A Primer on Duration, Convexity, and Immunization

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Abstract[§]

The concepts of duration, convexity, and immunization are fundamental tools of asset-liability management. This paper provides a theoretical and practical overview of the concepts, largely missing in the existing literature on the subject, and fills some holes in the body of research on the subject. We not present new research, but rather we provide a new presentation of the underlying theory, which we believe to be of value in the new North American actuarial education system.

Key words and phrases: *duration, convexity, M-squared, immunization, yield curve, term structure of interest rates.*

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1   Introduction

The concepts of duration and convexity are commonly used in the field of asset-liability management. They are important because they provide key measures of sensitivity of the price of a financial instrument to changes in interest rates and they help develop methodologies in interest rate risk management. Traditional approaches used by financial intermediaries often allowed for borrowing at short-term interest rates, relatively lower, and investing at longer-term interest rates, relatively higher, hoping to earn substantial profits from the difference in the level of the two interest rates. Interest rate risk management utilizing the concepts of duration and convexity helps point out the dangers of such a simplistic approach and develops alternatives to it. Thus, the thorough understanding of these two concepts must be an important part of the education of today's actuaries. In North America, the introduction of the concepts of duration and convexity now occur fairly early in the actuarial examination process. The new Society of Actuaries examination system starting in May 2005 will introduce these concepts in the new Financial Mathematics (FM) examination at the level of old Course 2 Society of Actuaries examination. They also are presented in the Society of Actuaries Course 6 examination, as well as Casualty Actuarial Society Examination 8, based on the more theoretical approach of Panjer (1998) and the more practical ones of Fabozzi (2000) or Bodie, Kane, and Marcus (2002).

There is a split in the way duration and convexity are generally presented in the finance and actuarial literature: from a theoretical perspective as rates of change or from a practical perspective as weighted average time to maturity (for duration) or weighted average square of time to maturity (for convexity). These two perspectives are naturally connected, but the nature of connection are not explicitly discussed in the educational actuarial literature.

The objective of this paper is to fill the existing void and give a general overview of the two fundamental concepts. This paper is presented at the level where it is accessible to students who have completed three semesters of calculus and one or two semesters of probability, i.e., at the level of the current Course P Society of Actuaries examination on probability, and have a working knowledge of the theory of interest as presented in the text by Kellison (1991). We hope that this paper will allow future actuaries to combine the theoretical and the practical approaches in their education and training.

2 Duration

2.1 Duration as Derivatives

Duration is a measure of the sensitivity of a financial asset to changes in interest rates. It is based on the assumption of using only one interest rate, which commonly is interpreted as a flat yield curve assumption. As a change in an interest rate amounts to a parallel shift in a flat yield curve, use of duration also commonly is said to assume a parallel shift in the yield curve.

For a given interest rate i, let δ denote the corresponding force of interest, which satisfies $\delta = \ln(1+i)$. Thus if P is the price of a financial asset, we often write P as a function of the interest rate i as $P(i)$ or as a function of the force of interest δ, $P(\delta)$. This notation is necessitated by the simultaneous use of the interest rate and the force of interest in our presentation.

Definition 1. *The duration of a security with price P is*

$$D(P) = -\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}i} = -\frac{d}{\mathrm{d}i}\ln(P).\tag{1}$$

We should emphasize the following features of this definition: (i) it makes no assumptions about the type or structure of the security; (ii) it applies whether or not the cash flows of the security are dependent on interest rates; (iii) it applies whether or not the security is risk-free; and (iv) it applies whether or not the security contains interest rate options. This definition applies to *all* securities, including bonds, mortgages, options, stocks, swaps, interest-only strips, etc. Later in this paper we will analyze this definition under some specific assumptions about the security.

The term $-\mathrm{d}P/\mathrm{d}i$ usually is termed the *dollar duration* of the security. We propose to abandon this term for a less restrictive one: *monetary duration*, which we believe to be better because of lack of reference to a specific national currency.

Because of the standard approximation of the derivative with a difference quotient, we see that for sufficiently small Δi:

$$D(P) \approx \frac{P(i-\Delta i) - P(i)}{P(i)\,\Delta i} = \frac{\dfrac{P(i-\Delta i) - P(i)}{P(i)}}{\Delta i}.\tag{2}$$

Equation (2) means that duration gives us the approximate ratio of the percentage loss in the value of the security per unit of interest rates, a

commonly used approximation. Note also that because the loss in the value of the security $[P(i - \Delta i) - P(i)]/P(i)$ is expressed as percentage and Δi is in percent per year (if the interest rate used is annual, a common standard), the unit for duration is a year (or, in general, the time unit over which the interest rate is given).

