THE MINIMUM SIZE OF A FINITE SUBSPACE PARTITION

ESMERALDA L. NĂSTASE† AND PAPA A. SISSOKHO‡

Abstract. A subspace partition of $\mathcal{P} = \text{PG}(n, q)$ is a collection of subspaces of $\mathcal{P}$ whose pairwise intersection is empty. Let $\sigma_q(n, t)$ denote the minimum size (i.e., minimum number of subspaces) in a subspace partition of $\mathcal{P}$ in which the largest subspace has dimension $t$. In this paper, we determine the value of $\sigma_q(n, t)$ for $n \leq 2t + 2$. Moreover, we use the value of $\sigma_q(2t + 2, t)$ to find the minimum size of a maximal partial $t$-spread in $\text{PG}(3t + 2, q)$.

1. Introduction

A subspace partition (or partition) of $\mathcal{P} = \text{PG}(n, q)$ is a collection of subspaces of $\mathcal{P}$ whose pairwise intersection is empty. Alternatively, we can think of $\mathcal{P}$ as the vector space of dimension $n + 1$ over $\text{GF}(q)$, denoted by $V = V(n + 1, q)$. Then, a subspace partition of $\mathcal{P}$ is equivalent to a partition of $V$ into a collection $\mathcal{S}$ of subspaces in such a way that each nonzero vector of $V$ occurs in exactly one subspace in $\mathcal{S}$. The collection $\mathcal{S}$ is said to be a vector space partition (or simply a partition) of $V$. There is a rich literature about partitions of $V$ (e.g., see [1, 3, 5, 14, 22] and the references therein).

Let $\sigma_q(n, t)$ denote the minimum size (i.e., minimum number of subspaces) in a subspace partition of $\mathcal{P}$ in which the largest subspace has dimension $t$. Since $\sigma_q(n, n) = 1$ and $\sigma_q(n, 0) = (q^{n+1} - 1)/(q - 1)$, we will focus on the case $0 < t < n$. Also note that if $t + 1$ divides $n + 1$, then $\sigma_q(n, t)$ is just the size of a $t$-spread of $\mathcal{P}$, i.e., a subspace partition of $\mathcal{P}$ in which all the subspaces have dimension $t$. For $0 < t < n$, A. Beutelspacher [2] established the following general lower bound:

$$\sigma_q(n, t) \geq q^{\lceil \frac{n+1}{t} \rceil} + 1.$$ 

In a recent manuscript, O. Heden and J. Lehmann [16] established new necessary conditions for the existence of certain subspace partitions. In particular they proved conditions for $\text{PG}(2t - 1, q)$ to admit partitions with subspaces of dimensions $t$ and $d < t$ (see Theorem 11 in [16]). In the process, they also prove that for any partition $\Pi$ of $\text{PG}(n, q)$ such that $t$ is the highest dimension that occurs in $\Pi$ and $d < n - t$ is another dimension that occurs in $\Pi$,

$$|\Pi| \geq q^{t+1} + q^{d+1} + 1.$$ 

Their result is an improvement on a result of G. Spera [22] who proved that if $\Pi$ is a partition of $\text{PG}(n, q)$ such that $s$ is the smallest dimension that occurs in $\Pi$, then $|\Pi| \geq q^{s+1} + 1$. In another related paper, A. Khare [20] established a sharp bound for the minimum number of subspaces needed to cover (not necessary partition) a given vector space $V$ (finite or infinite) into subspaces with fixed co-dimension.

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$k < \infty$. As observed above, if $V \cong \text{PG}(n, q)$ and $k + 1$ divides $n + 1$, then a $k$-spread provides a minimum covering of $V$.

Let $n$ and $t$ be fixed integers such that $0 < t < n$. In this paper, we prove that (see Corollary 7)

$$\sigma_q(n, t) = q^{t+1} + 1 \quad \text{for } n < 2t + 2,$$

and

$$\sigma_q(n, t) = q^{t+2} + q^\lceil t/2 \rceil + 1 \quad \text{for } n = 2t + 2.$$

We combine this result with a construction of P. Govaerts [13] to show (see Theorem 11) that the minimum size of a maximal partial $t$-spread in $\text{PG}(3t + 2, q)$ is given by $\sigma_q(2t + 2, t)$.

