monthly payments for three years and then decides to pay the loan off, how much principal is still owed at the end of year 3, and how much in total interest has been paid over this three-year period?

Three years into the loan's life, with 17 years (204 months) of payments to go, the remaining outstanding principal is the present value of the remaining scheduled payment stream:

$$\$1{,}012.24\left(\frac{1-\left(\frac{1}{1.00375}\right)^{204}}{.00375}\right) = \text{TOT}$$

$1,012.24 x 142.399945 = $144,142.78

So a "balloon payment" (meaning one considerably larger than the other payments in a repayment plan), consisting of the $1,012.24 regular monthly payment + the $144,142.78 in remaining principal owed, would be paid by the borrower to retire the loan after three years, at the end of month 36. [On a financial calculator, with all the information from above still entered (or re-enter if need be), hit 204 N (204 payments remain on the initial repayment schedule after 36 months); CPT PV, and the screen should show -$144,142.78.] Then with $144,142.78 still owed, the principal that has been repaid over the first 36 months is $160,000 – $144,142.78 = $15,857.22. So we can show

Total payments during years 1 – 3: 36 x $1,012.24	$36,440.60
Minus principal portion	15,857.22
Equals interest paid during first 3 years	$20,583.38

With the financial calculator's amortization tool: starting with the $1,012.24 payment still showing on the screen type 2nd AMORT 1 ENTER ↓ 36 ENTER ↓ (should show $144,142.78 principal balance still owed 36 months into the loan's life), then ↓ again (should show $15,857.22 repaid over those 36 months), then ↓ again (should show $20,583.38 as the interest paid over those 36 months).

e. If the borrower did not repay the remaining balance at the end of year 3, but rather continued to make payments for the remainder of the 20-year term, how much interest would be paid in year 14?

Recall our supremely important computational lesson: the amount owed on a loan at any time is the present value of the stream of remaining scheduled payments, discounted at the loan's periodic contract interest rate. So first we must consider how much is owed coming into the year and how much is owed coming out of the year. At the end of year 13 = start of year 14, with 156 months of payments completed and seven years = 84 scheduled months remaining, the borrower owes:

$$\$1{,}012.24\left(\frac{1-\left(\frac{1}{1.00375}\right)^{84}}{.00375}\right) = \text{TOT}$$

$$\$1{,}012.24 \times 71.941611 = \$\underline{72{,}822.10}$$

Then a year later, at the end of year 14 = start of year 15, with 168 months of payments completed and six years = 72 months of payments remaining, the borrower owes:

$$\$1{,}012.24\left(\frac{1-\left(\frac{1}{1.00375}\right)^{72}}{.00375}\right) = \text{TOT}$$

$$\$1{,}012.24 \times 62.995976 = \$\underline{63{,}766.98}$$

If principal owed drops from $72,822.10 at the start of year 14 to $63,766.08 at the year's end, then principal repaid during the year is $72,822.10 – $63,766.98 = $9,055.12. So we can show

Total payments during year 14: 12 x $1,012.24	$12,146.87
Minus principal portion	9,055.12
Equals interest paid during year 14	$3,091.75

In part c above we found the interest portion of year 1's $12,146.87 payment total to be a much higher $7,096.68, and the remainder that repays principal to be a much smaller $5,050.18.
Of course interest can be charged in any period only on principal that still is owed, so with only $72,822.10 rather than $160,000 owed at the start of the year, the component of year 14's twelve total payments attributable to interest is much smaller than its year 1 counterpart (and the portion attributable to principal much bigger). On the TI BA II Plus we can enter 2nd AMORT 157 ENTER ↓ 168 ENTER ↓ (year 13 ended with month 156 so year 14 contains months 157 – 168; the screen should show $63,766.98 as the principal balance still owed at the end of year 14, which is 168 months into the loan's life), then ↓ again (should show $9,055.12 as principal repaid over year 14's

twelve months), then ↓ again (should show \$3,091.75 as the interest paid over months 157 - 168).

5. Nine years ago a borrower obtained a fixed-rate, fixed-payment mortgage (FRM) loan. Interest was quoted as a 5.10% Annual Percentage Rate (APR), and equal payments of \$1,629.59 were to be made at the end of each month for 25 years. What was the amount borrowed? How much remains owed today? Show how we can compute the unchanging monthly payment if we know what remains owed today even if we do not know the amount originally borrowed; use the example of knowing \$213,586.62 is still owed nine years into the 25-year amortization period for this 5.10% APR loan.

Computing the answers actually is quite simple. Once more we draw on the fact that the amount of principal owed on a loan at any time is the present value of the stream of remaining scheduled payments. So on the day the loan was obtained nine years ago, the amount that was borrowed and therefore still owed at that time would have been the PV of the stream of all 25 x 12 = 300 scheduled payments, discounted to a PV at the .051 ÷ 12 = .00425 or .425% monthly r:

$$\$1{,}629.59\left(\frac{1-\left(\frac{1}{1.00425}\right)^{300}}{.00425}\right) = \text{TOT}$$

$$\$1{,}629.59 \times 169.367780 = \$\underline{276{,}000}$$

[Note that on the day the borrower got the loan the regular monthly payment would have been computed with the \$276,000 principal borrowed known and the payment as the unknown to solve for:

