AVOIDING ZERO-SUM SUBSEQUENCES OF PRESCRIBED LENGTH
OVER THE INTEGERS

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Abstract. Let \( t \) and \( k \) be a positive integers, and let \( I_k = \{ i \in \mathbb{Z} : -k \leq i \leq k \} \). Let \( s'_t(I_k) \) be the smallest positive integer \( \ell \) such that every zero-sum sequence \( S \) over \( I_k \) of length \( |S| \geq \ell \) contains a zero-sum subsequence of length \( t \). If no such \( \ell \) exists, then let \( s'_t(I_k) = \infty \).

In this paper, we prove that \( s'_t(I_k) \) is finite if and only if every integer in \([1, D(I_k)]\) divides \( t \), where \( D(I_k) = \max\{2, 2k - 1\} \) is the Davenport constant of \( I_k \). Moreover, we prove that if \( s'_t(I_k) \) is finite, then \( t + k(k - 1) \leq s'_t(I_k) \leq t + (2k - 2)(2k - 3) \). We also show that \( s'_t(I_k) = t + k(k - 1) \) holds for \( k \leq 3 \) and conjecture that this equality holds for any \( k \geq 1 \).

1. Introduction and Main results

We shall follow the notation in [17], by Grynkiewicz. Let \( \mathbb{N} \) be the set of positive integers. Let \( G_0 \) a subset of an abelian group \( G \). A sequence over \( G_0 \) is an unordered list of terms in \( G_0 \), where repetition is allowed. The set of all sequences over \( G_0 \) is denoted by \( F(G_0) \). A sequence with no term is called trivial or empty. If \( S \) is a sequence with terms \( s_i, 1 \leq i \leq n \), we write \( S = s_1 \cdot \ldots \cdot s_n = \prod_{i=1}^{n} s_i \). We say that \( R \) is a subsequence of \( S \) if any term in \( R \) is in \( S \). If \( R \) and \( T \) are subsequences of \( S \) such that \( S = R \cdot T \), then \( R \) is the complementary sequence of \( T \) in \( S \), and vice versa. We also write \( T = S \cdot R^{-1} \) and \( R = S \cdot T^{-1} \). For every sequence \( S = s_1 \cdot \ldots \cdot s_n \) over \( G_0 \),

- \( -S = (-s_1) \cdot \ldots \cdot (-s_n) \)
- the length of \( S \) is \( |S| = n \);
- the sum of \( S \) is \( \sigma(S) = s_1 + s_2 + \ldots + s_n \);
- the subsequence-sum of \( S \) is \( \Sigma(S) = \{ \sigma(R) : R \text{ is a subsequence of } S \} \).

For any sequence \( R \) over \( G_0 \) and any integer \( d \geq 0 \),

\[ R^{[0]} \text{ is the trivial sequence, and } R^{[d]} = \underbrace{R \cdot \ldots \cdot R}_{d} \text{ for } d > 0. \]

A sequence with sum 0 is called zero-sum. The set of all zero-sum sequences over \( G_0 \) is denoted by \( B(G_0) \). A zero-sum sequence is called minimal if it does not contain

Key words and phrases. zero-sum sequence over \( \mathbb{Z} \); no zero-sum subsequence of a given length.
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a proper zero-sum subsequence. The Davenport constant of $G_0$, denoted by $D(G_0)$ is the maximum length of a minimal zero-sum sequence over $G_0$. The research on zero-sum theory is quite extensive when $G$ is a finite abelian groups (e.g., see [4, 7, 9, 10] and the references therein). However, there is less activity when $G$ is infinite (e.g., see [1, 5] and the references therein). The study of the particular case $G = \mathbb{Z}^r$ was explicitly suggested by Baeth and Geroldinger [2] due to their relevance to direct-sum decompositions of modules. In a recent paper, Baeth et al. [3] studied the Davenport explicitly suggested by Baeth and Geroldinger [2] due to their relevance to direct-sum decompositions of modules. In a recent paper, Baeth et al. [3] studied the Davenport constant of $G_0 \subseteq \mathbb{Z}^r$. The Davenport constant of an interval in $\mathbb{Z}$ was first derived (see Theorem 1) by Lambert [15] (also see [6, 19, 20] for related work.) In a recent paper, Plagne and Tringali [16], studied the Davenport constant of the cartesian product of intervals of $\mathbb{Z}$.

For any integers $x$ and $y$ with $x \leq y$, let $[x, y] = \{i \in \mathbb{Z} : x \leq i \leq y\}$. For $k \in \mathbb{N}$, let $I_k = [-k, k]$.

**Theorem 1** (Lambert [15]). $D(I_k) = \max\{2, 2k - 1\}$ for any $k \in \mathbb{N}$.

For $G$ finite and $G_0 \subseteq G$, let $s_t(G_0)$ be the smallest integer $\ell$ such that any sequence $S \in \mathcal{F}(G_0)$ of length $|S| \geq \ell$ contains a zero-sum subsequence of length $t$. If $t = \exp(G)$, then $s_t(G_0)$ is called the Erdős–Ginzburg–Ziv constant and is denoted by $s(G)$. In 1961, Erdős–Ginzburg–Ziv [7] proved that $s(\mathbb{Z}_n) = 2n - 1$. Reider [18] proved that $s(\mathbb{Z}_p \oplus \mathbb{Z}_p) = 4p - 3$ for any prime $p$. In general, if $G$ has rank two, say $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$ with $1 \leq n_1 \mid n_2$, then $s(G) = 2n_1 + 2n_2 - 3$ (see Theorem 5.8.3 in Geroldinger–Halter–Koch [11]). For groups of higher rank, we refer the reader to Fan–Gao–Zhong [8]. More recently, Gao et al [13] proved that for any integer $k \geq 2$ and any finite $G$ with exponent $n = \exp(G)$, if the difference $n - |G|/n$ is large enough, then $s_{kn}(G) = kn + D(G) - 1$.