2. Main results

In our proofs, we use several results of Heden and Lehmann [16]. We start with some preliminary definitions introduced in [16].

Let $n \geq 2$ be an integer and let $\Pi$ be a subspace partition of $\mathbb{P} = \text{PG}(n, q)$ with $m_i$ subspaces of dimension $i$, $0 \leq i \leq n - 1$. Let $H$ be any $(n - 1)$-subspace of $\mathbb{P}$ and let $b_i \leq m_i$ be the number of subspaces of $\Pi$ that are contained in $H$. We say that $(m_{n-1}, \ldots, m_0)$ is the type of $\Pi$ and $b = (b_{n-1}, \ldots, b_0)$ is the type of the hyperplane $H$ (with respect to $\Pi$). Let $s_b$ denote the number of hyperplanes in $\mathbb{P}$ with type $b$ and define the set

$$B = \{ b : s_b > 0 \}.$$

For $0 \leq i \leq n$, let

$$\theta_i = \frac{q^{i+1} - 1}{q - 1}$$

denote the number of points in an $i$-space of $\Pi$, and let

$$h_q(n, i) = \max \left\{ 0, \frac{q^{n-i} - 1}{q - 1} \right\}$$

be the number of $(n - 1)$-spaces (or hyperplanes) in $\mathbb{P}$ that contain a fixed $i$-space of $\mathbb{P}$. Finally, for $n = 2t + 2$, we define

$$\mu_q(n, t) = q^{t+2} + 1.$$

By using a construction of A. Beutelspacher [1] (which was rediscovered by T. Bu [5]), it is easy to see that there is a partial $t$-spread in $\text{PG}(n, q)$ of size $\mu_q(n, t)$.

**Lemma 1** (Heden and Lehmann [16]). Let $\Pi$ be a subspace partition of $\text{PG}(n, q)$ and let $(b_{n-1}, \ldots, b_0)$ be the type of the hyperplane $H$ with respect to $\Pi$. Then the number of subspaces in $\Pi$ is

$$|\Pi| = 1 + \sum_{i=0}^{n-1} b_i q^{i+1}.$$

**Lemma 2** (Heden and Lehmann [16]). Let $\Pi$ be a subspace partition of $\text{PG}(n, q)$ of type $(m_{n-1}, \ldots, m_0)$ and let $b = (b_{n-1}, \ldots, b_0)$ be the type of the hyperplane $H$ with respect to $\Pi$. Let $s_b$ denote the number of hyperplanes in $\text{PG}(n, q)$ with type $b$ and suppose that $0 \leq d, \ell \leq n - 2$. Then

$$(i) \sum_{b \in B} s_b = \frac{2^{n+1} - 1}{q - 1}.$$
\[(ii) \sum_{b \in B} b_{q} s_b = m_{d} h_{q}(n, d) ,\]
\[(iii) \sum_{b \in B} (\frac{b_{q}}{2}) s_b = \left(\frac{m_{d}}{2}\right) h_{q}(n, 2d + 1) ,\]
\[(iv) \sum_{b \in B} b_{q} b_{q} s_b = m_{t} m_{d} h_{q}(n, d + \ell + 1) .\]

We will also use the next lemma of Beutelspacher [1] (also see Bu [5]).

**Lemma 3** (Beutelspacher [1]). Let \(n, d\) be integers such that \(0 \leq d \leq (n - 1)/2\). Then \(\text{PG}(n, q)\) admits a partition with one subspace of dimension \(n - d - 1\) and \(q^{n-d}\) subspaces of dimension \(d\).

We can now prove the following easy observation for the value of \(\sigma_{q}(n, t)\) when \(n < 2t + 2\).

