On $\lambda$-fold Partitions of Finite Vector Spaces and Duality

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Abstract

Vector space partitions of an $n$-dimensional vector space $V$ over a finite field are considered in [5], [10], and more recently in the series of papers [3], [8], and [9]. In this paper, we consider the generalization of a vector space partition which we call a $\lambda$-fold partition (or simply a $\lambda$-partition). In particular, for a given positive integer, $\lambda$, we define a $\lambda$-fold partition of $V$ to be a multiset of subspaces of $V$ such that every nonzero vector in $V$ is contained in exactly $\lambda$ subspaces in the given multiset. A $\lambda$-fold spread as defined in [12] is one example of a $\lambda$-fold partition. After establishing some definitions in the introduction, we state some necessary conditions for a $\lambda$-fold partition of $V$ to exist, then introduce some general ways to construct such partitions. We also introduce the construction of a dual $\lambda$-partition as a way of generating $\lambda'$-partitions from a given $\lambda$-partition. One application of this construction is that the dual of a vector space partition will, in general, be a $\lambda$-partition for some $\lambda > 1$. In the last section, we discuss a connection between $\lambda$-partitions and some designs over finite fields.

We denote by $V_n(q)$ the vector space of dimension $n$ over the field $\mathbb{F}_q$ with $q$ elements, where $q$ is a power of a prime. In a series of papers ([3], [8], [9]), we extended the results of T. Bu ([5]) and O. Heden ([10] and [11]) on partitioning $V$ into subspaces. (More precisely, we considered finding a set of subspaces of $V = V_n(q)$ such that every nonzero vector is in exactly one subspace in this set.)

One natural extension of our previous work is to examine the idea of a $\lambda$-fold partition of $V$. As in the vector space partition, we define a $\lambda$-fold partition to be a multiset of subspaces such that every nonzero vector in $V$ is contained in exactly $\lambda$ subspaces in our multiset. A $\lambda$-fold partition generalizes the idea of a $\lambda$-fold spread defined in Section 4.2 of J.W.P. Hirschfeld’s book on projective geometries over finite fields [12]. In fact, Corollary 8 of this paper extends Theorem 4.16 of [12]. The purpose of this note is to construct certain $\lambda$-fold partitions and consider some questions that naturally arise from our treatment of these partitions.

We start with a more precise definition of $\lambda$-fold partition which will be specially useful to prove our duality theorem (Theorem 15).

Definition 1 Let $\lambda$ be a positive integer. A $\lambda$-fold partition of the vector space $V$ is an ordered pair $(A, \alpha)$ such that $A$ is a set and $\alpha$ is a map from $A$ to $2^V$, the set of subsets of $V$, such that

1. if $a \in A$, then $\alpha(a)$ is a nonzero subspace of $V$,
2. if $0 \neq v \in V$, then the cardinality of the set $\{a \in A : v \in \alpha(a)\}$ is $\lambda$.

We call the cardinality of $A$ the size of the partition and say two $\lambda$-partitions $(A, \alpha)$ and $(B, \beta)$ are equal if there exists a bijection $\tau : A \to B$ such that $\alpha = \beta \tau$.

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Note that using this definition, a 1-fold partition of $V$ is just a vector space partition in the sense mentioned above. For brevity, we will henceforth refer to a $\lambda$-fold partition simply as a $\lambda$-partition. We will use the term 1-partition of $V$ when we are referring to a standard vector space partition.

We also make the observation that two $\lambda$-partitions $(A, \alpha)$ and $(B, \beta)$ are equivalent if and only if their multiset images $\{\alpha(a) : a \in A\}$ and $\{\beta(b) : b \in B\}$ are equal as multisets. As a result, sometimes we will identify a $\lambda$-partition with its multiset image.