Observe that if $G$ is torsion-free and $G_0 \subseteq G$, then for any nonzero $g \in G_0$ and for any $d \in \mathbb{N}$, the sequence $g^{[d]} \in \mathcal{F}(G_0)$ does not contain a zero-sum subsequence. Thus, we will work with the following analogue of $s_t(G_0)$.

**Definition 2.** 1 For any subset $G_0 \subseteq G$, let $\mathcal{S}_t(G_0)$ be the smallest positive integer $\ell$ such that any sequence $S \in \mathcal{B}(G_0)$ of length $|S| \geq \ell$ contains a zero-sum subsequence of length $t$. If no such $\ell$ exists, then let $\mathcal{S}_t(G_0) = \infty$.

If $t = \exp(G)$ is finite, then we denote $s_t(G_0)$ by $s(G)$. Let $r \in \mathbb{N}$ and assume that $G \cong \mathbb{Z}_n^r$. Then $G$ has Property D if any sequence $S \in \mathcal{F}(G)$ of length $s(G) - 1$ that does not admit a zero-sum subsequence has the form $S = T^{[n-1]}$ for some $T \in \mathcal{F}(G)$. Zhong found the following interesting connections between $s(G)$ and $\mathcal{S}(G)$ (see the Appendix for their proofs).

**Lemma 3** (Zhong [21]). Let $G$ be a finite abelian group.

(i) If $\gcd(s(G) - 1, \exp(G)) = 1$, then $\mathcal{S}(G) = s(G)$.

(ii) If $G \cong \mathbb{Z}_n^r$, with $n \geq 3$ and $r \geq 2$. Suppose that $s(G) = c(n - 1) + 1$ and $G$ has Property D. If $\gcd(s(G) - 1, n) = c$, then $\mathcal{S}(G) < s(G)$.

1This formulation was suggested to us by Geroldinger and Zhong [14].
Remark 4 (Zhong [21]).

(i) If \(G \cong \mathbb{Z}_n^2\) with \(n\) odd, then \(s'(G) = s(G)\).
(ii) If \(G \cong \mathbb{Z}_n^2\) with \(h \geq 2\), then \(s'(G) = s(G) - 1\).

In this paper, we prove the following results about \(s'_t(I_k)\), where \(I_k = [-k, k]\).

**Theorem 5.** Let \(k\) and \(t\) be positive integers.

(i) \(s'_t(I_k)\) is finite, then every integer in \([1, D(I_k)]\) divides \(t\).
(ii) If every integer in \([1, D(I_k)]\) divides \(t\), then

\[
t + k(k - 1) \leq s'_t(I_k) \leq t + (2k - 2)(2k - 3).
\]

**Corollary 1.** Let \(t \in \mathbb{N}\) and \(k \in [1, 3]\). Then \(s'_t(I_k) = t + k(k - 1)\) if and only if every integer in \([1, D(I_k)]\) divides \(t\).

**Conjecture 6.** Corollary 1 holds for any \(k \in \mathbb{N}\).

2. **Proofs of the main results**

For any integers \(a\) and \(b\), we denote \(\gcd(a, b)\) by \((a, b)\). We use the abbreviations z.s.s and z.s.s.b for zero-sum sequence(s) and zero-sum subsequence(s), respectively. The letters \(k\) and \(t\) will denote positive integers throughout the paper.

The following lemma gives a lower bound for \(s'_t(I_k)\).

**Lemma 7.** Consider the z.s.s \(U = k \cdot (-1)^{|k|}\) and \(V = (k - 1) \cdot (-1)^{|k-1|}\). Then, \(S = U^{\lfloor \frac{k}{2} \rfloor - 1} \cdot V^{|k|}\) and \(R = U^{k-1} \cdot V^{\lfloor \frac{k}{2} \rfloor - 1}\) are z.s.s that do not contain a z.s.s.b of length \(t\). Thus, \(s'_t(I_k) \geq t + k(k - 1)\).

**Proof.** We prove the lemma for \(S\) only since the proof for \(R\) is similar. By contradiction, assume that \(S\) contains a z.s.s.b of length \(t\). Since \(\sigma(S) = 0\), it follows that \(S\) also contains a z.s.s.b \(S'\) of length \(|S| - t = k(k - 1) - 1\). Moreover, \(S'\) can be written as \(S' = k[a] \cdot (k - 1)[b] \cdot (-1)[c]\) for some nonnegative integers \(a\), \(b\), and \(c\). Hence \(\sigma(S') = ak + b(k - 1) - c = 0\) and \(a + b + c = |S'| = k^2 - k - 1\). Thus

\[
(a + 1)(k + 1) = k(k - b).
\]

Since \(a, b, k \geq 0\), we have \(0 < k - b \leq k\). Since \((k, k + 1) = 1\), we obtain that \(k + 1\) divides \(k - b\), which is a contradiction. Thus \(s'_t(I_k) \geq |S| + 1 = t + k(k - 1)\). \(\square\)

**Example 8.** For \(k = 3\), \(S = (3 \cdot -1 \cdot -1 \cdot -1)^{[14]} \cdot (2 \cdot -1 \cdot -1)^{[3]}\) is a z.s.s of length 65 over \([-3, 3]\) which does not contain a z.s.s.b of length \(t = 60\).