**Proposition 4.** Let \(n\) and \(t\) be fixed integers such that \(0 < t < n \leq 2t + 1\). Then

\[\sigma_{q}(n, t) = q^{t+1} + 1.\]

**Proof.** Since \(0 < t < n \leq 2t + 1\), we have \(n = t + a + 1\) with \(0 \leq a \leq t\). Let \(\Pi\) be an arbitrary subspace partition of \(\mathbb{P} = \text{PG}(n, q)\) whose largest subspace \(U\) has dimension \(t\). Since \(n > t\), we have \(|\Pi| > 1\). So let \(U' \in (\Pi \setminus \{U\})\) be a subspace of largest possible dimension. Then \(\dim(U') \leq a\) since \(n = t + a + 1\). Since \(\Pi\) is arbitrarily chosen, counting the number of subspaces in \(\Pi\) yields

\[(2) \quad \sigma_{q}(n, t) \geq |\Pi| \geq 1 + \frac{\theta_{n} - \theta_{t}}{\theta_{a}} = 1 + q^{t+1}.\]

Now the proposition follows from (2) and the existence (see Lemma 3) of a partition \(\Pi_{0}\) of \(\text{PG}(n, q)\) with one subspace of dimension \(t\) and \(q^{t+1}\) subspaces of dimension \(a\).

To prove our main result, Theorem 6, we first prove the following lemma which may be of independent interest.

**Lemma 5.** Let \(n\) and \(t \geq 1\) be fixed integers such that \(n = 2t + 2\). Let \(\Pi\) be a subspace partition of \(\text{PG}(n, q)\) with no subspace of dimension higher than \(t\). Assume furthermore that \(\Pi\) contains two subspaces of dimensions \(t\) and \(d\) with \(0 \leq d < t\). Then

\[|\Pi| \geq q^{t+2} + q^{d+1} + 1.\]

**Proof.** Let \(\Pi\) be a subspace partition of \(\text{PG}(n, q)\) containing subspaces of dimension \(t\) and \(d\) with \(0 \leq d < t\). Define

\[(3) \quad L = \frac{\theta_{n} - \theta_{t-1} (\mu_{q}(n, t) + q^{d+1})}{\theta_{t} - \theta_{t-1}}.\]

We first show that the lemma holds if \(m_{t} \leq L\). Note that \(\Pi\) is the disjoint union of \(A = \{W \in \Pi : \dim(W) = t\}\) and \(B = \{W \in \Pi : \dim(W) \leq t - 1\}\). Since \(|A| = m_{t}\), we have

\[|\Pi| = |A| + |B| \geq m_{t} + \frac{\theta_{n} - m_{t} \cdot \theta_{t}}{\theta_{t-1}} = \frac{\theta_{n} - m_{t} (\theta_{t} - \theta_{t-1})}{\theta_{t-1}} \geq \frac{\theta_{n} - L (\theta_{t} - \theta_{t-1})}{\theta_{t-1}}\]
\[ \text{This shows that the lemma holds for } m_t \leq L. \]

Now suppose that \( m_t > L \). Since there exists a subspace of dimensions \( t \) and \( d \) in \( \Pi \), we have \( m_t > 0 \) and \( m_d > 0 \). It follows from Lemma 2(iv) that

\[ \sum_{b \in B} b_t b_d s_b = m_t m_d h_q(n, t + d + 1) \neq 0. \]

Moreover,

\[ \sum_{b \in B} b_t b_d s_b = \sum_{b \in B, 0 \leq b_t \leq q - 1} b_t b_d s_b + \sum_{b \in B, b_t \geq q} b_t b_d s_b. \]