Given a 1-partition of $V$, one easy way to construct a $\lambda$-partition of $V$ is to replicate the 1-partition $\lambda$ times. If one has $\lambda$ different 1-partitions, then we could also take the union (as multisets) of these 1-partitions to form another $\lambda$-partition of $V$. The $\lambda$-partitions generated in this way do not add much to our knowledge, but there are more interesting $\lambda$-partitions that do not come from 1-partitions in this way. One such example is the $q$-Grassmanian $G(n, n - 1)$ consisting of the set of all $(n - 1)$-dimensional subspaces of $V$ when $n \geq 3$, which forms a $\binom{q^{n-1}-1}{q-1}$-partition. More generally, we can consider the $q$-Grassmanian $G(n, r)$ consisting of all $r$-dimensional subspaces of the $n$-dimensional vector space $V$. In this case $G(n, r)$ consists of

$$
\binom{n}{r}_q = \frac{(q^n - 1)(q^{n-1} - \cdots)(q^{n-r+1} - 1)}{(q^r - 1)(q^{r-1} - \cdots)(q^1 - 1)} \text{ subspaces of dimension } r,
$$

each containing $q^r - 1$ nonzero vectors, so that each of the $q^n - 1$ nonzero vectors in $V$ are included in $\binom{n-1}{r-1}_q$ of these subspaces.

Therefore, $G(n, r)$ forms a $\binom{n-1}{r-1}_q$-partition of $V$.

If $(A, \alpha)$ is a $\lambda$-partition, then we define a $\lambda_0$-subpartition of $(A, \alpha)$ to be a $\lambda_0$-partition $(B, \beta)$ of $V$ where $B \subseteq A$, $\beta = \alpha|_B$, and $0 < \lambda_0 \leq \lambda$. We say that the $\lambda_0$-subpartition $(B, \beta)$ is proper if $0 < \lambda_0 < \lambda$. Note that if $(B, \beta)$ is a proper $\lambda_0$-partition of $(A, \alpha)$, then the complement of $(B, \beta)$, or $(A \setminus B, \alpha|_{A \setminus B})$, also forms a $(\lambda - \lambda_0)$-subpartition of $(A, \alpha)$. We say a $\lambda$-partition $(A, \alpha)$ is irreducible if it has no proper $\lambda_0$-subpartitions for any $0 < \lambda_0 < \lambda$ and reducible otherwise. Note that a 1-partition is always irreducible. Clearly, the $\lambda$-partitions built as unions of 1-partitions are reducible.

Note that not all irreducible $\lambda$-partitions are 1-partitions. For example, consider the 2-partition of $V = V_3(2)$ given by $\alpha : \{1, 2, 3, 4, 5, 6\} \to 2^V$, where the nonzero vectors of $\alpha(i)$ for $1 \leq i \leq 6$ are

$$
\alpha(1) = \{100, 011, 111\}, \quad \alpha(2) = \{010, 001, 011\}, \quad \alpha(3) = \{001, 110, 111\},
$$

$$
\alpha(4) = \{110, 010, 100\}, \quad \alpha(5) = \{101\}, \quad \alpha(6) = \{101\}.
$$

(Here we abbreviate the nonzero vector $(a, b, c)$ by the string of digits $abc$, where $a, b, c \in \{0, 1\}$.) Since a 1-partition of $V_3(2)$ can contain at most one 2-dimensional subspace, this 2-partition cannot be written as the union of two 1-partitions since it contains more than two 2-dimensional subspaces. Therefore, this 2-partition must be irreducible. This turns out to be a special case of Corollary 3 in the next section.