**Lemma 9.** Let \(a, b, x \in \mathbb{N}\). If \(S = a^{\lfloor \frac{k}{x} \rfloor} \cdot (-b)^{\lfloor \frac{k}{x} \rfloor}\) is a z.s.s, then the length of any z.s.s.b of \(S^{[x]}\) is a multiple of \(|S|\).
Proof. Let $S'$ be a z.s.s of $S^{[x]}$. Since the terms of $S$ are $a$ and $-b$, there exist nonnegative integers $h$ and $r$ such that $S' = a^h \cdot (-b)^r$ and

\[
\sigma(S') = ha - rb = 0 \Rightarrow h \frac{a}{(a,b)} = r \frac{b}{(a,b)}.
\]

Since $\left( \frac{b}{(a,b)}, \frac{b}{(a,b)} \right) = 1$, we obtain $\frac{b}{(a,b)}$ divides $h$ and $\frac{a}{(a,b)}$ divides $r$. Thus, $h = p \frac{b}{(a,b)}$ and $r = q \frac{b}{(a,b)}$ for some integers $p$ and $q$. Substituting $h$ and $r$ back into (1) yields $p = q$. Thus,

\[
|S'| = h + r = p \frac{b}{(a,b)} + q \frac{a}{(a,b)} = p|S|.
\]

\[\square\]

Lemma 10. If $s'_t(I_k)$ is finite, then every odd integer in $[1, D(I_k)]$ divides $t$.

Proof. Since the lemma is trivial for $k = 1$, we assume that $k \geq 2$. Then $D(I_k) = 2k - 1$ by Theorem 1. Let $\ell = 2c - 1$ be an odd integer in $[3, D(I_k)]$, and consider the minimal z.s.s $S = c^{[c-1]} \cdot (-c + 1)^{[c]}$. Then, for any $x \in \mathbb{N}$, it follows from Lemma 9 that for any z.s.s $R$ of $S^{[x]}$, $|R|$ divides $|S| = 2c - 1 = \ell$. Since $\ell$ does not divide $t$, there is no z.s.s of $S^{[x]}$ whose length is equal to $t$. Since $x$ is arbitrary, it follows that $s'_t(I_k)$ can be arbitrarily large. This proves the lemma. \[\square\]

To prove the upper bound in Theorem 5(ii), we will use the following lemma which is a directly application a well-known fact: “Any sequence of $n$ integers contains a nonempty subsequence whose sum is divisible by $n$”.

Lemma 11. Let $\beta \in \mathbb{N}$ and $X \in \mathcal{F}(\mathbb{Z})$. If $|X| \geq \beta$, then there exists a factorization $X = X_0 \cdot X_1 \cdot \ldots \cdot X_r$ such that

(i) $|X_0| \leq \beta - 1$ and no subsequence of $X_0$ has a sum that is divisible by $\beta$.

(ii) $|X_j| \leq \beta$ and $\sigma(X_j)$ is divisible by $\beta$ for any $j \in [1, r]$.

We will also use the following lemmas.

Lemma 12. Assume that $k \geq 2$ and that every integer in $[1, D(I_k)]$ divides $t$. Let $S$ be a z.s.s over $I_k = [-k, k]$ that does not contain a z.s.s of length $t$. Let $S = S_1 \cdot \ldots \cdot S_h$ be a factorization of $S$ into minimal z.s.s $S_i$, $1 \leq i \leq h$. If $|S| \geq t + k(k - 1)$, then there exists some length $\beta$ such that $n_\beta = |\{S_i : |S_i| = \beta, 1 \leq i \leq h\}|$ satisfies:

\[
n_\beta > (2k - 2)(2k - 3).
\]

Proof. Recall that $(a, b)$ denotes $\gcd(a, b)$. It is easy to see that

\[
(2k - 3, 2k - 2) = (2k - 2, 2k - 1) = (2k - 3, 2k - 1) = 1.
\]

Since $k > 1$ and every integer in $[1, D(I_k)] = [1, 2k - 1]$ is a factor of $t$, it follows from (2) that $t = p(2k - 1)(2k - 2)(2k - 3)$, for some $p \in \mathbb{N}$. By definition, we have
max$_{1 \leq i \leq h} |S_i| \leq D(I_k) = 2k - 1$. Thus, it follows from the pigeonhole principle that there exists some length $\beta$ such that

$$n_\beta \geq \frac{t + k(k - 1)}{\max_{1 \leq i \leq h} |S_i|} \geq \frac{t + k(k - 1)}{2k - 1} > p(2k - 2)(2k - 3).$$

\[\square\]

**Lemma 13.** Assume that $k \geq 2$ and that every integer in $[1, D(I_k)]$ divides $t$. Let $S$ be a z.s.s over $I_k = [-k, k]$ of length $|S| \geq t + k(k - 1)$ such that $S$ does not contain a z.s.s of length $t$. Let $S = S_1 \cdot \ldots \cdot S_h$ be a factorization of $S$ into minimal z.s.s $S_i$, $1 \leq i \leq h$. Let $L = \{|S_i| : 1 \leq i \leq h\}$, $\alpha = \max_{\ell \in L} \ell$, and let $n_\ell = |\{S_i : |S_i| = \ell, 1 \leq i \leq h\}|$.

If there exists $\beta \in L$ such that $n_\beta \geq \alpha - 1$, then

$$|S| \leq t - \beta + (\beta - 1) \max_{\ell \in L \setminus \{\beta\}} \ell.$$

**Remark 14.** By Lemma 12, there exists $\beta \in L$ such that $n_\beta > (2k - 2)(2k - 3)$. Moreover, $\alpha = \max_{\ell \in L} \ell \leq D(I_k) \leq (2k - 2)(2k - 3) + 1$ for $k \geq 2$. Thus, $n_\beta \geq \alpha$, i.e., the hypothesis of Lemma 13 always holds.

**Proof.** By hypothesis, there exists $\beta \in L$ such that $n_\beta \geq \alpha - 1$. Given a factorization $S = S_1 \cdot \ldots \cdot S_h$ into minimal z.s.s $S_i$, $1 \leq i \leq h$, consider the sequence of lengths in $L \setminus \{\beta\}$:

$$X = \prod_{i=1, |S_i| \neq \beta}^h |S_i| = \prod_{\ell \in L \setminus \{\beta\}} \ell^{[n_\ell]}.$$

It follows from Lemma 11 that there exists a factorization $X = X_0 \cdot X_1 \ldots X_r$ such that

1. $|X_0| \leq \beta - 1$, and no subsequence $X_0$ has a sum that is divisible by $\beta$.