Instead of defining duration in terms of the derivative with respect to the interest rate, one could define duration with respect to the force of interest as follows:

Definition 2. *The Macaulay duration of a security with price P is*

$$D_M(P) = -\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}\delta} = -\frac{d}{\mathrm{d}\delta}\ln P. \tag{3}$$

Clearly these two definitions of duration are connected because

$$\frac{\mathrm{d}P}{\mathrm{d}i} = \frac{\mathrm{d}P}{\mathrm{d}\delta}\frac{\mathrm{d}\delta}{\mathrm{d}i}.$$

Hence it follows that

$$D(P) = \frac{1}{1+i}D_M(P). \tag{4}$$

Suppose we have n securities, and let $\mathrm{Dur}(P_k)$ be either the duration or Macaulay duration of the kth security whose price is $P_k > 0$, for $k = 1, 2, \ldots, n$. If a security has price $P > 0$ that is a linear combination of the prices of these n securities, i.e.,

$$P = b_1 P_1 + b_2 P_2 + \cdots + b_n P_n \tag{5}$$

where the b_ks are constants, then it follows directly from the definition of duration or Macaulay duration that:

$$\mathrm{Dur}(P) = \sum_{k=1}^{n} b_k \frac{P_k}{P}\mathrm{Dur}(P_k). \tag{6}$$

2.2 Duration as Weighted Averages

Let A_t denote the known non-zero cash flow at time t produced by a security under consideration, and let $\mathcal{T}$ denote the set of future time points at which the security's cash flow occurs. For simplicity we further assume that A_t does not depend on i. Throughout this paper we say a security has deterministic cash flows when its cash flows do not depend on the interest rate. At first, we will assume that the cash

flows are discrete and that there is only one interest rate regardless of maturity (i.e., the yield curve is flat).

Then the present value of the security, i.e., its price, is:

$$P = \sum_{t \in \mathcal{T}} \frac{A_t}{(1 + i)^t}. \tag{7}$$

In this case, monetary duration is given by:

$$-\frac{\mathrm{d}P}{\mathrm{d}i} = \sum_{t \in \mathcal{T}} \frac{tA_t}{(1 + i)^{t+1}} = \frac{1}{(1 + i)} \sum_{t \in \mathcal{T}} t\,\mathrm{PV}\,(A_t), \tag{8}$$

where

$$\mathrm{PV}\,(A_t) = \frac{A_t}{(1 + i)^t}$$

is the present value of the cash flow A_t. The duration of this security is therefore:

$$D\,(P) = \frac{1}{P} \sum_{t \in \mathcal{T}} \frac{tA_t}{(1 + i)^{t+1}} = \frac{1}{(1 + i)} \sum_{t \in \mathcal{T}} t\,w_t \tag{9}$$

where w_t is the weight function

$$w_t = \mathrm{PV}\,(A_t)/P. \tag{10}$$

Thus duration turns out to be a weighted average time to maturity, modified by the factor $1/(1 + i)$. For this reason, the concept of duration as introduced in equation (1) is commonly called *modified duration* for securities with deterministic cash flows. For securities with cash flows that are dependent on interest rates, which causes the cash flows to be random in nature if interest rates are random, duration is most often termed *effective duration*. For such securities, however, duration still is defined as in equation (1).

The weighted average time to maturity concept is actually the original idea of duration. For a security with deterministic cash flows, Macaulay (1938) defined duration as

$$D_M\,(P) = \frac{1}{P} \sum_{t \in \mathcal{T}} \frac{tA_t}{(1 + i)^t} = \sum_{t \in \mathcal{T}} t\,w_t. \tag{11}$$

If the weights w_t are positive, we can introduce a discrete random variable T with probability distribution with $\mathbb{P}\,(T = t) = w_t$. It then becomes clear from equation (11) that the Macaulay duration is the expected value of T for this probability distribution, i.e., $D_M\,(P) = \mathbb{E}\,(T)$.

We quickly can see from equation (11) that the duration of a single payment at a future time t is $t/(1+i)$ and its Macaulay duration is t.

2.3   Duration Using Nominal Rates

Duration can be calculated with respect to nominal interest rates, such as semi-annual rates ($i^{(2)}$), quarterly rates ($i^{(4)}$), or ($i^{(12)}$). Recall the definition of a nominal rate, $i^{(m)}$, as given in Kellison (1991):

$$\left(1 + \frac{i^{(m)}}{m}\right)^m = (1+j)^m = 1 + i = e^\delta$$

where $j = i^{(m)}/m$. Therefore

$$\frac{\mathrm{d}i}{\mathrm{d}i^{(m)}} = \left(1 + \frac{i^{(m)}}{m}\right)^{m-1} = \frac{1+i}{1 + \frac{i^{(m)}}{m}} \tag{12}$$

and

$$\frac{\mathrm{d}\delta}{\mathrm{d}i^{(m)}} = \frac{\mathrm{d}\delta}{\mathrm{d}i} \cdot \frac{\mathrm{d}i}{\mathrm{d}i^{(m)}} = \frac{1}{1 + \frac{i^{(m)}}{m}}. \tag{13}$$

The definition of duration with respect to $i^{(m)}$ is

$$D^{(m)}(P) = -\frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}i^{(m)}}. \tag{14}$$

It is easy to prove that

$$D^{(m)} = \frac{1+i}{1 + \frac{i^{(m)}}{m}}D = \frac{1}{1 + \frac{i^{(m)}}{m}}D_M. \tag{15}$$

It is not common to consider the case of continuous stream of payments for calculation of duration, because such securities do not exist in reality. We briefly consider such hypothetical securities for purely theoretical purposes. Suppose a security has continuous cash flows of $A_t\mathrm{d}t$ in $(t, t+\mathrm{d}t)$, with a constant force of interest δ. The security's price, P, is given by

$$P(\delta) = \int_0^\infty e^{-t\delta}A_t\mathrm{d}t \tag{16}$$

assuming the integral exists. Its Macaulay duration is:

$$D_M\left(P\left(\delta\right)\right) = -\frac{P'\left(\delta\right)}{P\left(\delta\right)} = \frac{\int\limits_0^\infty te^{-t\delta}A_t \mathrm{d}t}{\int\limits_0^\infty e^{-t\delta}A_t \mathrm{d}t}. \tag{17}$$

2.4   Some Examples

Thus far we have assumed that the security's cash flows do not depend on the interest rate. What if there is such dependence? We will now consider a few examples of such securities.