One goal would be to classify all irreducible $\lambda$-partitions for a given $V$. We note that the problem of classifying all irreducible $\lambda$-partitions includes the classification of all vector space partitions as a subproblem. To aid us in classifying $\lambda$-partitions, we introduce the following terminology. Let $(A, \alpha)$ be a $\lambda$-partition of $V$, where $V$ has dimension $n$. We say the $\lambda$-partition $(A, \alpha)$ is of type
[(t_1, n_1), \ldots, (t_s, n_s)] if for all 1 \leq k \leq n we have

\[ |\{a : \dim(\alpha(a)) = k\}| = \sum_{n_i = k} t_i. \]

Note that this notation does not exclude \( t_i = 0 \) for some \( i \) nor do the \( n_i \) need to be distinct. We will consider two partition types \([(t_s, n_s), \ldots, (t_1, n_1)]\) and \([(c_r, m_r), \ldots, (c_1, m_1)]\) to be the same if for all \( 1 \leq k \leq n \) we have

\[ \sum_{n_i = k} t_i = \sum_{m_j = k} c_j. \]

Sometimes it will be convenient to use the more compact notation \( n_1^{t_1} \cdots n_s^{t_s} \) for the type \([(t_s, n_s), \ldots, (t_2, n_2), (t_1, n_1)]\).

Before continuing, we prove the following analogy to [5, Lemma 1].

**Lemma 1** Let \((A, \alpha)\) be a \( \lambda \)-partition of \( V \) and let \( W \) be a subspace of \( V \). Define \( A_W = \{a \in A : \alpha(a) \cap W \neq \{0\}\} \) and \( \alpha_W : A_W \rightarrow 2^W \) by \( \alpha_W(a) = \alpha(a) \cap W \). Then \((A_W, \alpha_W)\) is a \( \lambda \)-partition of \( W \).

**Proof.** We verify the two conditions for \((A_W, \alpha_W)\) to be a \( \lambda \)-partition of \( W \). Indeed, for every \( a \in A_W \) we have \( \alpha_W(a) = \alpha(a) \cap W \), which is a nonzero subspace. Also, for any \( 0 \neq w \in W \) we have \( \{a \in A : w \in \alpha(a)\} = \{a \in A : w \in \alpha(a) \cap W\} = \{a \in A_W : w \in \alpha_W(a)\} \), where the last equality follows because if \( 0 \neq w \in \alpha(a) \cap W \) then \( a \in A_W \). Hence, \( |\{a \in A_W : w \in \alpha_W(a)\}| = |\{a \in A : w \in \alpha(a)\}| = \lambda \). Therefore, \((A_W, \alpha_W)\) is a \( \lambda \)-partition of \( W \) as claimed.

Note, when \( \dim(W) = \dim(V) - 1 \), we have for any \( a \in A \) either \( \dim(\alpha(a) \cap W) = \dim(\alpha(a)) \) or \( \dim(\alpha(a) \cap W) = \dim(\alpha(a)) - 1 \), hence we can use this observation to determine the type of \((A_W, \alpha_W)\) from \((A, \alpha)\).

For example, this lemma can be applied to the \( \left(\frac{q^{n-1} - 1}{q - 1}\right) \)-partition of \( V \) consisting of all the \((n - 1)\)-dimensional subspaces by intersecting with one of those \((n - 1)\)-dimensional subspaces \( W \) to get a \( \left(\frac{q^{n-1} - 1}{q - 1}\right) \)-partition of type \( \left(1, n - 1, \frac{q^n - q}{q - 1}, n - 2\right) \).

In Section 1, we first discuss some necessary conditions for a \( \lambda \)-partition to exist. In Section 2, we create some further examples. In Section 3, we introduce the concept of a dual \( \lambda \)-partition. This allows us to construct \( \lambda \)-partitions from known 1-partitions in a nontrivial way as well as to create new \( \lambda \)-partitions from those constructed in Section 2.

**1 Necessary conditions**

In this section, we prove a series of necessary conditions for \( \lambda \)-partitions to exist. For 1-partitions, there are two immediate necessary conditions. The first of these is the usual diophantine equation counting the nonzero vectors. So for a 1-partition of \( V_n(q) \) of type \([(a_1, n_1), \ldots, (a_t, n_t)]\) to exist, we must have

\[ \sum_{i=1}^t a_i(q^{n_i} - 1) = q^n - 1. \]
The second condition is a simple dimension consideration that can be stated as follows:

if \( a_i \neq 0 \neq a_j \) with \( i \neq j \), then \( n_i + n_j \leq n \) and if \( a_i \geq 2 \), then \( n_i \leq n/2 \).