2. $|X_j| \leq \beta$ and $\beta$ divides $\sigma(X_j)$ for all $j \in [1, r]$.

Thus,

$$\sigma(X_j) = \sum_{x \in X_j} x \leq |X_j| \cdot \max_{x \in X_j} x \leq \beta \alpha$$

for all $j \in [1, r]$.

Note that (4), (5), and the hypothesis on $\beta$ imply that:

$\beta$ divides $t$; $n_\beta \geq \alpha - 1$; $\sigma(X_j) \leq \alpha \beta$; and $\beta$ divides $\sigma(X_j)$ for all $j \in [1, r]$.

Thus, if

$$\beta n_\beta + \sum_{j=1}^r \sigma(X_j) \geq t,$$

then there exists a nonnegative integer $n'_\beta \leq n_\beta$ and a subset $Q \subseteq [1, r]$ such that

$$\beta n'_\beta + \sum_{q \in Q} \sigma(X_q) = t.$$
Then $S$ would contain a z.s.s of length $t$ obtained by concatenating $n_β'$ minimal z.s.s of $S$ of length $β$ and all the z.s.s of $S$ whose lengths are in $X_q$ for all $q ∈ Q$. This contradicts the hypothesis of the theorem. Thus, $βn_β + \sum_{j=1}^{r} σ(X_j) < t$ must hold. Since $β$ divides both $t$ and $\sum_{j=1}^{r} σ(X_j)$, we obtain

$$βn_β + \sum_{j=1}^{r} σ(X_j) ≤ t - β.$$ 

Thus, it follows from the definition of $X$ and $X_j$, $0 ≤ j ≤ r$, that

$$|S| = \sum_{ℓ ∈ L} ℓn_ℓ = βn_β + σ(X)$$

$$= βn_β + \sum_{j=1}^{r} σ(X_j) + σ(X_0)$$

$$≤ t - β + σ(X_0).$$

(6)

Next, it follows from (3) and (6) that

$$|S| ≤ t - β + σ(X_0) ≤ t - β + |X_0| \max_{ℓ ∈ L \backslash \{β\}} ℓ ≤ t - β + (β - 1) \max_{ℓ ∈ L \backslash \{β\}} ℓ.$$ 

□

Proof of Theorem 5. We first prove part (i). Suppose that $s'_t(I_k)$ is finite. Then it follows from Lemma 10 that every odd integer in $[1, D(I_k)]$ divides $t$. Thus, it remains to show that if $a$ is an even integer in $[1, D(I_k)]$, then $a$ divides $t$.

Case 1: $a = 2^e$ for some integer $e ≥ 1$.

Lemma 9 implies that for any $p ∈ N$, the sequence $S = (1 \cdot -1)^[p]$ is a z.s.s whose z.s.s have lengths that are multiples of 2. Therefore, if 2 does not divide $t$, then $s'_t(I_k) ≥ |S| = 2p$, where $p$ can be chosen to be arbitrarily large. Thus, 2 divides $t$ if $s'_t(I_k)$ is finite.

Now assume that $e > 1$. Since the gcd of two numbers divides their difference, $(a/2 - 1, a/2 + 1) ≤ 2$. But 2 does not divide $a/2 - 1$ or $a/2 + 1$; and so $(a/2 - 1, a/2 + 1) = 1$.Lemma 10 implies that for any $p ∈ N$, the sequence $S^{[p]}$ with $S = (a/2 - 1)^{(a/2+1)} \cdot (-a/2 - 1)^{(a/2-1)}$ is a z.s.s whose z.s.s have lengths that are multiples of $|S| = (a/2 + 1) + (a/2 - 1) = a$. Thus, if $a$ does not divide $t$, we can obtain arbitrarily long z.s.s over $I_k = [-k, k]$ that do not contain z.s.s of length $t$, because $p$ can be chosen to be arbitrarily large. Thus, $a$ divides $t$ if $s'_t(I_k)$ is finite.

Case 2: $a$ is not a power of 2.

Then $a = 2^e j$, where $e$ and $j$ are nonnegative integers such that $j$ is odd. By Lemma 10, $j$ divides $t$, and if follows from Case 1 that $2^e$ divides $t$. Since $j$ is odd, $(2^e, j) = 1$. Since $2^e$ and $j$ are factors of $t$, it follows that $2^e j$ divides $t$.

Thus, it follows from Case 1, Case 2, and Lemma 10 that every integer in $[1, D(I_k)]$ divides $t$. 

}\end{proof}
Since the lower bound of $s'_t(I_k)$ in Theorem 5(ii) follows from Lemma 7, it remains to prove its upper bound. Let $k, t \in \mathbb{N}$ be such that every integer in $[1, D(I_k)]$ divides $t$. In particular, $t$ is even. Let $S$ be an arbitrary z.s.s over $I_k = [-k, k]$ that does not contain a z.s.s of length $t$.

If $k = 1$, then it follows from Theorem 1 that $D(I_k) = 2$. Thus, $2$ divides $t$ and $|S| = x_1 + 2x_2$ for some nonnegative integers $x_1$ and $x_2$. If $|S| \geq t$, then $x_1 \geq 2$ or $x_2 \geq t/2$ (because $t$ is even). This implies that there exist nonnegative integers $x'_1 \leq x_1$ and $x'_2 \leq x_2$ such that $x'_1 + 2x'_2 = t$. Thus $S' = (1 \cdot -1)^{|x'_2|} \cdot 0^{|x'_1|}$ is a z.s.s of $S$ of length $t$, which contradicts the fact that $S$ does not contain a z.s.s of length $t$. Hence $|S| \leq t - 1$, and $s'_t(I_k) \leq |S| + 1 = t$.