Example 1. Consider a discrete security paying a cash flow $A_t = e^{t\delta}$ at a single time t. Its price (present value) is $P = 1$. As $-\mathrm{d}P/\mathrm{d}\delta = 0$, and the duration of this security is zero. From equation (6), any linear combination of instruments like this, paying the accumulated value of a monetary unit at a given interest rate, also will have duration of zero.

Example 2. Similarly, if a discrete security with a single cash flow of $A_t = e^{(t-1)\delta}$ at time t, its price is $P = e^{-\delta}$ and duration of 1.

These two examples illustrate the well-known fact that floating rate securities[1] indexed to a short term rate (i.e., rate that resets somewhere between times 0 and 1 year) have durations between 0 and 1. By using the same argument, one can show that the duration of a floating rate security that resets every n-years and with no restrictions on the level of the new rate after reset (so that the new rate can fully adjust to the market level of the interest rates) is the same as the duration on an otherwise identical n-year bond.

Example 3. Consider a security that is an n-year certain annuity-immediate with level payments of $1/m$ made m times per year for n years, i.e., payments are made at times $1/m, 2/m, ...,(nm-1)/m$. Assuming a constant interest rate to maturity of i, the price of this security is $P = a_{\overline{n}|i}^{(m)}$. It follows that the Macaulay duration of this security is:

$$D_M\left(a_{\overline{n}|i}^{(m)}\right) = \frac{1}{a_{\overline{n}|i}^{(m)}}\sum_{k=1}^{nm}\frac{k}{m}\cdot\frac{1}{m}\cdot(1+i)^{-\frac{k}{m}}$$

$$= \frac{1}{d^{(m)}} - \frac{n}{((1+i)^n - 1)}, \tag{18}$$

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[1]Floating rate securities are securities whose coupons *reset*, i.e., change in a manner consistent with the market level of interest rates.

where $d^{(m)} = 1 - (1 + i)^{-\frac{1}{m}}$.

Example 4. On the other hand, if the security under consideration is a continuous annuity paid for n years, then its price is $P = \bar{a}_{\overline{n}|i}$ and its Macaulay duration is

$$D_M\left(\bar{a}_{\overline{n}|i}\right) = \frac{1}{\delta} - \frac{n}{e^{n\delta} - 1}. \tag{19}$$

The price and duration follow directly from those in example 3 above by letting $m \to \infty$ in $a^{(m)}_{\overline{n}|i}$ and in equation (18).

Note that the second terms in (18) and in (19) are identical. When $n \to \infty$, the limit is $1/d^{(m)}$, which is the price of a discrete perpetuity-immediate, and $1/\delta$, which is the price of a continuous perpetuity. Note that the duration of a continuous perpetuity is its price.

Example 5. What would be the Macaulay duration of a perpetuity-due? As every payment of such a perpetuity arrives exactly an mth of a year before the corresponding payment of a perpetuity-immediate, its Macaulay duration is

$$\frac{1}{d^{(m)}} - \frac{1}{m} = \frac{1}{i^{(m)}}.$$

Thus the Macaulay duration of a perpetuity-due is the price of the corresponding perpetuity-immediate, while the Macaulay duration of a perpetuity-immediate is the price of the corresponding perpetuity-due.

Example 6. Finally, consider a security that is a risk-free bond with principal value of one dollar, maturing n years from now, paying an equal coupon of $r^{(m)}/m$ per unit of principal value m times a year at the end of each mth of a year, with $i^{(m)}$ being the nominal annual yield interest rate compounded m times a year at the time of bond issue and i being the annual effective interest rate. The price of this bond is

$$P = r^{(m)} a^{(m)}_{\overline{n}|i} + (1 + i)^{-n}$$

and its Macaulay duration, calculated here as a weighted-average time to maturity as in equation (11), is:

$$D_M(P) = \frac{1}{P}\left[\sum_{k=1}^{nm}\left(\frac{k}{m}\cdot\frac{r^{(m)}}{m}\cdot(1+i)^{-\frac{k}{m}}\right) + n\cdot(1+i)^{-n}\right]$$

$$= \frac{1}{P}\left[r^{(m)}(Ia)^{(m)}_{\overline{n}|\,i} + n(1+i)^{-n}\right]$$

$$= \frac{1}{P}\left[\frac{r^{(m)}}{i^{(m)}}\ddot{a}^{(m)}_{\overline{n}|\,i} + \left(\frac{r^{(m)}}{i^{(m)}}-1\right)n(1+i)^{-n}\right]. \tag{20}$$

If the bond is currently trading at par then $r^{(m)} = i^{(m)}$ so that the price of the bond is $P = 1$ and its Macaulay duration reduces to

$$D_M(P) = \ddot{a}^{(m)}_{\overline{n}|\,i}. \tag{21}$$

2.5 Effective Duration

In the above examples there was a direct functional relationship between the cash flows and interest rate. In practice, however, securities have complex relationships between cash flows and interest rates, and one cannot generally write a direct functional relationship between the cash flows and interest rate. In such cases duration is usually estimated rather than directly calculated.