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.00425}\right)^{300}}{.00425}\right) = \$276{,}000$$

$$\text{PMT} \times 169.367780 = \$276{,}000$$

$$\text{So PMT} = \$276{,}000 \div 169.367780 = \$1{,}629.59 \quad \text{OR}$$

$$\text{PMT} = \$276{,}000 \times .005904 = \$\underline{1{,}629.59}$$]

And the amount of principal that remains owed today, with 25 – 9 = 16 years or 192 months of payments remaining, is not a simple 16/25 x \$276,000 = \$176,640.00, but a much larger

$$\$1{,}629.59\left(\frac{1-\left(\frac{1}{1.00425}\right)^{192}}{.00425}\right) = \text{TOT}$$

$$\$1{,}629.59 \times 131.067724 = \$\underline{213{,}586.62}$$

Finally, we can easily compute the regular monthly payment on a fixed-rate, fixed-payment (FRM) home mortgage loan if we know r, the amount of principal still owed, and the number of remaining scheduled payments – even if we do not know the original amount borrowed, or even the number of originally scheduled payments (our computations are based on how many payments remain, not how many are past). To compute the equal end-of-month payments for this 5.1% APR loan nine years into its 25-year term (with 16 years = 192 months of payments remaining), when \$213,586.62 still is owed, we would note yet again that the amount owed at any time (here, today) is the PV of the stream of the remaining scheduled payments, and just reverse the process shown above to find

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.00425}\right)^{192}}{.00425}\right) = \$213{,}586.62$$

$$\text{PMT} \times 131.067724 = \$213{,}586.62$$
$$\text{So PMT} = \$213{,}586.62 \div 131.067724 = \$1{,}629.59 \text{ OR}$$
$$\text{PMT} = \$213{,}586.62 \times .007630 = \$\underline{1{,}629.59}$$

Check your understanding with a different example: $179,203.49 is still owed on a loan with a 7.38% APR or .0738 ÷ 12 = .0065 or .615% monthly interest rate, and seven years = 84 months of end-of-month scheduled payments remaining. Because the amount of principal owed on a loan at any time is the PV of the stream of scheduled remaining payments (we do not have to know the original negotiated maturity period), we compute the monthly payment as

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.00615}\right)^{84}}{.00615}\right) = \$179{,}203.49$$
$$\text{PMT} \times 65.448801 = \$179{,}203.49$$
$$\text{So PMT} = \$179{,}203.49 \div 65.448801 = \$2{,}738.07 \text{ OR}$$
$$\text{PMT} = \$179{,}203.49 \times .015279 = \$\underline{2{,}738.07}$$

[And if we know a loan's regular payment and periodic interest rate, we can compute the amount of principal that initially was borrowed IF we also know the originally agreed maturity period; for this $2,738.07 monthly payment loan with a .615% monthly r, if the original maturity was 20 years = 240 months then principal borrowed was

$$\$2{,}738.07\left(\frac{1-\left(\frac{1}{1.00615}\right)^{240}}{.00615}\right) = \text{TOT}$$
$$\$2{,}738.07 \times 125.270652 = \$\underline{343{,}000}\ ]$$

6. What is the monthly payment on a $236,000 fixed-rate, fixed-payment (FRM) home mortgage loan with an interest rate represented as a 3.8835% effective annual rate (EAR), if equal payments are to be made at the end of each month for 15 years? What would the accompanying annual percentage rate (APR) of interest be? This question is similar to the first part of question 2, except now the interest rate is presented in EAR rather than APR terms. (Discussing the interest rate in EAR terms is conceptually useful but admittedly not real-world realistic, because while EAR is a more comprehensive measure of a borrower's yearly opportunity cost of the debt financing, the interest rate that federal law requires lenders to quote in making home mortgage loans is an APR application.) Show why we can not instead just compute the annual payment for a 15-year loan with that APR (or EAR) interest rate, and then divide the resulting annual payment by 12 to get the correct monthly payment.

Recall that general practice is for interest rates to be *discussed* in annual terms to provide some consistency in presentation across different types of financial transactions, whereas in computing values in situations with non-annual payments we have to *work* with a periodic r that relates to the timing of the payments and compounding. So if payments and compounding do not occur annually we must convert the annual rate talked about to a periodic (often semi-annual, quarterly, or monthly) r to work with. The complication is that the annual interest rate can be presented either as a simpler ("convenient," as I like to say) annual percentage rate, or APR, or else as the more complicated ("accurate," as I like to say) effective annual rate, or EAR. An APR is just the periodic r multiplied by the number of periods is a year; if 1.5% interest is earned every quarter on an investment plan the accompanying APR is just .015 x 4 = .06, or 6%. Now turn it around: if told in the talking phase that payments and compounding occur quarterly and the APR is 6%, we find the quarterly r to use in computing by just dividing the APR by 4: .06 ÷ 4 = .015 or 1.5%; you can do the math in your head.