Now assume $k \geq 2$. Since $S$ was arbitrarily chosen, it follows that if $|S| \leq t + k(k - 1) - 1$, then
\[
s'_t(I_k) \leq |S| + 1 \leq t + k(k - 1) \leq t + (2k - 2)(2k - 3),
\]
and the upper bound in Theorem 5(ii) follows. So we may assume that $|S| \geq t + k(k - 1)$. Let $S = S_1 \cdots S_h$ be a factorization of $S$ into minimal z.s.s. Let $L = \{|S_i| : 1 \leq i \leq h\}$, $\alpha = \max_{\ell \in L} \ell$, and let $n_\ell = \{|S_i| : |S_i| = \ell, 1 \leq i \leq h\}$. Then Remark 14 implies that there exists $\beta \in L$ such that $n_\beta \geq \alpha - 1$. If $\beta = \alpha$, then Lemma 13 yields
\[
|S| \leq t - \alpha + (\alpha - 1) \max_{\ell \in L \setminus \{\alpha\}} \ell \leq t - \alpha + (\alpha - 1)^2.
\]
If $1 \leq \beta \leq \alpha - 1$, then Lemma 13 also yields
\[
|S| \leq t + \max_{1 \leq \beta \leq \alpha - 1} \left( -\beta + (\beta - 1) \max_{\ell \in L \setminus \{\beta\}} \ell \right)
\]
\[
\leq t + \max_{1 \leq \beta \leq \alpha - 1} (-\beta + (\beta - 1)\alpha)
\]
\[
= t + (-\alpha + 1 + (\alpha - 2)\alpha)
\]
\[
= t - \alpha + (\alpha - 1)^2.
\]
So in all cases, we obtain
\[
|S| \leq t - \alpha + (\alpha - 1)^2 \leq t - (2k - 1) + (2k - 2)^2,
\]
where we used the fact $\alpha \leq D(I_k) = 2k - 1$. Since $S$ was chosen to be an arbitrary z.s.s over $I_k = [-k, k]$ which does not contain a z.s.s of length $t$, it follows that
\[
s'_t(I_k) \leq |S| + 1 \leq t - (2k - 1) + (2k - 2)^2 + 1 = t + (2k - 2)(2k - 3).
\]

Proof of Corollary 1. For $k \in \{1, 2\}$, the corollary holds since the upper and lower bounds of $s'_t(I_k)$ given by Theorem 5 are both equal to $t + k(k - 1)$.

For $k = 3$, it also follows from Theorem 5 that $t + 6 \leq s'_t(I_3) \leq t + 12$. Thus, it remains to show that if $S$ is an arbitrary z.s.s over $I_3$ which does not contain a z.s.s of length $t$, then $|S| \neq t + d$ for all $d \in [6, 11]$. 

\[\square\]
Consider a factorization $S = S_1 \cdots S_h$ into minimal z.s.s $S_i$, $i \in [1, h]$. Let $L = \{|S_i|: 1 \leq i \leq h\}$, $\alpha = \max_{\ell \in L} \ell$, and let $n_\alpha = |\{S_i: |S_i| = \ell, 1 \leq i \leq h\}|$. Thus, $\alpha \leq D(I_3) = 5$. If $\alpha \leq 4$, then Lemma 13 yields

$$|S| \leq t + \max_{\ell \in L} ((\alpha - 1)^2 - \alpha) = t + (4 - 1)^2 - 4 = t + 5.$$ 

Thus, we may assume that $\alpha = \max \ell = 5$ for any factorization of $S$.

If $\beta \in \{1, 2\}$ and $n_\beta \geq 4$, then Lemma 13 yields

$$|S| \leq t + \max_{\beta \in \{1, 2\}} ((\beta - 1)\alpha - \beta) = t + (2 - 1)5 - 2 = t + 3.$$

Next, suppose that $R$ is a z.s.s of $S$ with length at least 4. Then $R \cdot -R$ can be trivially factorize into $|W| \geq 4$ z.s.s of length 2. This would yields a new factorization $S = S_1' \cdots S_h'$ with $n_2 \geq 4 \geq n_5 - 1$, which would imply that $|S| < t + 5$ by the above analysis.

Also note that if $n_\ell \geq t/\ell$ holds for some length $\ell \in L$, then we obtain a z.s.s of $S$ of length $t$ by concatenating $t/\ell$ z.s.s of length $\ell$ in $S$. This would contradict the definition of $S$. Thus, we can assume that $n_\ell \leq t/\ell - 1$ for all $\ell \in L \subseteq [1, 5]$.

To recapitulate, we may assume that for any factorization $S = S_1 \cdots S_h$, with $S_L = \prod_{i=1}^h |S_i|$ and $n_\ell = |\{S_i: |S_i| = \ell, 1 \leq i \leq h\}|$, we have:

(i) $S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 3^{[n_3]} \cdot 2^{[n_2]} \cdot 1^{[n_1]}$, where $0 \leq n_\ell \leq t/\ell - 1$ for $\ell \in [1, 5]$; $n_5 \geq 1$; and $n_1, n_2 \leq 3$.

(ii) There is a one-to-one correspondence between the subsequences $S_L'$ of $S_L$ and the z.s.s $S'$ of $S$ with length $\sigma(S_L')$.

(iii) If $R$ is z.s.s over $I_3$ such that $|R| \geq 4$, then $R$ and $-R$ cannot both be subsequences of $S$.

(iv) If $R$ is a minimal z.s.s of $S$ such that $|R| = 5$, then $R = 3^2[-2]^3$.

(This follows from (iii) and the fact $A = 3^2[-2]^3$ and $-A$ are the only minimal z.s.s of length 5 over $I_3 = [-3, 3]$. Thus, if $-A$ is the z.s.s of $S$, then we can analyze $-S$ instead of $S$.)

We now prove the following claims.