The standard approximation approach is to use the Taylor series expansion of the price as a function of interest rate:

$$P(i+\Delta i) = P(i) + \frac{\mathrm{d}P}{\mathrm{d}i}\Delta i + \frac{1}{2}\frac{d^2P}{di^2}(\Delta i)^2 + \cdots \tag{22}$$

Ignoring terms involving $(\Delta i)^2$ and higher yields

$$-\frac{\mathrm{d}P}{\mathrm{d}i}\frac{1}{P} \approx \frac{P(i)-P(i+\Delta i)}{(\Delta i)\,P(i)} \tag{23}$$

and

$$-\frac{\mathrm{d}P}{\mathrm{d}i}\frac{1}{P} \approx \frac{P(i-\Delta i)-P(i)}{(\Delta i)\,P(i)}. \tag{24}$$

We obtain a commonly used approximation of duration by averaging the right side of equations (23) and (24) yields

$$D_E(P) \approx \frac{P(i-\Delta i)-P(i+\Delta i)}{2P(i)\,(\Delta i)}. \tag{25}$$

Because this approximation can deal with any interest rate and/or any default options embedded in the security, $D_E(P)$ is often called an *option-adjusted duration* or *effective duration.*

3   Convexity

For any security with price P, the quantity:

$$C(P) = \frac{1}{P} \frac{d^2 P}{di^2} \tag{26}$$

is called the *convexity* of the security, and

$$C_M(P) = \frac{1}{P} \frac{d^2 P}{d\delta^2} \tag{27}$$

is called the *Macaulay convexity* of the security. As $PD_M(P) = -dP/d\delta$, the monetary duration of the security, we also have:

$$C_M(P) = -\frac{1}{P} \frac{d}{d\delta} (P \cdot D_M(P)) = D_M^2(P) - \frac{dD_M(P)}{d\delta}. \tag{28}$$

The quantity

$$M^2(P) = \frac{d^2 (\ln(P))}{di^2} = -\frac{dD(P)}{di} \tag{29}$$

is called the *M-squared* of the security, while

$$M_M^2(P) = \frac{d^2 (\ln P)}{d\delta^2} = -\frac{dD_M(P)}{d\delta} = C_M(P) - D_M^2(P) \tag{30}$$

will be termed the *Macaulay M-squared.*

For a security with discrete deterministic cash flows so that $P = \sum_{t \in \mathcal{T}} A_t e^{-\delta t}$, we have

$$
\begin{aligned}
C_M(P) &= \frac{1}{P} \sum_{t \in \mathcal{T}} t^2 e^{-\delta t} A_t \\
&= \sum_{t \in \mathcal{T}} t^2 w_t
\end{aligned} \tag{31}
$$

where w_t is defined in equation (10),

$$
\begin{aligned}
M_M^2(P) &= \frac{1}{P} \sum_{t \in \mathcal{T}} (t - D_M(P))^2 e^{-\delta t} A_t \\
&= \sum_{t \in \mathcal{T}} w_t (t - D_M(P))^2,
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
\frac{dM_M^2(P)}{d\delta} &= -\frac{1}{P} \sum_{t \in \mathcal{T}} (t - D_M(P))^3 e^{-\delta t} A_t \\
&= - \sum_{t \in \mathcal{T}} w_t (t - D_M(P))^3 .
\end{aligned}
\tag{33}
$$

Similar expressions can be developed for $C(P)$, $M^2(P)$, and $dM^2(P)/di$.

Equation (32) allows for a relatively simple and intuitive interpretation of Macaulay duration, Macaulay convexity, and Macaulay M-squared of a deterministic security. As we stated before, assuming cash flows are positive, Macaulay duration is the expected time to cash flow with respect to the probability distribution whose probability function (or probability density function, in the case of continuous payments) is $f_T(t) = w_t$. Macaulay convexity is the second moment of this random variable, and Macaulay M-squared is the variance of it. This means that Macaulay duration can be interpreted intuitively as the expected time until maturity of cash flows of a security, Macaulay M-squared is the measure of dispersion of the cash flows of the said security, and Macaulay convexity is a sum of Macaulay M-squared and the square of Macaulay duration.

By the chain rule of calculus,

$$
\frac{dP}{di} = \frac{1}{(1+i)} \frac{dP}{d\delta}
\tag{34}
$$

and

$$
\begin{aligned}
\frac{d^2 P}{di^2} &= \frac{d}{di} \left(\frac{1}{1+i} \frac{dP}{d\delta} \right) \\
&= -\frac{1}{(1+i)^2} \frac{dP}{d\delta} + \frac{1}{(1+i)^2} \frac{d^2 P}{d\delta^2},
\end{aligned}
\tag{35}
$$

which means that

$$
C = \frac{1}{(1+i)^2} D_M + \frac{1}{(1+i)^2} C_M.
\tag{36}
$$

For $M^2 = C - D^2$, we easily can prove that

$$M^2 = \frac{1}{(1+i)^2}C_M + \frac{1}{(1+i)^2}D_M - \frac{1}{(1+i)^2}D_M^2$$
$$= \frac{1}{(1+i)^2}M_M^2 + \frac{1}{(1+i)^2}D_M. \tag{37}$$

For a security with discrete deterministic cash flows A_t (at time t) and price P given in equation (7), then

$$C = \frac{1}{(1+i)^2}\sum_{t\geq 0} t\,(t+1)\,w_t = \frac{1}{(1+i)^2}C_M + \frac{1}{(1+i)^2}D_M. \tag{38}$$

If this security consists of a single payment at time t, then its Macaulay convexity is t^2 and its convexity is

$$C = \frac{t^2}{(1+i)^2} + \frac{t}{(1+i)^2} = \frac{t\,(t+1)}{(1+i)^2}, \tag{39}$$

its M_M^2 is 0, and its M^2 is $t/(1+i)^2$.