The diophantine equation for 1-partitions has an easy generalization to \( \lambda \)-partitions. In particular, if \((A, \alpha)\) is a \( \lambda \)-partition of \( V_n(q) \) and \( n_a = \dim \alpha(a) \), then

\[
\sum_{a \in A} (q^{n_a} - 1) = \lambda(q^n - 1). \tag{1}
\]

Therefore, if \((A, \alpha)\) is a \( \lambda \)-partition of type \( n_1^{c_1} \cdots n_t^{c_t} \), we must have

\[
\sum_{i=1}^{t} c_i (q^{n_i} - 1) = \lambda(q^n - 1). \tag{2}
\]

The next theorem is a generalization of the dimension condition for 1-partitions.

**Theorem 2** Let \((A, \alpha)\) be a \( \lambda \)-partition of the \( n \)-dimensional vector space \( V \) over \( \mathbb{F}_q \), and suppose that \( a_1, a_2, \ldots, a_{\lambda+1} \in A \) are distinct elements of \( A \). Then

\[
\sum_{i=1}^{\lambda+1} \dim \alpha(a_i) \leq \lambda n.
\]

**Proof.** Let \( W_j = \alpha(a_1) \cap \alpha(a_2) \cap \cdots \cap \alpha(a_j) \) for \( 1 \leq j \leq \lambda + 1 \). We will prove by induction that

\[
\dim W_j \geq \left( \sum_{i=1}^{j} \dim \alpha(a_i) \right) - (j - 1)n, \quad 1 \leq j \leq \lambda + 1.
\]

This is trivial for \( j = 1 \). Assume it holds for \( j \). Then

\[
\dim W_{j+1} = \dim(W_j \cap \alpha(a_{j+1})) = \dim W_j + \dim \alpha(a_{j+1}) - \dim(W_j + \alpha(a_{j+1}))
\]

\[
\geq \left( \sum_{i=1}^{j} \dim \alpha(a_i) \right) - (j - 1)n + \dim \alpha(a_{j+1}) - n = \left( \sum_{i=1}^{j+1} \dim \alpha(a_i) \right) - jn.
\]

Therefore, the \( j + 1 \) case is established, hence \( \dim W_{\lambda+1} \geq \left( \sum_{i=1}^{\lambda+1} \dim \alpha(a_i) \right) - \lambda n \).

Now if \( \sum_{i=1}^{\lambda+1} \dim \alpha(a_i) > \lambda n \), then \( \dim W_{\lambda+1} > 0 \) and hence \( W_{\lambda+1} \) contains a nonzero vector \( w \). Since \( w \) is in each set \( \alpha(a_i) \) for all \( 1 \leq i \leq \lambda + 1 \), the set \( \{ a \in A : w \in \alpha(a) \} \) has cardinality at least \( \lambda + 1 \). This contradicts the assumption that \((A, \alpha)\) is a \( \lambda \)-partition of \( V \).

We can use the above theorem to determine some irreducible \( \lambda \)-partitions, as pointed out by a referee for this paper. We are grateful for this observation.