**Claim 1:** If $5 \cdot 3^4$ is a subsequence of $S_L$, then $|S| \neq t + d$ for all $d \in [6, 11]$.

If $n_4 + n_2 + n_1 \geq 1$, then either $5 \cdot 4 \cdot 3^4$, or $5 \cdot 3^4 \cdot 2$, or $5 \cdot 3^4 \cdot 1$ is a subsequence of $S_L$, which implies that $\Sigma(S_L)$ contains all the integers in $[6, 11]$. Thus, $n_4 = n_2 = n_1 = 0$, which implies that $S_L = 5^{[n_5]} \cdot 3^{[n_3]}$. If $n_5 \leq 1$, then

$$|S| = \sigma(S_L) = 5n_5 + 3n_3 \leq 5 + 3(t/3 - 1) < t + 5.$$ 

Thus, we may assume that $S_L = 5^{[n_5]} \cdot 3^{[n_3]}$, where $n_5 \geq 2$ and $n_3 \geq 4$.

Then $\Sigma(S_L)$ contain all the integers in $[6, 11] \setminus \{7\}$; and so $|S| \neq t + d$ for $d \in [6, 11] \setminus \{7\}$. It remains to show that $|S| \neq t + 7$. 

Note that the only minimal z.s.s of length 3 over $[-3, 3]$ are (up to sign) $B_1 = 2 \cdot (-1)[2]$ and $B_2 = 3 \cdot -2 \cdot -1$. Since $5 \cdot 3[4]$ is a subsequence of $S_L$, it follows from the assumptions (i)–(iv) (see above) that $S' = A \cdot X \cdot Y \cdot Z \cdot W$ is a subsequence of $S$, where $A = 3[2] \cdot (-2)[3]$ and $X, Y, Z, W \in \{-B_1, B_1, -B_2, B_2\}$. By inspecting the sequence $S'$ for all possible choices of $X, Y, Z,$ and $W$; we see that $S'$ admits a z.s.s of length 7. For instance, if $X = Y = Z = B_2$, then

$$S' = A \cdot B_2[3] \cdot W = A[2] \cdot 3 \cdot (-1)[3] \cdot W$$

contains the subsequence $3 \cdot (-1)[3] \cdot W$, which is a z.s.s of length $4 + |W| = 7$. Hence, $|S| \neq t + 7$. Thus, $|S| \neq t + d$ for all $d \in [6, 11]$.

**Claim 2:** If $5 \cdot 4[2] \cdot 3$ is a subsequence of $S_L$, then $|S| \neq t + d$ for all $d \in [6, 11]$.

If $n_3 \geq 2$, or $n_2 \geq 1$, or $n_1 \geq 1$, then either $5 \cdot 4[2] \cdot 3[2]$, or $5 \cdot 4[2] \cdot 3 \cdot 2$, or $5 \cdot 4[2] \cdot 3 \cdot 1$ is a subsequence of $S_L$, which implies that $\Sigma(S_L)$ contains all the integers in $[6, 11]$. In these cases, $|S| \neq t + d$ for $d \in [6, 11]$, we are done. Thus, we may assume that $n_2 = n_1 = 0$ and $n_3 = 1$, which implies that $S_L = 5[n_5] \cdot 4[n_4] \cdot 3$. If $n_5 \leq 1$, then

$$|S| = \sigma(S_L) = 5n_5 + 4n_4 + 3 \leq 5 + 4(t/4 - 1) + 3 < t + 5.$$  

Thus, we may assume that

$$S_L = 5[n_5] \cdot 4[n_4] \cdot 3,$$

where $n_5 \geq 2$ and $n_4 \geq 2$.

Thus, $5[2] \cdot 4[2] \cdot 3$ is a subsequence of $S_L$, which implies that $\Sigma(S_L)$ contain all the integers in $[7, 11]$. Thus $|S| \neq t + d$ for $d \in [7, 11]$. It remains to show that $|S| \neq t + 6$.

Note that the only minimal z.s.s of length 4 over $[-3, 3]$ are (up to sign) $C_1 = 3 \cdot (-1)[3]$ and $C_2 = 3 \cdot 1 \cdot (-2)[2]$. Since $5 \cdot 4[2] \cdot 3$ is a subsequence of $S_L$, it follows from the assumptions (i)–(iv) that $S' = A \cdot X \cdot Y \cdot Z$ is a subsequence of $S$, where $A = 3[2] \cdot (-2)[3]$, $X, Y \in \{-C_1, C_1, -C_2, C_2\}$, and $Z \in \{-B_1, B_1, -B_2, B_2\}$. By inspecting the sequence $S'$ for all possible choices of $X, Y,$ and $Z$; we see that $S'$ admits a z.s.s of length 6. For instance, if $X = C_1$ and $Y = C_2$, then

$$S' = A \cdot C_1 \cdot C_2 \cdot Z = A \cdot (3 \cdot -1 \cdot -2)[2] \cdot (1 \cdot -1) \cdot Z$$

contains the subsequence $(3 \cdot -1 \cdot -2) \cdot Z$, which is a z.s.s of length $3 + |Z| = 6$. Hence, $|S| \neq t + 5$. Thus, $|S| \neq t + d$ for all $d \in [6, 11]$.