Again, we suppose there are n securities. This time, however, we let Conv (P_k) be either the convexity or Macaulay convexity of the kth security whose price is $P_k > 0$, for $k = 1, 2, \ldots, n$. If a security has price $P > 0$ given by equation (5), where the b_ks are constants, then it follows directly from the definition of convexity or Macaulay convexity that:

$$\text{Conv}\,(P) = \sum_{k=1}^{n} b_k \frac{P_k}{P}\text{Conv}\,(P_k). \tag{40}$$

If a security has embedded options (such as direct interest rate options, prepayment option, or the option to default), then the only practical calculation of convexity is as an approximation. Using the Taylor series expansion of equation (22) and ignoring terms in powers of $(\Delta i)^3$ and higher yields

$$P\,(i+\Delta i) - P\,(i) \approx \frac{dP}{di}\Delta i + \frac{1}{2}\frac{d^2P}{di^2}\,(\Delta i)^2$$
$$P\,(i-\Delta i) - P\,(i) \approx \frac{dP}{di}\,(-\Delta i) + \frac{1}{2}\frac{d^2P}{di^2}\,(-\Delta i)^2$$

which are summed to give the following approximation to

$$\frac{d^2P}{di^2}(\Delta i)^2 \approx P(i-\Delta i) - 2P(i) + P(i+\Delta i).$$

It follows that

$$C = \frac{d^2P}{di^2}\frac{1}{P} \approx \frac{P(i-\Delta i) - 2P(i) + P(i+\Delta i)}{P(i)(\Delta i)^2}, \tag{41}$$

which is a popular approximation to C that is used for securities with interest sensitive cash flows.

For nominal interest rates, the convexity measure with respect to $i^{(m)}$ is based on the following result:

$$\begin{aligned}
\frac{d^2P}{d(i^{(m)})^2} &= \frac{d}{di^{(m)}}\left(\frac{dP}{di^{(m)}}\right) = \frac{d}{di^{(m)}}\left(\frac{1}{1+\frac{i^{(m)}}{m}}\frac{dP}{d\delta}\right) \\
&= \frac{1}{\left(1+\frac{i^{(m)}}{m}\right)^2}\frac{d^2P}{d\delta^2} + \frac{1}{\left(1+\frac{i^{(m)}}{m}\right)^2}\frac{1}{m}\left(-\frac{dP}{d\delta}\right).
\end{aligned}$$

Therefore, convexity with respect to $i^{(m)}$, $C^{(m)}$, is

$$C^{(m)} = \frac{1}{\left(1+\frac{i^{(m)}}{m}\right)^2}C_M + \frac{1}{\left(1+\frac{i^{(m)}}{m}\right)^2}\frac{1}{m}D_M. \tag{42}$$

It is worthwhile to note that for $m \to \infty$, equation (42) becomes equation (36). For $m = 1$, the right side of equation (42) reduces to C_M, indicating consistency in both boundary cases.

Let us illustrate the concepts of duration and convexity with a simple example.

Example 7. Consider a bond whose current price is 105 and whose derivative with respect to the yield to maturity is -525. The yield to maturity is an annual effective interest rate of 6%. Then the duration of the bond is:

$$-\frac{1}{P}\cdot\frac{dP}{di} = -\frac{1}{105}\times(-525) = 5.$$

Because the effective measure of duration is equal to the Macaulay duration divided by $1 + i$, we also can calculate the Macaulay duration of this bond as $5 \times 1.06 = 5.30$. Now suppose that for the same bond, the second derivative of the price with respect to the interest rate is 6720. Then its convexity is:

$$\frac{1}{P}\cdot\frac{d^2P}{di^2} = \frac{1}{105}\times 6720 = 64.$$

4　Classical Immunization

Assume that a financial intermediary has assets, $A(i)$, and liabilities, $L(i)$, that depend on the interest rates. Then the surplus, or capital, of the intermediary, $S(i)$, is defined as

$$S(i) = A(i) - L(i).$$

Though in practice the surplus value may be established not by the market, but by the regulatory or accounting principles, it is important that managers of a financial intermediary understand the relationship of surplus value (market value) to interest rate changes.

Redington (1952) proposed an integrated treatment of assets and liabilities through the study of the surplus function $S(i)$. Suppose the objective of the financial intermediary is to prevent the surplus level from changing solely due to interest rate changes. One possible approach to achieving this objective is to structure the assets and liabilities so that the change in the value of S to be close to zero for infinitesimal changes in interest rates, i.e., to have $\Delta S \approx 0$ for $\Delta i \approx 0$. This implies that the financial intermediary must set

$$\frac{dS}{\mathrm{d}i} = \frac{dA}{\mathrm{d}i} - \frac{dL}{\mathrm{d}i} = 0, \tag{43}$$

i.e., the monetary duration of assets must be equal to the monetary duration of liabilities. If, additionally, the financial intermediary wants to ensure that slight interest rate changes yield an increase in the level of its surplus, the following condition must hold:

$$\frac{d^2 S}{\mathrm{d}i^2} > 0, \tag{44}$$

i.e., the surplus is a convex function of the interest rate. This convexity can be achieved by having assets of greater monetary convexity than that of liabilities.