**Corollary 3** Suppose \((A, \alpha)\) is a \( \lambda \)-partition of \( V = V_n(q) \) and \( n > \lambda \). If there exists an integer \( 0 < k < n/\lambda \) such that \( |\{a \in A : \dim \alpha(a) = n - k\}| > \lambda \), then \((A, \alpha)\) is irreducible.
Proof. Let \( k \) be as in the statement of the Corollary and assume \((A,\alpha)\) is reducible. Let \((A_1,\alpha_1)\) be a proper \( \lambda_1 \)-subpartition and let \((A_2,\alpha_2)\) be its complement, which is a \( \lambda_2 \)-partition. By the Pigeonhole principle, for either \( i = 1 \) or \( i = 2 \) we know \((A_i,\alpha_i)\) must contain at least \( \lambda_i + 1 \) subspaces of dimension \( n - k \). By Theorem 2
\[
\lambda_i n \geq (\lambda_i + 1)(n - k) = (\lambda_i + 1)n - (\lambda_i + 1)k > (\lambda_i + 1)n - n = \lambda_i n,
\]
which is a contradiction. Therefore, \((A,\alpha)\) must be irreducible. \( \square \)

**Theorem 4** Let \((A,\alpha)\) be a \( \lambda \)-partition of \( V = V_n(q) \). Assume \( r = \max \{ \dim \alpha(a) : a \in A \} < n \) and \( \dim \alpha(a) \geq n - r \) for all \( a \in A \). Then
\[
|A| \geq \lambda + q^r.
\]

Proof. We have the usual diophantine equation
\[
\sum_{a \in A}(q^{\dim \alpha(a)} - 1) = \lambda(q^n - 1),
\]
and so
\[
\sum_{a \in A}q^{\dim \alpha(a)} = \lambda(q^n - 1) + |A|.
\]

Choose \( a_0 \in A \) with \( \dim \alpha(a_0) = r \). Taking \( W_i \) to be \( \alpha(a_0) \), we note for \( a \neq a_0 \) we have
\[
\dim(\alpha(a_0) \cap \alpha(a)) = \dim(\alpha(a_0)) + \dim(\alpha(a)) - \dim(\alpha(a_0) + \alpha(a)) \geq \dim(\alpha(a_0)) + \dim(\alpha(a)) - n.
\]

Let \( t \) count the elements \( v \) of \( \alpha(a_0) \setminus \{0\} \), each counted as many times as there exists an \( a \in A \setminus \{a_0\} \) such that \( v \in \alpha(a) \). Then
\[
t = \sum_{a_0 \neq a \in A} |(\alpha(a_0) \cap \alpha(a)) \setminus \{0\}| \geq \sum_{a_0 \neq a \in A} (q^{\max(0,\dim(\alpha(a)) + r - n)} - 1).
\]

But each element of \( \alpha(a_0) \setminus \{0\} \) must be in \( \alpha(a) \) for \( \lambda - 1 \) elements of \( A \setminus \{a_0\} \), so \( t = (\lambda - 1)(q^r - 1) \). Hence we get
\[
\sum_{a \in A \setminus \{a_0\}} (q^{\dim \alpha(a) + r - n - 1} + q^r - 1) \leq (q^r - 1)\lambda.
\]

The left side is
\[
\sum_{a \in A}(q^{\dim \alpha(a) + r - n - 1} - (q^{2r - n} - 1) + q^r - 1)
\]
\[
= q^{r - n} \sum_{a \in A} q^{\dim \alpha(a)} - |A| - q^r(q^{r - n} - 1)
\]
\[
= q^{r - n}[\lambda(q^n - 1) + |A|] - |A| - q^r(q^{r - n} - 1)
\]
\[
= \lambda q^{r - n}(q^n - 1) + (q^{r - n} - 1)|A| - q^r(q^{r - n} - 1).
\]
Since this is less than or equal to the right hand side, \((q^r - 1)\lambda\), we have
\[(q^{r-n} - 1)|A| - q^r(q^{r-n} - 1) \leq \lambda[q^r - 1 - q^{r-n}(q^n - 1)] = \lambda(q^{r-n} - 1).
\]
Dividing by the negative number \(q^{r-n} - 1\) reverses the sense of the inequality, and the theorem follows.