But within that APR measure we conveniently ignore the impact of interest-on-interest earned within each year, *i.e.*, the idea that 1.5% interest is earned in quarter 2 on the 1.5% that already was earned in quarter 1, *etc.* The EAR does accurately include that interest-on-interest impact; EAR is computed as $(1 + r)^{\text{number of periods in a year}} - 1$. So both the APR and EAR include the 1.5% per quarter r and the four quarterly periods that constitute a year, but the APR does not incorporate the impact of compounding within a year – making it convenient to deal with, but not an accurate representation of a money provider's gain in wealth or money user's opportunity cost over a full year. Because the EAR accurately includes the benefit or cost of the intra-year interest buildup (and thus is a better measure to use in comparing an investment with competing investments) it is always greater than the APR – unless payments and compounding occur annually, in which case APR and EAR are the same value. So with 1.5% earned each quarter the EAR, an accurate measure of the wealth gain for an investor who stays in the plan for an entire year, is not .015 x 4 = .06 or 6%, but rather the slightly higher $(1.015)^4 - 1 = .061364$, or 6.1364%. Now turn it around – if told in the talking phase that payments and compounding occur quarterly and the EAR is 6.1364%, we must undo the compounding to find the quarterly r by taking a fourth root: $\sqrt[4]{1.061364} - 1 = .015$; you need a calculator that can handle exponents. Note that you do not take the fourth root of just the EAR; the 1 must be included. That 1 represents 100% of the initial principal; each quarter the investor earns 1.5% interest on the 1.5% interest earned in earlier quarters, but also each quarter earns 1.5% interest on the initial principal held. Taking that fourth root yields 1 + the quarterly r, so after subtracting 1 we are left with just the desired quarterly r.

So now after that long-winded explanation of APRs and EARs we can address the question at hand. If there are monthly payments (twelve payments and compounding periods per year) we find the monthly r to work with by dividing a given APR by 12 or, as in this example, taking the 12th root of (1 + the given EAR) and subtracting 1: $\sqrt[12]{1.038835} - 1 = .00318$. With 15 years of monthly payments, we have 15 x 12 = 180 payment periods. Thus with \$236,000 borrowed we compute a monthly payment of

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.00318}\right)^{180}}{.00318}\right) = \$236{,}000$$

PMT x 136.892323 = \$236,000

So PMT = \$236,000 ÷ 136.892323 = \$1,723.98 OR

PMT = \$236,000 x .007305 = \$<u>1,723.98</u>

On the TI BA II Plus type in \$236,000 +/- PV, \$0 FV, 15 x 12 = N, 1.038835 y^x (1 ÷ 12) = - 1 = x 100 = I/Y (should show a .318% periodic rate; remember the calculator's time value function keys work with full percentages); CPT PMT. It should show \$1,723.98, just as we computed manually above.

What is the corresponding APR? With monthly r of .00318 the more accurate EAR is $(1.00318)^{12} - 1 = .038835$, but the simpler, convenient APR is just .00318 x 12 = .038160, such that if we were told the APR of 3.8160% we would find the monthly r to work with as just .038160 ÷ 12 = .00318. Notice how the simpler 3.8160% APR is slightly less than the compounded 3.8835% EAR, as should always be the case when payments and compounding occur more frequently than once per year.

Finally, as noted we can not find the correct monthly payment by simply computing as though the loan carried annual payments and dividing the computed annual payment by 12. With a 3.8160% APR the annual payment would be

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.038160}\right)^{15}}{.038160}\right) = \$236{,}000$$

PMT x 11.262813 = \$236,000

PMT = \$236,000 x .088788 = \$20,953.91, and \$20,953.91 ÷ 12 = \$<u>1,746.16</u> > \$1,723.98

[Using EAR as the interest rate and dividing the resulting annual payment by 12 gives an even less accurate estimate;

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.038835}\right)^{15}}{.038835}\right) = \$236{,}000$$

$$\text{PMT} \times 11.209510 = \$236{,}000$$

$$\text{PMT} = \$236{,}000 \times .089210 = \$21{,}053.55, \text{ and } \$21{,}053.55 \div 12 = \$1{,}754.46 > \$1{,}723.98]$$

Treating an FRM loan as annual and then dividing the computed annual payment by 12 *always yields an answer greater* than the correctly computed monthly payment. For a loan with annual payments, interest is owed on the full borrowed principal for an entire year before a payment is made that reduces principal still owed, and thus reduces interest owed the next period. But with monthly FRM payments some principal is repaid after just one month, reducing interest owed even in month 2; far less interest is paid over time. Or viewed from a slightly different angle, if you are willing to pay $^1/_{12}$ of the computed annual payment each month you can afford to take out a bigger loan.

7. A $242,000 fixed-rate, fixed-payment (FRM) home mortgage loan carries scheduled payments of $1,870.09 at the end of each month for 30 years. Set up the equations to compute the effective cost of borrowing, first with zero discount points, and then with the lender charging two discount points as a condition of granting the loan. The payments and number of periods are monthly figures, so the r's in the equations will be monthly figures that you should convert to annual percentage rate (APR) and effective annual rate (EAR) values. (After setting up the equations, it is sensible to compute the monthly r's with Excel® or your financial calculator's I/Y function key.)