Suppose, instead, the intermediary was more concerned with protecting the ratio of its assets to liabilities, rather than protecting the actual surplus level.[2] In such a case, the intermediary would be interested in setting the derivative with respect to the interest rate of the ratio of assets and liabilities to zero, while keeping its second derivative positive. As the natural logarithm is a strictly increasing function, however, we can transform this ratio as follows:

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[2]This may be a result of the common regulatory concern with capital ratio (i.e., ratio of surplus to assets) or management's desire to control risk by monitoring the capital ratio.

$$R\,(i) = \ln\left(\frac{A\,(i)}{L\,(i)}\right).$$

To protect the surplus ratio level, we set $dR/di = 0$, i.e.,

$$\frac{d\ln(A(i))}{di} = \frac{d\ln(L(i))}{di}, \tag{45}$$

or, equivalently, set the duration of assets equal to the duration of liabilities and simultaneously set $d^2R/di^2 > 0$, i.e.,

$$\frac{d^2\ln(A(i))}{di^2} > \frac{d^2\ln(L(i))}{di^2}, \tag{46}$$

i.e., ensure that the M^2 for the assets is greater than the M^2 for the liabilities. When durations of assets and liabilities are equal, greater M^2 is equivalent to greater convexity, so this condition can be restated as convexity of assets exceeding convexity of liabilities. The approach of equations (45) and (46) is the most common form of classical immunization and is considered to be the standard for applications of immunization.

We should note that classical immunization has many critics, including the present authors, because it violates the no-arbitrage principle of pricing capital assets (Gajek and Ostaszewski, 2002, 2004; Ostaszewski, 2002; and Ostaszewski and Zwiesler, 2002, as well as Panjer, 1998, Chapter 3). The more commonly quoted criticisms of classical immunization include the following:

- Immunization assumes one interest rate, i.e., flat yield curve, which only moves in parallel shifts;

- Immunization assumes only instantaneous infinitely small change in the yield curve, and, of course, such changes are not usually experienced in practice; and

- Immunization requires continuous costly rebalancing due to the continuous changes in the underlying values of the assets and liabilities that result in changes in durations and convexities.

Interestingly, many problems with immunization can be avoided with relatively small modification of the idea. Instead of trying to unrealistically assure that $\Delta S = S\,(i + \Delta i) - S\,(i)$ is always nonnegative, one can instead try to bound ΔS from below by a (possibly negative) quantity that can be made as large as possible via a proper choice of the asset portfolio. We will briefly outline this approach. Note that

$$\Delta S = S\,(i + \Delta i) - S\,(i) = \sum_{t>0} \frac{S_t}{(1+i)^t}\left(\frac{(1+i)^t}{(1+i+\Delta i)^t} - 1\right), \qquad (47)$$

where S_t is the net surplus cash flow at time t. Hence, by the Schwartz Inequality, we have:

$$\Delta S \geq -\left(\sum_{t>0} \frac{S_t^2}{(1+i)^{2t}}\right)^{\frac{1}{2}} \left(\sum_{t>0}\left(\frac{(1+i)^t}{(1+i+\Delta i)^t} - 1\right)^2\right)^{\frac{1}{2}}. \qquad (48)$$

Therefore, the change in surplus value is bounded from below by a product of two quantities: the first one depending on the portfolio structure, and the second one depending only on the change in the interest rate. It is clear from (48) that ΔS might be negative, but if we find a way to decrease the quantity:

$$\sum_{t>0} \frac{S_t^2}{(1+i)^{2t}},$$

which can be termed the immunization risk measure, then we can reduce the risk of decline in surplus value, at least in the worst case scenario. This approach is analyzed in detail by Gajek and Ostaszewski (2004).

Suppose that your company is planning to fund a liability of $1 million to be paid in five years. Assume that the current yield on bonds of all maturities is 4%. Your company can invest in a one-year zero-coupon bond or a ten-year zero-coupon bond to fund this liability. Find the amounts of the two bonds that should be purchased in order to match the duration of the liability. Will such duration-matched portfolio immunize the liability?

The present value of the liability is:

$$\frac{1000000}{1.04^5} \approx 821927.11.$$

The Macaulay duration of the liability is five. Its duration is

$$\frac{5}{1.04} \approx 4.76190476.$$

Let us write w for the portion of the asset portfolio invested in the one-year zero-coupon bond. Then $1 - w$ is the portion invested in the ten-year zero-coupon bond. The duration of the asset portfolio is the weighted average of durations of those two zero-coupon bonds, i.e.,

$$\frac{1}{1.04} \times w + \frac{10}{1.04} \times (1 - w) = \frac{10}{1.04} - \frac{9w}{1.04}.$$

In order to match the duration of the liability, we must have

$$\frac{10}{1.04} - \frac{9w}{1.04} = \frac{5}{1.04}.$$

Therefore, $9w = 5$, and

$$w = \frac{5}{9} \approx 55.56\%.$$

In order to match durations, we must invest 55.56% of the portfolio in the one-year zero-coupon bond and 45.44% in the ten-year zero-coupon bond.