**Claim 3:** If $5 \cdot 4[3]$ is a subsequence of $S_L$, then $|S| \neq t + d$ for all $d \in [6, 11]$.

If $n_3 \geq 1$, then $5 \cdot 4[2] \cdot 3$ is a subsequence of $S_L$, and we are back in Case 2. Thus, we may assume that $n_3 = 0$. If $n_2 \geq 1$ or $n_1 \geq 2$, then either $5 \cdot 4[3] \cdot 2$ or $5 \cdot 4[3] \cdot 1[2]$ is a subsequence of $S_L$, which implies that $\Sigma(S_L)$ contains all the integers in $[6, 11]$. Thus, $S$ contains z.s.s of length $\ell$ for all $\ell \in [6, 11]$. Hence, $|S| \neq t + d$ for $d \in [6, 11]$. Thus, we may assume that $n_2 = 0$ and $n_1 \leq 1$. Thus, $S_L = 5[n_5] \cdot 4[n_4] \cdot 1[n_1]$. Moreover, if $n_5 \leq 1$, then

$$|S| = \sigma(S_L) = 5n_5 + 4n_4 + n_1 \leq 5 + 4(t/4 - 1) + 1 < t + 5.$$  

Thus, we may assume that

$$|S| = \sigma(S_L) = 5n_5 + 4n_4 + n_1 \leq 5 + 4(t/4 - 1) + 1 < t + 5.$$
\[ S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 1^{[n_1]}, \] where \( n_5 \geq 2, n_4 \geq 3, \) and \( n_1 \leq 1. \)

Since \( 5^{[2]} \cdot 4^{[3]} \) is a subsequence of \( S_L, \) it follows that \( \Sigma(S_L) \) contain all the integers in \([8, 10]. \) Thus, \( S \) admits z.s.s of length \( \ell \) for all \( \ell \in [8, 10]. \) Hence, \( |S| \neq t + d \) for all \( d \in [8, 10]. \) Moreover, it follows from the assumptions (i)–(iv) that \( S' = A \cdot X \cdot Y \cdot Z \) is a subsequence of \( S, \) where \( A = 3^{[2]} \cdot (-2)^{[3]} \) and \( X, Y, Z \in \{-C_1, C_1, -C_2, C_2\}. \) By inspecting the sequence \( S' \) for all possible choices of \( X, Y, \) and \( Z; \) we see that \( S' \) admits a z.s.s of length 7. Hence, \( |S| \neq t + 7. \) Overall, we obtain \( |S| \neq t + d \) for any \( d \in [7, 10]. \)

If \( 5 \cdot 4^{[4]} \) is a subsequence of \( S_L, \) it again follows from the assumptions (i)–(iv) that \( S' = A \cdot X \cdot Y \cdot Z \cdot W \) is a subsequence of \( S, \) where \( A = 3^{[2]} \cdot (-2)^{[3]} \) and \( X, Y, Z, W \in \{-C_1, C_1, -C_2, C_2\}. \) By inspecting the sequence \( S' \) for all possible choices of \( X, Y, Z, \) and \( W; \) we see that \( S' \) admits z.s.s of lengths 6 and 11. In this case, \( |S| \neq t + d \) for all \( d \in [6, 11]. \) Thus, we may assume that

\[ S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 1^{[n_1]}, \] where \( n_5 \geq 2 \) and \( n_1 \leq 1. \)

Now, it remains to show that \( |S| \neq t + a \) for \( a \in \{6, 11\}. \) However, if \( |S| = t + a, \) then

\[ 5n_5 + 4(3) + n_1 = \sigma(S_L) = |S| = t + a \Rightarrow 5n_5 = t + a - 12 - n_1. \]

This is a contradiction since 5 divides \( t \) (by hypothesis) and 5 does not divide \( a - 12 - n_1 \) for \( a \in \{6, 11\} \) and \( n_1 \in \{0, 1\}. \) Thus, \( |S| \neq t + d \) for all \( d \in [6, 11]. \)

Based on Claim 1–Claim 3, we may assume the following:

(v) \( S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 3^{[n_3]} \cdot 2^{[n_2]} \cdot 1^{[n_1]}, \) where 0 \( \leq n_\ell \leq t/\ell - 1 \) for all \( \ell \in [1, 5]; \)

\( n_1, n_2, n_3 \leq 3; n_4 \leq 2; (n_4, n_3) \neq (2, 1); \) and \( n_5 \geq 1. \)

We will use this assumption in the following cases.

**Case 1:** \( |S| \neq t + 6. \)

Assume that \( |S| = t + 6. \) If \( n_1 \geq 1, \) then 5 \cdot 1 is a subsequence of \( S_L, \) which implies that \( S \) contains a z.s.s of length 5 + 1 = 6 whose complementary sequence in \( S \) is a z.s.s of length 6. Thus, \( n_1 = 0. \) By a similar reasoning, we infer that the following conditions hold: \( n_3 \leq 1; \) and \( n_4 \geq 1 \Rightarrow n_2 = 0. \) Moreover, it follows from condition (v) that \( (n_4, n_3) \neq (2, 1). \) Consequently, either \( S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 3^{[n_3]} \cdot 2^{[n_2]} \) with \( n_3 \leq 1, n_4 \leq 2, \) and \( (n_4, n_3) \neq (2, 1); \) or \( S_L = 5^{[n_5]} \cdot 3^{[n_3]} \cdot 2^{[n_2]} \) with \( n_3 \leq 1 \) and \( n_2 \leq 2. \) Thus,

\[ |S| = \sigma(S_L) \leq 5n_5 + 8 \leq 5(t/5 - 1) + 8 < t + 6, \]

which is a contradiction. Thus, \( |S| \neq t + 6. \)

**Case 2:** \( |S| \neq t + 7. \)

Assume that \( |S| = t + 7. \) If \( n_2 \geq 1, \) then 5 \cdot 2 is a subsequence of \( S_L, \) which implies that \( S \) contains a z.s.s of length 5 + 2 = 7 whose complementary sequence in \( S \) is a z.s.s of length 7. Thus, \( n_2 = 0. \) By a similar reasoning, we infer that the following conditions hold: \( n_1 \leq 1; n_4 \geq 1 \Rightarrow n_3 = 0; n_3 \geq 1 \Rightarrow n_4 = 0; \) and \( n_3 \geq 2 \Rightarrow n_1 = 0. \)
Consequently, either $S_L = 5^{[n_5]} \cdot 4^{[n_4]} \cdot 1^{[n_1]}$ with $n_4 \leq 2$ and $n_1 \leq 1$; or $S_L = 5^{[n_5]} \cdot 3 \cdot 1$, or $S_L = 5^{[n_5]} \cdot 3^{[n_3]}$ with $n_3 \leq 3$. Thus, 

$$|S| = \sigma(S_L) \leq 5n_5 + 9 \leq 5(t/5 - 1) + 9 < t + 7,$$

which is a contradiction. Thus $|S| \neq t + 7$.