Here we use the same skeleton equation as before, but now the interest rate charged on the loan is the unknown to solve for. Because the payments occur monthly, we should think in terms of having 360 monthly payment periods and a periodic interest rate of r; the zero discount points case shows

$$\text{PMT} \times \text{FAC} = \text{TOT}$$

$$\$1{,}870.09\left(\frac{1-\left(\frac{1}{1+r}\right)^{360}}{r}\right) = \$242{,}000$$

Because the present value of a level ordinary annuity factor's denominator contains r and its numerator contains r to a power (here $r^{1/360}$), there is no way to isolate r on one side of the equals sign and have only terms without r on the other side. Thus there is no way to directly solve for r; we must use trial and error. We find the monthly periodic interest rate r to be .0071285763; let's double-check:

$$\$1{,}870.09\left(\frac{1-\left(\frac{1}{1.0071285763}\right)^{360}}{.0071285763}\right) = \$242{,}000$$

$$\$1{,}870.09 \times 129.4055408 = \$242{,}000 \checkmark$$

Now to solve what the question asks for: with a monthly periodic rate of .0071285763, we would have interest quoted as an annual percentage rate (APR) of .0071285763 x 12 = .08554292, or about 8.554%, while the effective annual rate (EAR) would be $(1.0071285763)^{12} - 1 =$ .08897780, or about 8.898%. [APR and EAR are explained in more detail in the solution to problem 6 above.]

On a financial calculator, which is programmed to handle the trial and error attempts quickly, we would enter $242,000 +/- PV, $0 FV, $1,870.09 PMT, 360 N; CPT I/Y. The screen goes blank for a couple of noticeable seconds while the calculator does trial and error, then it should show .71285763; that is a monthly rate because the $1,870.09 payment and 360 time periods are monthly figures. (It is a whole percentage, so you will likely want to divide it by 100 to get the decimal equivalent .0071285763 monthly r and store it before going on.) Then multiply the stored r solution by 12 to get the APR, or take (1 + r) to the 12^{th} power to compute 1 + the EAR.

Finally, if the lender charges two discount points, it means the payments are computed based on an APR of ≈ 8.554% and the full $242,000 nominal amount borrowed/lent, but the borrower must give the lender .02 x $242,000 = $4,840 at the closing, and thus the borrower leaves the closing with only $242,000 - $4,840 or .98 x $242,000 = $237,160 net of what she arrived at the closing with. The monthly periodic rate of return to the lender (which equals the monthly periodic rate of cost to the borrower, unless a middle party brings the borrower and lender together and takes away some of what the borrower pays), if the two points are charged, is computed with the equation

$$\$1{,}870.09 \left(\frac{1 - \left(\frac{1}{1+r} \right)^{360}}{r} \right) = \$237{,}160$$

That rate turns out to be a slightly higher .0073130660 (the lender gives up a smaller amount but gets the same $1,870.09 monthly amounts back as in the no-points case, so the rate of return is higher – recall that a periodic rate of return relates what an investor receives each period, or receives on average over multiple periods, relative to what initially was given up). Double-check:

$$\$1{,}870.09 \left(\frac{1 - \left(\frac{1}{1.0073130660} \right)^{360}}{.0073130660} \right) = \$237{,}160$$

$$\$1{,}870.09 \times 126.8174264 = \$237{,}160 \checkmark$$

With a monthly periodic rate of .0073130660, we have an APR of .0073130660 x 12 = .08775679, or about 8.776%, while the EAR would be $(1.0073130660)^{12} - 1 =$ .09137401, or about 9.137% (a bit higher than the 8.554% APR & 8.898% EAR values found in the case above with no points charged).

On a financial calculator, enter .98 x $242,000 = +/- PV, $0 FV, $1,870.09 PMT, 360 N; CPT I/Y. It should show .73130660; again that is a monthly r because the $1,870.09 payment and 360 time periods are monthly figures. (You probably should divide that whole percentage figure by 100 to get the decimal equivalent .0073130660, and store before going on.) Then multiply the stored calculator solution by 12 to get the APR, or take (1 + r) to the 12^{th} power to compute 1 + the EAR.

Caution with your calculator: watch the negative signs when solving for the rate (I/Y, r, i, or however the calculator designates it). In a loan situation the lender parts with money up front and then receives money as payments are made (borrower is in just the opposite position). So either the PV or the PMT (but only one of the two, not both) must be shown as a negative or the calculator can not solve for r, because a party that gets money up front and then continues to get payments later is earning an infinite rate of return, which the calculator can not find as a possible solution. A similar problem arises when we try to solve for an unknown number of periods n and fail to correctly enter either the PV or PMT (but not both) as negative. It is not a problem when we solve for the payment (or the loan amount) and forget to enter the loan amount (or the payment) as a negative, because the calculator will simply attach a negative sign to whatever dollar value it computes.

8. How long would it take to pay off a $150,000 fixed-rate, fixed-payment home mortgage loan (FRM) with equal end-of-month payments of $1,200 and a 3.3% annual contract interest rate (APR)? What about a $125,000 FRM with unchanging end-of-month payments of $437.50 and a 4.2% annual contract interest rate (APR)? [Find the answers with logarithms; we want everyone to see what your financial calculator is doing when you hit CPT-N.]