If \( \sum_{b \in B, b_t \geq q} b_t b_d s_b \neq 0 \), then there exists \( b \in B \) such that \( b_t \geq q, b_d \geq 1, \) and \( s_b \geq 1 \). In this case, Lemma 1 yields

\[ |\Pi| = \sum_{i=0}^{n-1} b_t q^{t+1} + 1 \geq b_t q^{t+1} + b_d q^{d+1} + 1 \geq q^{t+2} + q^{d+1} + 1, \]

and the lemma follows. So we may assume that \( \sum_{b \in B, b_t \geq q} b_t b_d s_b = 0 \). We will show that this contradicts the assumption \( m_t > L \). From (6) and Lemma 2(ii), we obtain

\[ (q - 1)m_d h_q(n, d) = \sum_{b \in B} (q - 1) \cdot b_d s_b \]
\[ = \sum_{b \in B, 0 \leq b_t \leq q - 1} (q - 1) \cdot b_d s_b + \sum_{b \in B, b_t \geq q} (q - 1) \cdot b_d s_b \]
\[ \geq \sum_{b \in B, 0 \leq b_t \leq q - 1} b_t \cdot b_d s_b + \sum_{b \in B, b_t \geq q} b_t \cdot b_d s_b \]
\[ = \sum_{b \in B} b_t b_d s_b \]
\[ = m_t m_d h_q(n, t + d + 1) \]

Since \( m_d > 0 \), dividing both sides of (8) by \( m_d \) yields

\[ m_t \leq \frac{(q - 1) h_q(n, d)}{h_q(n, t + d + 1)} = \frac{(q - 1)(q^{2t+2-d} - 1)}{q^{t+1-d} - 1}. \]

Since \( 0 \leq d \leq t - 1 \), the right side (9) is maximized when \( d = t - 1 \). Hence

\[ m_t \leq \frac{(q - 1)(q^{2t+2-(t-1)} - 1)}{q^{t+1-(t-1)} - 1} = \frac{(q - 1)(q^{t+3} - 1)}{q^2 - 1} = q^{t+3} - 1. \]

Also, since \( \mu_q(n, t) = q^{t+2} + 1 \) (see (1)) and \( L \) (defined in (3)) is minimized when \( d = t - 1 \), the assumption \( m_t > L \) yields

\[ m_t > L \geq \frac{\theta_{2t+2} - \theta_{t-1} \cdot (\mu_q(n, t) + q^t)}{\theta_t - \theta_{t-1}} \]
\[ = \frac{(q^{2t+3} - 1) - (q^t - 1)(q^{t+2} + q^t + 1)}{(q^{t+1} - q^t)}. \]
Theorem 6. Let $n$ and $t \geq 1$ be fixed integers such that $n = 2t + 2$. Then

$$\sigma_q(n, t) = q^{t+2} + q^{[t/2] + 1} + 1.$$  

Proof. Let $\Pi$ be a subspace partition of $\text{PG}(n, q)$ in which the largest subspace has dimension $t$. Let $\beta = [t/2]$ and define the set

$$G = \{\dim(W) : W \in \Pi \text{ and } \beta \leq \dim(W) \leq t - 1\}.$$  

First, suppose that $G \neq \emptyset$. Then for any $d \in G$, Lemma 5 yields

$$|\Pi| \geq q^{t+2} + q^{s+1} + 1 \geq q^{t+2} + q^t + 1 \geq q^{t+2} + q^{s+1} + 1.$$  

So, we may assume that $G = \emptyset$. Hence, all other subspaces in $\Pi$ have dimensions at most $\beta - 1$. Recall from (1) that $\mu_q(n, t) = q^{t+2} + 1$. We consider the following two cases based on whether $m_t = \mu_q(n, t)$ or not.

Case 1: $m_t = \mu_q(n, t)$.  