Immunization requires that the asset portfolio has convexity in excess of that of the liability. The convexity of the liability is:

$$\frac{5 \times 6}{1.04^2} \approx 27.7366864.$$

The convexity of the asset portfolio is:

$$\frac{5}{9} \times \frac{1 \times 2}{1.04^2} + \frac{4}{9} \times \frac{10 \times 11}{1.04^2} \approx 46.2278107.$$

Therefore, the asset portfolio has convexity in excess of that of the liability, and the portfolio is immunized.

5   Yield Curve and Multivariate Immunization

5.1   The Yield Curve

So far we have assumed the same interest rate for discounting cash flows for all maturities. In practice, however, the rates used for discounting cash flows for various maturities differ. This can be seen by comparing the actual interest rates for *pure discount bonds*, also known as *zero coupon bonds*, i.e., bonds that make only one payment at maturity, and no intermediate coupon payments. These bonds are discounted at different rates that depend on their remaining term to maturity.

The *yield curve* or *term structure of interest rates* is the pattern of interest rates for discounting cash flows of different maturities. The specific functional relationship between the time of maturity and the corresponding interest rate is usually called the *yield curve*, especially

when represented graphically, while *term structure of interest rates* is the general description of the phenomenon of rates varying for different maturities. When longer term bonds offer higher yield to maturity rates than shorter term bonds (as is usually the case in practice) the pattern of yield rates is termed an *upward sloping yield curve*. If yield to maturity rates are the same for all maturities, we call this pattern a *flat yield curve*. Finally, a rare, but sometimes occurring, situation when longer term yield to maturity rates are lower than shorter-term ones, is termed an *inverted yield curve*.

When practitioners estimate the yield curve, they begin with the yield rates of bonds that are perceived to be risk-free. In the United States, the most common bonds utilized as risk-free bonds are those issued by the federal government, i.e., United States Treasury Bills (those with maturities up to a year), Treasury Notes (those with maturities between one and ten years), and Treasury Bonds (those with maturities of ten years or more). But this explanation does not make it clear what interest rate is used in the yield curve for each maturity. There are three ways to define the yield curve (and term structure of interest rates):

1. Assign to each term to maturity the yield rate of a risk-free bond with that term to maturity and trading at par, i.e., trading at its redemption value. The resulting yield curve is termed the *bond yield curve*;

2. Assign to each maturity the yield rate on a risk-free zero-coupon bond of that maturity. This yield curve is called the *spot curve*, and the interest rates given by it are called *spot rates*; and

3. Use the short-term interest rates in future time periods implied by current bond spot rates.

Let us explain the concepts of short-term interest rates and forward rates. A *short-term interest rate* (or *short rate*) refers to an interest rate applicable for a short period of time, up to one year, including the possibility of an instantaneous rate over the next infinitesimal period of time. A *spot interest rate* (or *spot rate*) for maturity n periods, s_n, is an interest rate payable on a loan of maturity n periods that starts immediately and accumulates interest to maturity, , $n = 1, 2, \ldots$. A single period *forward interest rate* (or *forward rate*), f_t, is an interest rate payable on a future loan that commences at time t until time $t + 1$, $t = 0, 1, 2, \ldots$.

If we use the one-year rate as the short rate for the purpose of deriving forward rates, we have the following relationship:

$$(1 + s_n)^n = (1 + f_0)(1 + f_1)\ldots(1 + f_{n-1}). \tag{49}$$

We also have

$$1 + f_{n-1} = \frac{(1 + s_n)^n}{(1 + s_{n-1})^{n-1}}. \tag{50}$$

The yield curve also can be studied for the continuously compounded interest rate, i.e., for the force of interest, δ_t, which is expressed as a function of time.

The distinction between the spot rate and the forward rate is best explained by presenting their mathematical relationship. If δ_t is the spot force of interest for time t and φ_t is the forward force of interest at time t, then the accumulated value at time t of a monetary unit invested at time 0 is:

$$\left(e^{\delta_t}\right)^t = e^{\int_0^t \varphi_s ds}. \tag{51}$$

Therefore we have

$$\delta_t = \frac{1}{t} \int_0^t \varphi_s ds,$$

i.e., the spot rate for time t is the mean value of the forward rates between times 0 and t. By the fundamental theorem of calculus,

$$\varphi_t = t\frac{d\delta_t}{dt} + \delta_t. \tag{52}$$

This shows us that $\varphi_t > \delta_t$ if and only if $d\delta_t/dt > 0$.

We will illustrate the use of spot and forward rates with a simple example. Suppose a 4%, 1000 par, annual coupon bond with a four-year maturity exists in a market in which the spot rates are:

- 1 year spot rate is $s_1 = 3.0\%$,

- 2 year spot rate is $s_2 = 3.5\%$,

- 3 year spot rate is $s_3 = 4.0\%$,

- 4 year spot rate is $s_4 = 4.5\%$.

Then the value of this bond is the present value of its cash flows discounted using the spot rates:

$$\frac{40}{1.03} + \frac{40}{1.035^2} + \frac{40}{1.04^3} + \frac{1040}{1.045^4} \approx 983.84.$$

For the same date, we also can calculate the corresponding one-year forward rates at times 0, 1, 2, 3 (i.e., from time 0 to time 1, from time 1 to time 2, from time 2 to time 3, and from time 3 to time 4) as follows:

- The forward rate from time 0 to time 1 is $f_1 = 3.0\%$, same as the one year spot rate.