Here we are just trying to solve for the number of time periods in our now-familiar loan repayment structure. For the first loan, with a 3.3% ÷ 12 = .2750% or .00275 monthly periodic interest rate, we solve for n in

$$\$1{,}200\left(\frac{1-\left(\frac{1}{1.00275}\right)^{n}}{.00275}\right) = \$150{,}000$$

$$\left(\frac{1-\left(\frac{1}{1.00275}\right)^{n}}{.00275}\right) = 125$$

$$1-\left(\frac{1}{1.00275}\right)^{n} = .343750$$

$$-\left(\frac{1}{1.00275}\right)^{n} = -.656250 \text{ so } \left(\frac{1}{1.00275}\right)^{n} = .656250$$

$$(.997258)^{n} = .656250 \text{ so } \ln(.997258)^{n} = \ln .656250$$

$$n \ln(.997258) = \ln .656250$$

$$n(-.002746) = -.421213$$

n = 153.379043, just over 153 months (just under 13 years)

Double check as $$\$1{,}200\left(\frac{1-\left(\frac{1}{1.00275}\right)^{153.379043}}{.00275}\right) = \$150{,}000 \checkmark$$

(If it were structured as a 20 year loan the borrower could pay a minimally required $854.60 per month for 240 months, but by paying a larger $1,200 per month could have the obligation met in just over 153 months.) Financial calculator solution: enter $150,000 +/- PV, $0 FV, $1,200 PMT, 3.3 ÷ 12 = I/Y; CPT N. The calculator screen should show 153.379043.

For the second loan, with a 4.2% ÷ 12 = .35% or .0035 monthly periodic interest rate, we solve for n in

$$\$437.50\left(\frac{1-\left(\frac{1}{1.0035}\right)^{n}}{.0035}\right) = \$125{,}000$$

$$\left(\frac{1-\left(\frac{1}{1.0035}\right)^{n}}{.0035}\right) = 285.714286$$

$$1-\left(\frac{1}{1.0035}\right)^{n} = 1 \Rightarrow -\left(\frac{1}{1.0035}\right)^{n} = 0 \Rightarrow \left(\frac{1}{1.0035}\right)^{n} = 0$$

The algebra here is not telling us that the answer is zero periods (the loan is not being paid back immediately). A fraction less than 1 taken to a power approaches a value of 0 only as the exponent approaches infinity; an infinite number of periods would be needed for this loan to be repaid. The first month's interest payment should be (.042 ÷ 12) x $125,000 = $437.50. So if only $437.50 is paid at the end of month 1, the entire $125,000 principal remains outstanding going into month 2 – and this situation repeats every month. So principal owed is never reduced; the loan is never repaid. (This interest-only loan would require, at some point, a $125,437.50 balloon payment that included the final month's interest and 100% of the principal borrowed.) If you tried to use your financial calculator, you would enter $125,000 +/- PV, $0 FV, $437.50 PMT, 4.2 ÷ 12 = I/Y; CPT N. The calculator can not compute an infinite repayment period, so the screen displays an Error message.

9. "Two-step" mortgage loans frequently are "5/25" or "7/23" (the same interest rate and monthly payment for the first five or seven years of a 30-year amortization period, followed by year-to-year changes based on each year's new market interest rate). Let's pretend we are borrowing $290,000 on a "6/24" plan (not commonly seen but certainly not inappropriate); the monthly payments will remain the same for all of years 1-6, based on a 5.16% stated APR interest rate, and then year 7's monthly payment will be computed based on a new, higher predicted 6.12% APR. Compute the monthly payment that applies to years 1 – 6, and the expected monthly payment for year 7.

We compute the initial payment for a two-step loan, or any adjustable rate mortgage (ARM) loan, the same way we compute the unchanging payment for a standard fixed-rate, fixed-payment mortgage (FRM) loan. Here the monthly periodic rate r for years 1 – 6 is .0516 ÷ 12 = .0043, and we compute:

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.0043}\right)^{360}}{.0043}\right) = \$290{,}000$$

PMT x 182.934916 = $290,000

So PMT = $290,000 ÷ 182.934916 = $1,585.26 OR

PMT = $290,000 x .005466 = $1,585.26

On a financial calculator type $290,000 +/- PV, $0 FV, 360 N, 5.16 ÷ 12 = I/Y (periodic interest rate is 5.16% ÷ 12 = .43%, or .0043); CPT PMT. Should show $1,585.26, as computed manually above. At the end of year 6 a new year 7 payment is computed based on a new stated annual interest rate that reflects market conditions at the end of the sixth year. (If the year 7 stated interest rate is the same as the year 1 – 6 rate the year 7 payment will stay at the year 1 – 6 level, but here the rate changes.) To compute the year 7 payment, first compute the principal that remains owed at the end of year 6 (after 72 months, when 360 – 72 = 288 months of scheduled payments remain):

$$\$1{,}585.26\left(\frac{1-\left(\frac{1}{1.0043}\right)^{288}}{.0043}\right) = \$1{,}585.26\ (164.972522) = \$261{,}524.88$$ ("goodbye, old loan")

(The loan is 6 years = 20% of the way to maturity, but principal repaid is not 20% of $290,000 = $58,000; only $290,000 - $261,524.88 = $28,475.12 has been repaid.) Then compute the payment for the "new" loan. This loan has a $261,524.88 original principal amount and a 24-year (288 month) life. With a new interest rate of 6.12% APR (for a .0612 ÷ 12 = .0051 monthly periodic rate r), the year 7 payment is computed as

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.0051}\right)^{288}}{.0051}\right) = \$261{,}524.88$$ ("hello, new loan")