**Case 3:** $|S| \neq t + 8$.

Assume that $|S| = t + 8$. If $n_3 \geq 1$, then $5 \cdot 3$ is a subsequence of $S_L$, which implies that $S$ contains a z.s.s of length $5 + 3 = 8$ whose complementary sequence in $S$ is a z.s.s of length $t$. Thus, $n_3 = 0$. By a similar reasoning, we infer that $n_4 \leq 1$; $n_2 \leq 3$; $n_1 \leq 2$; $n_2 \geq 1 \Rightarrow n_1 = 0$; and $n_3 \geq 1 \Rightarrow n_2 = 0$. Consequently either $S_L = 5^{[n_5]} \cdot 4 \cdot 2$, or $S_L = 5^{[n_5]} \cdot 4 \cdot 1^{[n_1]}$, or $S_L = 5^{[n_5]} \cdot 2^{[n_2]}$, or $S_L = 5^{[n_5]} \cdot 1^{[n_3]}$, where $n_2 \leq 3$ and $n_1 \leq 2$. Thus, 

$$|S| = \sigma(S_L) \leq 5n_5 + 6 \leq 5(t/5 - 1) + 6 < t + 8,$$

which is a contradiction. Thus, $|S| \neq t + 8$.

**Case 4:** $|S| \neq t + d$ for $d \in [9, 11]$.

Assume that $|S| = t + 9$. If $n_3 \geq 1$, then 3 is a subsequence of $S_L$ which implies that $S$ contain a z.s.s $T$ of length 3. Thus $S' = S \cdot T^{-1}$ is a z.s.s of length $|S| - 3 = t + 6$ which does not contain a z.s.s of length 6 and, equivalently, length $t$. This contradicts Case 1, where we showed that no such z.s.s exists. Thus, $n_3 = 0$. Similarly, $n_2 = 0$ (by Case 2) and $n_1 = 0$ (by Case 3). Consequently, $S_L = 5^{[n_5]} \cdot 4^{[n_4]}$ with $n_4 \leq 2$. Thus, 

$$|S| = \sigma(S_L) = 5n_5 + 4n_4 \leq 5(t/5 - 1) + 4(2) < t + 9,$$

which is a contradiction. Thus, $|S| \neq t + 9$.

Since $n_5 \geq 2$, $S$ contains a z.s.s of length $\sigma(5^{[n_5]}) = 10$. Thus, $|S| \neq t + 10$.

Since $n_5 \geq 1$, $S$ contains a z.s.s $T$ of length 5. Thus $S' = S \cdot T^{-1}$ is a z.s.s of length $|S| - 5 = t + 6$ which does not contain a z.s.s of length 6 and, equivalently, length $t$. This contradicts Case 1. Thus, $|S| \neq t + 11$.

In conclusion, we have shown that if $S$ is an arbitrary z.s.s over $I_3 = [-3, 3]$ which does not contain a z.s.s of length $t$, then $|S| = t + d$ for $d \in [6, 11]$. Thus, $\mathcal{s}(I_3) = t + 6$. □

3. APPENDIX

In this section, we include Zhong’s proofs of Lemma 3 and Remark 4.

**Proof of Lemma 3.**

(i) Since $s(G) \leq s'(G)$, it suffices to prove that $s'(G) \geq s(G)$. Let $S = \prod_{i=1}^{s(G)-1} g_i$ be a sequence in $\mathcal{F}(G)$ of length $|S| = s(G) - 1$ such that $S$ has no zero-sum subsequence of length $\exp(G)$. Assume that $\sigma(S) = h \in G$ and let $t \in \mathbb{N}$ be such that $(s(G) - 1)t \equiv 1 \pmod{\exp(G)}$. Then $(s(G) - 1)th = h$ in $G$. Define $S' = \prod_{i=1}^{s(G)-1} (g_i - th)$. Since $\sigma(S') = \sigma(S) - (s(G) - 1)th = 0$ and $S'$ does not contain a zero-sum subsequence of length $\exp(G)$, it follows that $s'(G) \geq s(G)$. 


(ii) Let $S \in \mathcal{B}(G)$ be such that $|S| = s(G) - 1$. We want to prove that $S$ contains a zero-sum subsequence of length $n = \exp(G)$. If we assume to the contrary that $S$ does not contain a zero-sum subsequence of length $n$, then Property D implies that there exists $T \in \mathcal{F}(G)$ such that $S = T^{[n-1]}$. Thus, $|T| = c$ and $\sigma(T) = 0$. Therefore $T^{[n/c]}$ is a zero-sum sequence of length $n$, a contradiction. □

Proof of Remark 4.

(i) Let $n$ be odd and $G \cong \mathbb{Z}_d^2$. Since $s(G) = 4n - 3$, then $\gcd(s(G) - 1, n) = 1$. Thus, $s(G) = s'(G)$ by Lemma 3(i).

(ii) Let $h \geq 2$ be an integer and $G \cong \mathbb{Z}_{2^h}$. Then $\exp(G) = 2^h$, $s(G) = 4(2^h - 1) + 1$, $\gcd(s(G) - 1, \exp(G)) = 4$, and $G$ has Property D (by [12, Theorem 3.2]). Thus, Lemma 3(ii) yields $s'(G) < s(G)$. Since $\gcd(s(G) - 2, \exp(G)) = 1$, the proof of Lemma 3(i) yields $s'(G) > s(G) - 2$. Thus, $s'(G) = s(G) - 1$. □

Acknowledgement: We thank Alfred Geroldinger for providing references and for his valuable comments which helped clarify the definitions and terminology. We also thank Qinghai Zhong for allowing us to include Lemma 3 and Remark 4.

References