- The forward rate from time 1 to time 2, denoted by f_2, is derived from the condition

$$(1 + 0.03)(1 + f_2) = 1.035^2,$$

  so that

$$1 + f_2 = \frac{1.035^2}{1.03} \approx 1.04002427,$$

  and

$$f_2 \approx 4.002427\%.$$

- The forward rate from time 2 to time 3, denoted by f_3, is derived from the condition

$$(1 + s_2)^2 (1 + f_3) = 1.035^2 \times (1 + f_3) = 1.04^3,$$

  so that

$$1 + f_3 = \frac{1.04^3}{1.035^2},$$

  and

$$f_3 \approx 5.007258\%.$$

- The forward rate from time 3 to time 4, denoted by f_4, is derived from the condition

$$(1 + s_3)^3 (1 + f_4) = 1.04^3 \times (1 + f_4) = 1.045^4,$$

  so that

$$1 + f_4 = \frac{1.045^4}{1.04^3},$$

  and

$$f_4 \approx 6.014469\%.$$

5.2 Multivariate Immunization

To address some of the weaknesses of classical immunization, Ho (1990) and Reitano (1991a, 1991b) developed a multivariate generalization of duration and convexity. They replaced the single interest rate parameter i by a yield curve vector $\vec{i} = (i_1, ..., i_n)$, where the coordinates of the yield curve vector correspond to certain set of key rates. Reitano (1991a) wrote: "For example, one might base a yield curve on observed market yields at maturities of 0.25, 0.5, 1, 2, 3, 4, 5, 7, 10, 20 and 30 years." The price function is then $P(i_1, ..., i_n)$. Instead of analyzing derivatives with respect to one interest rate variable, one could use multivariate calculus tools to study the price function.

There is one objection that could be raised with respect to this approach. For example, when analyzing a deterministic function of several variables $f(x_1, x_2, \ldots, x_n)$, it is implicitly assumed that the variables x_j and x_k are mutually independent, i.e., $\partial x_j / \partial x_k = 0$. This is definitely *not* the case when various maturity interest rates are considered. Nevertheless, one can study such multivariate models for the purpose of better understanding their properties.

The quantities $\partial \ln P / \partial i_k$ are termed *partial durations* (Reitano, 1991a, 1991b) or *key-rate durations* (Ho, 1990). The *total duration vector* is:

$$-\frac{P'(i_1, ..., i_n)}{P(i_1, ..., i_n)} = -\frac{1}{P(i_1, ..., i_n)} \left(\frac{\partial P}{\partial i_1}, ..., \frac{\partial P}{\partial i_n} \right). \qquad (53)$$

One also can introduce the standard notion of directional derivative of $P(i_1, ..., i_n)$ in the direction of a vector $\vec{v} = (v_1, ..., v_n)$:

$$P'_{\vec{v}}(i_1, ..., i_n) = \vec{v} \bullet \left(\frac{\partial P}{\partial i_1}, \ldots, \frac{\partial P}{\partial i_n} \right) \qquad (54)$$

where the "$\bullet$" refers to the dot product of the vectors. The second derivative matrix also can be used to define the *total convexity*:

$$\frac{P''(i_1, ..., i_n)}{P(i_1, ..., i_n)} = \frac{1}{P(i_1, ..., i_n)} \left[\frac{\partial^2 P}{\partial i_k \partial i_l} \right]_{1 \leq k, l \leq n}. \qquad (55)$$

One now can view the surplus of an insurance firm as a function of the set of key interest rates chosen. Applying multivariate calculus, we can obtain the two immunization algorithms that are directly analogous to the one-dimensional case:

- To protect the absolute surplus level, set the first derivative (gradient) of the surplus function to zero, i.e.,

$$S'(i_1, ..., i_n) = \vec{0}, \quad \text{or, equivalently} \quad A'(i_1, ..., i_n) = L'(i_1, ..., i_n) \tag{56}$$

where $\vec{0}$ is the zero vector, with all its components being zero and with the symbols A, L referring to assets and liabilities, respectively. In addition we must make the second derivative matrix, $S''(i_1, ..., i_n)$, positive definite.

- To protect the relative surplus level (i.e., surplus ratio), set:

$$-\frac{A'(i_1, ..., i_n)}{A(i_1, ..., i_n)} = -\frac{L'(i_1, ..., i_n)}{L(i_1, ..., i_n)} \tag{57}$$

and make the total convexity matrix positive definite.

It should be noted (Panjer, 1998, Chapter 3) that key-rate immunization with respect to a large number of key-rates, large enough to be effectively exhaustive of all possible rates determining the yield curve, forces the immunized portfolio toward an exact cash flow match for the corresponding liabilities. While such cash flow matching does provide complete protection against interest rate risk, it is generally more expensive than an immunizing portfolio; if cash flow matching were our objective, this entire analysis would have been unnecessary.

6   Closing Comments

Duration, convexity, and immunization too often are taught in a simplified or even simplistic way and from a perspective somewhat conflicting with that of actuarial practice. We hope that this primer will be a useful tool for practicing actuaries, and others interested in measures of sensitivity with respect to interest rates.

This paper covers some of the material currently included in the Financial Mathematics examination in the new actuarial education system in North America effective in 2005, and we hope that our work can be of value to candidates studying for that examination.

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