PMT x 150.772068 = $261,524.88

So PMT = $261,524.88 ÷ 150.772068 = $1,734.57 OR

PMT = $261,524.88 x .006633 = $1,734.57

(and then a new payment would be computed in a similar manner for each of years 8 to 30). The interest rate at the renewal date has risen, so year 7's payment is higher than the payment seen in each of years 1 – 6. [If the "new" rate were the original 5.16% APR = .0043 r, the "new" payment would be

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.0043}\right)^{288}}{.0043}\right) = \$261{,}524.88$$

PMT x 164.972522 = $261,524.88

So PMT = \$261,524.88 ÷ 164.972522 = \$1,585.26 OR
PMT = \$261,524.88 x .006062 = \$1,585.26

(if the interest rate does not change, then year 7's monthly payment should not change from the year 1-6 level). If the new rate charged in year 8 is a 5.64% APR = .47% monthly r, principal owed at the end of year 7/start of year 8 is \$1,734.57 x 147.920206 (PVA factor for .51%/month and 276 months) = \$256,578.12, and year 8's monthly payment is \$256,578.12 ÷ 154.440908 (factor for .47%/month and 276 months) = \$1,661.34 -- higher than year 1-6's \$1,585.26 monthly payments with the .43% monthly r, but lower than year 7's \$1,734.57 with the .51% monthly r. And if year 9's rate is back to .51% monthly r the amount owed will be \$1,661.34 x 151.064994 = \$250,969.59 and the new monthly payment will be \$250,969.59 ÷ 144.888829 = \$1,732.15 -- slightly less than year 7's \$1,734.57 because principal amortizes more quickly in year 8 with the lower monthly r.]

On a financial calculator, let's assume you have the original information already entered (or re-enter it): \$290,000 +/- PV, \$0 FV, 360 N, 5.16 ÷ 12 = I/Y; CPT PMT and it shows \$1,585.26. Enter 288 N, CPT PV and it shows \$261,524.88 in principal remaining owed when 288 payments remain based on the original loan terms ("goodbye, old loan"). Hit the +/- key to make it negative, and PV to enter this amount as the new loan balance to carry forward. Enter \$0 FV, 288 N, 6.12 ÷ 12 = I/Y, CPT PMT; it shows \$1,734.57 ("hello, new loan"), as we computed manually above.

10. Five years ago a home buyer obtained a \$185,000 mortgage loan, which carried a 6.6% stated annual percentage rate (APR) of interest and was to be repaid with equal end-of-month payments over 30 years. Today she would be able to get a 25-year loan with a 5.4% stated APR. Thus she might expect to make level payments over the next 25 years, regardless of whether she continues with the current loan or gets a new one. Origination costs, and other costs such as searching the market and lost time, in connection with getting a new loan would be approximately \$9,500. If she has extra cash she can place it in a low-risk investment to earn a 4.2% annual percentage rate (APR) of return. What net present value (NPV) would the borrower realize if she refinanced the loan? What would her NPV of refinancing be if she planned to remain in the local area for only seven more years before retiring, selling the house, and moving away?

First step in this Net Present Value analysis: compute the initial/current monthly loan payment, based on the original 360-month life and a monthly periodic interest rate of .066 ÷ 12 = .0055:

PMT x FAC = TOT

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.0055}\right)^{360}}{.0055}\right) = \$185{,}000$$

PMT x 156.578125 = \$185,000
So PMT = \$185,000 ÷ 156.578125 = \$1,181.52 OR
PMT = \$185,000 x .006387 = \$1,181.52

Second step: compute the monthly loan payment that would be made on a possible new loan, based on a 300-month life and monthly periodic interest rate of .054 ÷ 12 = .0045. Principal that remains unpaid 60 months into the original loan's 360-month life (with 360 – 60 = 300 months to go) is

$$\$1{,}181.52\left(\frac{1-\left(\frac{1}{1.0055}\right)^{300}}{.0055}\right) = \$1{,}181.52 \times 146.741797 = \$173{,}378.19$$ ("goodbye, old loan")

[Recall that on the day she took out the loan she owed $\$1{,}181.52\left(\frac{1-\left(\frac{1}{1.0055}\right)^{360}}{.0055}\right) = \$185{,}000.$]

The monthly payment on a new $173,378.19, 25-year, 5.4% ÷ 12 = .0045 monthly periodic rate r loan would be

$$\text{PMT}\left(\frac{1-\left(\frac{1}{1.0045}\right)^{300}}{.0045}\right) = \$173{,}378.19 \text{ ("hello, new loan")}$$

$$\text{PMT} \times 164.438547 = \$173{,}378.19$$

$$\text{So PMT} = \$173{,}378.19 \div 164.438547 = \$1{,}054.36 \text{ OR}$$

$$\text{PMT} = \$173{,}378.19 \times .006081 = \$\underline{1{,}054.36}$$

Third step: So by refinancing after five years with a new 25-year loan, the borrower would realize a reduction in her monthly payment of $1,181.52 – $1,054.36 = $<u>127.15</u>. Thus she would save $127.15 every month for 25 years (300 months); with a monthly periodic reinvestment or opportunity rate r of 4.2% ÷ 12 = .0035, the PV of this savings stream (the present value of an annuity) would be

$$\$127.15\left(\frac{1-\left(\frac{1}{1.0035}\right)^{300}}{.0035}\right) = \text{TOT}$$

$$\$127.15 \times 185.548614 = \$\underline{23{,}593.28}$$

Final step: The cost of going through a refinancing would be the up-front cost of getting the new loan, or about $9,500, which would occur today and thus already is a present value figure. So refinancing has a net present value of PV Savings Stream minus PV Costs of Refinancing = $23,593.28 – $9,500 = $<u>14,093.28</u>. NPV is positive, so <u>refinance</u> (refinancing to get the lower payments going forward creates just over $14,000 in wealth today for the borrower).