**Lemma 5** Let \((A, \alpha)\) be a \(\lambda\)-partition of \(V = V_n(q)\) such that \(n > m = \min\{\dim \alpha(a) : a \in A\}\). Let \(W \subseteq V\) be a subspace of dimension \(n - 1\). If \(k = |\{a \in A : \alpha(a) \not\subseteq W \text{ and } \dim \alpha(a) = m\}|\), then \(q\) divides \(k\).

**Proof.** First suppose that \((B, \beta)\) is a \(\lambda\)-partition of \(V_N(q)\) where the minimum dimension of any subspace in the partition is \(M\). Let \(B' = \{b \in B : \dim \beta(b) = M\}\), and suppose \(|B'| = R\). Then by Equation (1)
\[\lambda(q^N - 1) = R(q^M - 1) + \sum_{b \in B \setminus B'} (q^{\dim \beta(b)} - 1) = Rq^M + \sum_{b \in B \setminus B'} q^{\dim \beta(b)} - |B|,
\]
and so
\[|B| = \lambda - \lambda q^N + Rq^M + \sum_{b \in B \setminus B'} q^{\dim \beta(b)}. \quad (*)
\]
Thus
\[|B| \equiv \lambda \mod q^M \quad \text{and} \quad |B| \equiv \lambda \mod q.
\]
Applying this to \((A, \alpha)\) gives \(|A| \equiv \lambda \mod q^m\) and \(|A| \equiv \lambda \mod q\).

Let \((A_W, \alpha_W)\) be the \(\lambda\)-partition induced by \((A, \alpha)\) on \(W\). If \(m = 1\), then \(|A_W| = |A| - k\). Since \(|A_W| \equiv \lambda \mod q\) also, we see that \(q\) divides \(k\).

Now assume \(m > 1\) and \(k > 0\). Then \(A = A_W\) and the minimum dimension of a subspace of \((A_W, \alpha_W)\) is \(m - 1\). Applying \((*)\) to \((A_W, \alpha_W)\) gives
\[|A| = |A_W| = \lambda - \lambda q^{n-1} + kq^{m-1} + \sum_{a \in A_W} q^{\dim \alpha_W(a)}.
\]
Since \(|A| \equiv \lambda \mod q^m\) and \(n - 1 \geq m\), we see that \(q\) divides \(k\).

For any \(\lambda\)-partition \(\mathcal{P}\) of \(V_n(q)\), let \(\dim_{\min}(\mathcal{P})\) be the minimum dimension that occurs in \(\mathcal{P}\). Define
\[S(\mathcal{P}) = \{U \in \mathcal{P} : \dim(U) = \dim_{\min}(\mathcal{P})\},
\]
and let \(\tau(\mathcal{P})\) denote the number of subspaces of dimension \(\dim_{\min}(\mathcal{P})\) in \(\mathcal{P}\) (counting duplications).

**Corollary 6** Let \(\mathcal{P}\) be a \(\lambda\)-partition of \(V = V_n(q)\), and let \(m = \dim_{\min}(\mathcal{P}) < n\) and \(|S(\mathcal{P})| = 1\). Then \(q\) divides \(\tau(\mathcal{P})\).
Proof. If \( |S(P)| = 1 \), then \( S(P) = \{U\} \) for some subspace \( U \subseteq V \). Let \( W \subseteq V \) be an \((n-1)\)-dimensional subspace not containing the subspace \( U \). Then none of the \( k = \tau(P) \) subspaces of dimension \( m \) in \( P \) is contained in \( W \) (since they are all identical to \( U \)). Thus, it follows from Lemma 5 that \( q \) divides \( \tau(P) \) and our conclusion holds.

2 Some Initial Constructions

We start this section with a well-known example.

Example 1

Let \( V \) be an \( n \)-dimensional vector space over \( F = \mathbb{F}_q \) and identify \( V \) with \( \mathbb{F}_{q^n} \). Then \( V \) can be partitioned into 1-dimensional \( \mathbb{F}_q \)-subspaces to form the projective space \( \mathbb{P}(V) \). Let \( J \subseteq V \) be a subset consisting of one nonzero element from each one-dimensional subspace. Note \( |J| = q^n - 1 \).