If $b_i \geq q + 1$ for some $b \in B$, then

$$|\Pi| = \sum_{i=0}^{n-1} b_i q^{t+1} + 1 \geq b_t \cdot q^{t+1} + 1 \geq q^{t+2} + q^{t+1} + 1 \geq q^{t+2} + q^{s+1} + 1.$$  

If $b_t \leq q$ for all $b \in B$, then

$$q \sum_{b \in B} s_b = \sum_{b \in B} q \cdot s_b \geq \sum_{b \in B} b_t s_b = m_t h_q(n, t).$$  

Using Lemma 2(i) and (ii), we infer that (12) holds if and only if

$$q \left(\frac{q^{n+1} - 1}{q - 1}\right) = q \sum_{b \in B} s_b \geq m_t h_q(n, t) = (q^{t+2} + 1) \cdot \frac{q^{n-t} - 1}{q - 1}$$

$$\Leftrightarrow q^{n+2} - q \geq q^{n+2} - q^{t+2} + q^{n-t} - 1$$

$$\Leftrightarrow q^{n-t} + q = q^{t+2} + q \leq q^{t+2} + 1,$$

which is a contradiction since $q > 1$.

Case 2: $m_t \leq \mu_q(n, t) - 1$. In this case, each subspace in $\Pi$, other than the $m_t$ subspaces, has dimension at most $\beta - 1$ (so at most $\theta_{\beta-1}$ points). Therefore, we can estimate the number of subspaces in $\Pi$ as follows

$$|\Pi| \geq m_t + \frac{\theta_n - m_t \cdot \theta_t}{\theta_{\beta-1}}$$

$$= \frac{\theta_n - m_t (\theta_t - \theta_{\beta-1})}{\theta_{\beta-1}}$$

$$\geq \frac{\theta_n - (\mu_q(n, t) - 1) \cdot (\theta_t - \theta_{\beta-1})}{\theta_{\beta-1}}$$

$$= \frac{(q^{2t+3} - 1) - q^{t+2}(q^{t+1} - q^s)}{q^s - 1}$$

$$\geq q^{t+2} + q^{s+1} + q.$$  

(13)
Now it follows from (10), (11), and (13) that
\[ |\Pi| \geq q^{t+2} + q^{\beta+1} + 1 \]
holds in all cases. Since \( \Pi \) is an arbitrarily chosen subspace partition, we obtain
(14)
\[ \sigma_q(n,t) \geq q^{t+2} + q^{\beta+1} + 1. \]
Moreover, it follows from Lemma 3 that there exists a partition \( \Pi_0 \) of \( \text{PG}(2t+2,q) \)
into one subspace \( W \) of dimension \( t+1 \) and \( q^{t+2} \) subspaces of dimension \( t \). If \( t \)
is even, then \( t + 2 = 2(\beta + 1) \) and we can partition \( W \) into a \( \beta \)-spread containing
\( q^{\beta+1} + 1 \) subspaces. If \( t \) is odd then \( t + 2 = 2\beta + 1 \) and we use Lemma 3 again
to partition \( W \) into one subspace of dimension \( \beta \) and \( q^{\beta+1} \) subspaces of dimension \( \beta - 1 \). This shows that
(15)
\[ q^{t+2} + q^{\beta+1} + 1 = |\Pi_0| \geq \sigma_q(n,t). \]
Finally (14) and (15) yield
\[ \sigma_q(n,t) = q^{t+2} + q^{\beta+1} + 1. \]
\[ \square \]

Proposition 4 and Theorem 6 lead directly to the following corollary.

**Corollary 7.** Let \( n \) and \( t \) be fixed integers such that \( 0 < t < n \). Then
\[ \sigma_q(n,t) = q^{t+1} + 1 \quad \text{for} \quad n < 2t + 2, \]
and
\[ \sigma_q(n,t) = q^{t+2} + q^{\lceil t/2 \rceil + 1} + 1 \quad \text{for} \quad n = 2t + 2. \]

**Proof.** This follows directly from Proposition 4 and Theorem 6. \[ \square \]

We conclude this section by proposing the following conjecture.

**Conjecture 8.** Let \( n, k, \) and \( t \) be positive integers such that \( n = k(t+1) \). If \( k \geq 2 \) then
\[ \sigma_q(n,t) = \frac{q^{(t+1)+1}(q^{(k-1)(t+1)} - 1)}{q^{t+1} - 1} + q^{\lceil t/2 \rceil + 1} + 1. \]
Note that Conjecture 8 holds for \( k = 2 \) (see Theorem 6) and \( \sigma_q(n,t) = q^{t+1} + 1 \)
for \( k = 1 \) (see Proposition 4).