Financial calculator: Enter $185,000 +/- PV, $0 FV, 360 N, 6.6 ÷ 12 = I/Y; CPT PMT. It should show $1,181.52. Store that amount in memory register 1 by hitting STO 1. Then enter 300 N, CPT PV; it should show -$173,378.19 ("goodbye, old loan"). This becomes the principal amount for a new 25-year, 5.4% loan, so enter 5.4 ÷ 12 = I/Y, CPT PMT; it should show $1,054.36 ("hello, new loan"). Hit the +/- key, then hit the + key, RCL 1 (to retrieve the $1,181.52 from memory register 1), and the = key, to subtract $1,054.36 from the $1,181.52. It should show $127.15. Hit the PMT key to make $127.15 the regular payment, then enter 4.2 ÷ 12 = I/Y, CPT PV. (The needed $0 FV and 300 N values should not have changed, but if in doubt re-enter them.) It should show $-23,593.28. Hit the +/- key and then enter -$9,500 to subtract the $9,500 refinancing cost from the positive $23,593.28 refinancing benefit; the result is the $14,093.28 NPV computed above.

[Possible slight adjustment to financial calculator steps above: after entering CPT PMT to get the $1,181.52 payment and STO 1, type 60 N and CPT FV (instead of 300 N and CPT PV). The result is a $173,378.19 future value that lacks a negative sign, so hit +/- and PV to have the correct principal for computing the new 25-year loan's payment, and then proceed as on line 4 above. This approach reflects a different thought process: instead of directly viewing the amount still owed after five years as the PV of the stream of remaining all-equal payments scheduled under the original loan agreement, we look ahead from the origination date and ask: if $185,000 borrowed is by definition the PV of all payments to be received, and prepayment were to occur after five years, how much principal would still be owed, along with the regular payment made, at the end of month 60? The structure for computing manually is the same as we see in working with the value of a coupon bond:

$$\$185{,}000 = \$1{,}181.52\left(\frac{1-\left(\frac{1}{1.0055}\right)^{60}}{.0055}\right) + \text{Principal Owed}\left(\frac{1}{1.0055}\right)^{60}$$

$$\$185{,}000 = \$1{,}181.52\ (50.986533) + \text{Principal Owed}\ (.719574)$$

$$\$185{,}000 = \$60{,}241.55 + \text{Principal Owed}\ (.719574)$$

($185,000 - $60,241.55) = Principal Owed (.719574)
$124,758.45 ÷ (.719574) = Principal Owed = $173,378.19

This thought process is not limited to cases with early repayment expected; with no early principal repayment we would have

$$\$185{,}000 = \$1{,}181.52\left(\frac{1-\left(\frac{1}{1.0055}\right)^{360}}{.0055}\right) + \text{Principal Owed}\left(\frac{1}{1.0055}\right)^{360}$$

$185,000 = $1,181.52 (156.578125) + Principal Owed (.138820)
$185,000 = $185,000 + Principal Owed (.138820)
($185,000 - $185,000) = Principal Owed (.138820)
$0 ÷ (.138820) = Principal Owed = $0 after 360th regular payment has been made]

Note that textbooks typically recommend treating the expected reinvestment rate in a refinancing situation as being equal to the new mortgage lending rate, which here would be a 5.4% APR. But an argument can be made that our assumed reinvestment rate should be something closer to a risk-free rate, which always is lower than the mortgage lending rate, as shown above. After all, switching to a loan with a lower interest rate generates an *assured monthly savings stream* for the borrower, so we might want to find the present value of that assured stream based on a low-risk rate of return, as with 4.2% annually above. And because discounting is a form of penalizing, discounting at a lower rate penalizes expected future cash flows less for not being in our hands today, in turn generating a higher present value. But if instead we used the somewhat higher 5.4% APR = .45% monthly r reinvestment rate as the textbooks traditionally have shown, rather than the 4.2% APR = .35% low-risk monthly r we used earlier, the borrower's PV of receiving the stream of expected $127.15 inflows per month for 300 months would be computed as a somewhat smaller

$$\$127.15\left(\frac{1-\left(\frac{1}{1.0045}\right)^{300}}{.0045}\right) = \$127.15 \times 164.438547 = \$\underline{20{,}909.04}$$

So a more traditional approach might show a lower present value for the stream of $127.15 monthly savings benefits, and the NPV of refinancing would be a bit lower than the NPV we found earlier: $20,909.04 – $9,500 = $<u>11,409.04</u>. [The borrower's option to refinance will not be as far in-the-money if the PV of the savings stream from refinancing is lower, because of higher perceived risk.]