If \( U \) is a \( k \)-dimensional subspace of \( V \), then the multiset \( P(U) = \{\alpha U : \alpha \in J\} \) will have \( |J| \) elements and so \( P(U) \) will form a \( \left( \frac{q^k - 1}{q^r - 1}, k \right) \)-partition of \( V \) of type \( \left( \frac{q^n - 1}{q^r - 1}, k \right) \). Indeed, note that for any nonzero \( v \in V \) we have \( v \in \alpha U \iff \alpha^{-1}v \in U \), hence there are exactly \( \frac{q^k - 1}{q^r - 1} \) subspaces in our set that contain \( v \).

Next, we generalize the above example to examine homogeneous \( \lambda \)-partitions, i.e., \( \lambda \)-partitions of type \( n_1^{t_1} \).

Proposition 7 Let \( 1 \leq k \leq n = \dim V \) and let \( r = \gcd(k, n) \). There exists a \( \left( \frac{q^k - 1}{q^r - 1}, k \right) \)-partition of \( V \) of type \( \left( \frac{q^n - 1}{q^r - 1}, k \right) \).

Proof. If \( k \mid n \), we get the 1-partition given in [5, Lemma 2]. So assume \( k \) does not divide \( n \). Let \( r = \gcd(k, n) \) and \( V = V_{n/r}(q^r) \), hence \( V \) is an \( n \)-dimensional vector space over \( \mathbb{F}_q \). Then we can choose \( U \) to be a \((k/r)\)-dimensional \( \mathbb{F}_{q^r} \)-subspace of \( V \). Using Example 1, we can use \( U \) to create a \( \lambda = \left( \frac{(q^r)^{k/r} - 1}{q^r - 1} \right) \)-partition of \( V \) of type \((k/r)^t\) of \( \mathbb{F}_{q^r} \) subspaces where

\[
t = \left( \frac{(q^r)^{n/r} - 1}{q^r - 1} \right) = \frac{q^n - 1}{q^r - 1}.
\]

Since each \( \mathbb{F}_{q^r} \)-subspace of \( V \) of dimension \( k/r \) is also a \( k \)-dimensional \( \mathbb{F}_q \)-subspace of \( V \), this gives us the desired \( \left( \frac{q^k - 1}{q^r - 1}, k \right) \)-partition of \( V \) of type \( \left( \frac{q^n - 1}{q^r - 1}, k \right) \).

Corollary 8 Let \( 1 \leq k \leq n = \dim V \) and \( r = \gcd(k, n) \). Then there exists a \( \lambda \)-partition of \( V \) of type \( k^t \) if and only if

\[
\left( \frac{q^k - 1}{q^r - 1} \right) \mid \lambda.
\]
Proof. Let $\tau = \frac{q^k - 1}{q^r - 1}$ and $m = \frac{q^n - 1}{q^r - 1}$. If $\tau | \lambda$, we can just take $\lambda/\tau$ copies of the $\tau$-partition of $V$ from Proposition 7 to get the corresponding $\lambda$-partition.

Conversely, assume that there exists a $\lambda$-partition of type $k^l$. Then it follows from Equation (2) that
\[ t(q^k - 1) = \lambda(q^n - 1) \Rightarrow t\tau = \lambda m \Rightarrow \tau | \lambda m. \]
Therefore, since $\gcd(\tau, m) = 1$, we see that $\tau | \lambda$.

Next, we describe two methods that allow us to construct $\lambda$-partitions from 1-partitions. First, we introduce a technique for generating some $q^m$-partitions of $V$.