3. An application to maximal partial \( t \)-spreads

Let \( \mathbb{P} = \text{PG}(n,q) \) denote the projective space of dimension \( n \) over the Galois field
\( \text{GF}(q) \). A **partial \( t \)-spread** of \( \mathbb{P} \) is a collection \( \mathcal{S} = \{W_1, \ldots, W_k\} \) of \( t \)-dimensional
subspaces of \( \mathbb{P} \) such that \( W_i \cap W_j = \emptyset \) for \( i \neq j \). The number \( |\mathcal{S}| \) is called the **size**
of \( \mathcal{S} \). If \( \mathbb{P} = \bigcup_{W \in \mathcal{S}} W \), then \( \mathcal{S} \) is called a **spread**. It is well-known that a spread
exists if and only if \( t + 1 \) divides \( n + 1 \).

A **maximal** partial \( t \)-spread is one which cannot be extended to a larger one. The problem of classifying the maximal partial \( t \)-spreads of \( \mathbb{P} \) has been extensively studied (see [9, 11, 13, 15, 18, 19]). It has applications in the construction of error-correcting codes [6, 8], orthogonal arrays [7, 10], and factorial designs [21].

Let \( n \) and \( t \) be fixed integers and let \( k \) and \( r \) be the unique integers defined by
\( n - t = k(t + 1) + r - 1 \) and \( 0 \leq r \leq t \). We let \( \tau_q(n,t) \) denote the minimum number
of subspaces in any maximal partial \( t \)-spread of \( \mathbb{P} \). The maximal partial \( t \)-spread
Lemma 9 (Govaerts [13]). Let \( n, k, \) and \( t \geq 0 \) be fixed integers and write \( n = k(t + 1) + t \). If \( k \geq 2 \) then there exist (see page 610 in [13] for a construction) maximal partial \( t \)-spreads of \( \text{PG}(n, q) \) of size \( \tau_q(n, t) \leq \mu_q(n - t, t) + q^{\lceil t/2 \rceil + 1} \). Consequently,

\[
\tau_q(n, t) \leq \mu_q(n - t, t) + q^{\lceil t/2 \rceil + 1}.
\]

We can apply our main result, Theorem 6, to determine the value of \( \tau_q(3t + 2, t) \).

Our strategy is due to Govaerts but we replace his set-partition based analysis with the more appropriate subspace-partition analysis. We first introduce the relevant definitions. A set of points \( B \) of \( \mathbb{P} \) is called a blocking set with respect to the \( t \)-spaces of \( \mathbb{P} \) if \( W \cap B \neq \emptyset \) for any \( t \)-spaces \( W \) in \( \mathbb{P} \). Note that any \((n - t)\)-space of \( \mathbb{P} \) is a blocking set with respect to the \( t \)-spaces of \( \mathbb{P} \). Such blocking sets are called trivial. The following lemma follows from the results of Govaerts (see case 2, page 612 in [13]).

Lemma 10 (Govaerts [13]). Let \( n, k, \) and \( t \) be positive integers such that \( n = k(t + 1) + t \). If \( k \geq 2 \) and \( S \) is a minimum size maximal partial \( t \)-spread of \( \text{PG}(n, q) \), then \( \bigcup_{W \in S} W \) contains a trivial blocking set.

We can use Lemma 10 with \( k = 2 \) to prove the following theorem.