Finally, if our borrower plans to stay in the house for only seven more years = 84 months, whether she refinances or not, then the PV of the stream of reduced monthly payments from refinancing (based on the low-risk 4.2% annual = .0035 monthly discount rate) will be just

$$\$127.15\left(\frac{1-\left(\frac{1}{1.0035}\right)^{84}}{.0035}\right) = \text{TOT}$$

$$\$127.15 \times 72.668817 = \$\underline{9{,}240.14},$$

with the PV of the savings stream seemingly a bit too low to cover the $9,500 cost of refinancing.

But there is something more to the story. Right now the borrower owes $173,378.19, which she will repay through a loan with 25 years = 300 months of scheduled payments, either by continuing with the existing 30-year loan that is five years along, or by refinancing to a new 25-year loan. If she keeps her existing loan with the .55% monthly r and 300 remaining $1,181.52 scheduled monthly payments (as computed above), then seven years = 84 months from today (12 years = 144 months into the existing loan's amortization), when she sells the house, she will have to pay her current lender a final regular monthly payment plus all principal still owed (per the "alienation" or "due on sale" clause). That amount will be the PV of the stream of 360 – 144 (counting from the origination date) or 300 – 84 (measured from today) = 216 scheduled payments that will then remain on the existing loan:

$$\$1{,}181.52 \left(\frac{1-\left(\frac{1}{1.0055}\right)^{216}}{.0055} \right) = \$1{,}181.52 \times 126.213786 = \$\underline{149{,}123.96}$$

If instead she refinances the $173,378.19 currently owed with a new 25-year = 300 month loan that has a 5.4% annual = .45% monthly interest rate and $1,054.36 monthly payment (as computed above), then in seven years = 84 months, when she moves and repays any remaining principal owed along with making her month 84 payment, she will owe the new lender the present value of the stream of 300 – 84 = 216 remaining scheduled payments that will not be made:

$$\$1{,}054.36 \left(\frac{1-\left(\frac{1}{1.0045}\right)^{216}}{.0045} \right) = \$1{,}054.36 \times 137.966344 = \$\underline{145{,}466.84}$$

So if she refinances today, then in seven years when she sells the house she will owe the new lender $149,123.96 - $145,466.84 = $<u>3,657.12</u> less than if she had not refinanced and would have to pay off the bigger amount that would be owed to the original lender. (Remember that a loan with 300 months of payments and a .45% monthly interest rate amortizes more rapidly than an equally sized loan with 300 months of payments and a .55% monthly rate? See the discussion toward the end of the Topic 12 outline, and a related spreadsheet you can plug numbers into on the course web site.)

And the present value of a $3,657.12 benefit that would be received in 7 years = 84 months, if we use the low-risk discount rate since that smaller obligation would be assured if refinancing were to occur, is

$$\$3{,}657.12 \left(\frac{1}{1.0035} \right)^{84} = \$3{,}657.12 \times .745659 = \$\underline{2{,}726.97}$$

Thus the present value of the benefits of refinancing becomes $9,240.14 + $2,726.97 = $<u>11,967.11</u>. Based on these figures, the NPV of refinancing, when the cost of doing so (already in present value terms) is $9,500 and the PV of the benefits (paying $127.15 less per month for just 84 months and then settling up in 84 months at a cost $3,657.12 less than with no refinancing) is slightly positive, and thus in theory refinancing is wealth-enhancing/worth doing: $11,967.11 – $9,500 = $<u>2,467.11</u>.

When someone refinances a home mortgage loan, there is no requirement that the life of the new loan equal the remaining maturity on the initial loan, or that the principal borrowed on the new loan be exactly the amount still owed on the initial loan simply rolled over to a new lender. (In fact the "new" lender can be the "old" lender; when interest rates decline borrowers often approach their current lenders to arrange lower-rate replacement loans. The current lender may already have sold the note in the secondary mortgage market, e.g. to Fannie Mae or Freddie Mac, and thus does not care that the new loan will produce lower monthly payments. Or the existing lender may still hold the higher-rate note in its portfolio, and is not thrilled to see the monthly payments it collects go down, but the borrower could as easily refinance with a different lender, so the current lender might as well at least earn the fees the borrower pays when the new loan is created.)

In this example we keep the term and principal owed on a possible replacement loan equal to the remaining term and remaining principal owed on the initial loan, just to prevent complications from obscuring the main points that concern us in an introductory coverage. In the FIL 360 course we analyze the more difficult cases in which borrowers refinance to loans with maturities shorter or longer than the remaining lives of their initial loans, and cases in which replacement loans' initial principal balances are higher (perhaps a "cash-out refinance" that involves borrowing more, which does not imperil the lender's equity cushion if significant initial principal has been repaid and/or the home has gone up in value) or lower than what is owed on the loans they are replacing.

Note also that there is an NPV impact on the lender when a borrower repays a loan before the agreed term ends. Generally the lender loses wealth (realizes a negative NPV) if repayment occurs when current market interest rates are lower than they were at the time the loan was originated (perhaps the borrower is refinancing specifically to get a lower interest rate, and the lender must then re-lend the received principal at today's lower rate). But the lender gains wealth (realizes a positive NPV) if a borrower repays when interest rates are higher than they were at the time the loan was originated (perhaps the borrower has sold the house in connection with a job transfer that has no connection to interest rate levels, and the lender is able to re-lend the received principal at today's new, higher rate). We cover this issue in more detail in FIL 360. •