Theorem 11. For any positive integer \( t \), we have

\[
\tau_q(3t + 2, t) \geq \sigma_q(2t + 2, t).
\]

Proof. Let \( S \) be a minimum size maximal partial \( t \)-spread in \( \text{PG}(3t + 2, q) \). Then by Lemma 10, \( A = \bigcup_{W \in S} W \) contains a trivial blocking set. In other words, there exists a \((2t + 2)\)-space \( B \subseteq A \). Let

\[
\Pi_S = \{ W \cap B : W \in S \}.
\]

Since \( B \) is a blocking set with respect to \( t \)-spaces, we have \( W \cap B \neq \emptyset \) for any \( W \in S \). Thus, \( \Pi_S \) is a subspace partition of \( B \cong \text{PG}(2t + 2, q) \) containing subspaces of dimensions at most \( t \). If \( \Pi_S \) contains a \( t \)-subspace, then it follows from Theorem 6 and the minimality of \( S \) that

\[
\tau_q(3t + 2, t) = |S| = |\Pi_S| \geq \sigma_q(2t + 2, t).
\]

If \( \Pi_S \) contains no \( t \)-subspace, then each subspace in \( \Pi_S \) has dimension at most \( t - 1 \) (and contains at most \( \theta_{t-1} \) points). So we can estimate the number of subspaces in \( \Pi_S \) to obtain

\[
\tau_q(3t + 2, t) = |S| = |\Pi_S| \geq \left\lceil \frac{\theta_{2t+2}}{\theta_{t-1}} \right\rceil.
\]
\[ q^{(2t+2)+1} - 1 \]
\[ q^{t+1} - 1 \]
\[ q^{t+2} + q^{\left\lceil \frac{t}{2} \right\rceil +1} + 1 = \sigma_q(2t + 2, t). \]

This concludes the proof of the theorem. \( \square \)

We can now prove the following corollary which determines the number \( \tau_q(3t + 2, t) \) for all \( t \geq 1 \). The cases \( 1 \leq t \leq 2 \) were already known from the work of Govaerts [13].

**Corollary 12.** Let \( t \geq 1 \) be a fixed integer. Then
\[ \tau_q(3t + 2, t) = \sigma_q(2t + 2, t) = q^{t+2} + q^{\left\lceil \frac{t}{2} \right\rceil +1} + 1. \]

**Proof.** This is a direct consequence of Theorem 6, Lemma 9, and Theorem 11. \( \square \)

We believe that if Conjecture 8 is true, it can be combined with Lemma 9 to prove that
\[ \tau_q(n, t) = \sigma_q(n - t, t) = q^{(t+1)+1}(q^{(k-1)(t+1)} - 1) / q^{t+1} - 1 + q^{\left\lceil \frac{t}{2} \right\rceil +1} + 1, \]
for any integers \( k \geq 2 \) and \( t \geq 1 \) such that \( n = k(t + 1) + t \).

We remark that the cases for \( k = 1 \) and \( 1 \leq r \leq t \), i.e., \( 2t + 1 \leq n \leq 3t \), have proved to be difficult. In particular, for \( n = 3 \) and \( t = 1 \), Glynn [12] established the following lower bound
\[ \tau_q(3, 1) \geq 2q, \]
while Gács and Szönyi [11] later proved the following upper bound
\[ \tau_q(3, 1) \leq \begin{cases} (2 \ln q + 1)q + 1, & \text{if } q \text{ odd} \\ (6.1 \ln q + 1)q + 1, & \text{if } q > q_0 \text{ even}, \end{cases} \]

Although the gap between these bounds is somewhat considerable, they are (as far as we know) the best bounds for \( \tau_q(3, 1) \).

Furthermore, there are (e.g., see Hirschfeld [17]) maximal partial 1-spreads of \( \text{PG}(3, q) \) of size \( q^2 - q + 2 \) for any \( q > 3 \), and of size 7 for \( q = 3 \). For a while, it was generally believed that these maximal partial 1-spreads have largest possible size among all maximal partial 1-spreads which are not 1-spreads. However, for \( q = 7 \), Heden [15] constructed a maximal partial 1-spread of size 45. All the maximal partial 1-spreads of \( \text{PG}(3, q) \) of size 45 have subsequently been classified by Blokhuis, Brouwer, and Wilbrink [4].

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