**Proposition 9** Let $(A, \alpha)$ be a $\lambda$-partition of $V = V_n(q)$, and let $U, W$ be subspaces such that $V = U \oplus W$. If $\pi : V \to U$ is the projection onto $U$ associated with the above direct sum decomposition of $V$, then $\pi$ induces a $\lambda q^m$-partition $(B, \beta)$ of $U$ where $m = \dim(W)$, $B = \{(a, w) : a \in A, w \in W, a, w \in \alpha(a) \neq \alpha(a)\}$, and $\beta : B \to 2^U$ is given by $\beta(a, w) = \pi(\alpha(a))$.

**Proof.** Note that for any $a \in A$, $\pi(\alpha(a))$ is a subspace of $U$, so it is clear that $\beta(a, w) = \pi(\alpha(a))$ is a subspace of $U$ for all $(a, w) \in B$. Since $W \cap \alpha(a) \neq \alpha(a)$, we get $\beta(a, w) = \pi(\alpha(a)) \neq \{0\}$.

Let $u \in U^* = U \setminus \{0\}$ and let $B_u = \{(a, w) \in B : u \in \beta(a, w)\}$. We now show that $|B_u| = \lambda q^m$ by counting in two ways the cardinality of the set
\[ S = \{(u, w) : u \in U^*, w \in W, \text{ and } u \in \beta(a, w) \text{ for some } a \in A\}. \]

For each $u \in U^*$, there are exactly $|B_u|$ subspaces $\beta(a, w) \in B$ that contain $u$. So $|S| = |U^*| |B_u|$. On the other hand, for each of the $|U^*| |W|$ pairs $(u, w)$ with $u \in U^*$ and $w \in W$, the number of $a \in A$ such that $u \in \beta(a, w)$ is the same as the number of $a \in A$ such that the vector $v = u + w$ is in the subspace $\alpha(a)$. Since this latter number is $\lambda$, we also have $|S| = \lambda |U^*| |W|$. Combining these two counts of $|S|$ yields
\[ |U^*| |B_u| = |S| = \lambda |U^*| |W| \Rightarrow |B_u| = \lambda |W| = \lambda q^m, \]
which concludes the proof.

It follows from the above construction that the type of the $\lambda q^m$-partition will depend on the relationship between the subspaces $\alpha(a)$ and the subspace $W$. In particular, if $n_a = \dim \alpha(a)$ and $r_a = \dim(\alpha(a) \cap W)$, then this subspace will contribute $q^{n_a}$ copies of a subspace of dimension $n_a - r_a$ in the new partition $(B, \beta)$. In this way, we can decompose every subspace $\alpha(a)$ of $(A, \alpha)$ to determine a $\lambda q^m$-partition of $U$.

**Example 2**
Consider $V_5(2)$. We can identify $V_5(2)$ with a 5-dimensional subspace $V$ of $V_6(2)$ and let $W$ be a one-dimensional complement of $V$. Let $(A, \alpha)$ be a partition of $V_6(2)$ of type $[(21, 2)]$. Since $W$ is one-dimensional, it is contained in exactly one of the two-dimensional subspaces. Hence the
2-partition induced on \( V \) is of type \([(20, 2), (2, 1)]\). Similarly, we can see that a \([(9, 3)]\) partition of \( V_6(2) \) induces a 2-partition of \( V \) of type \([(8, 3), (2, 2)]\).

One important special case of the above is when \((A, \alpha)\) is a 1-partition and \( W = \bigcup_{a \in C} \alpha(a) \) for some proper subset \( C \subset A \). If this is the case, we can take \( B = A \setminus C \) and get a \( q^n\)-partition of \( V \).

A second technique for generating \( \lambda \)-partitions from 1-partitions is given in the theorem below.

**Theorem 10** Let \( V = V_n(q) \) and let \((A, \alpha)\) be a 1-partition of type \( n^t n \cdots 2^t 1 \). (Here we allow the possibility that \( t_j = 0 \) if \( j > 1 \).) Then for any integer \( 1 \leq k \leq n \), there exists a \( \lambda \)-partition \((B, \beta)\) of type

\[
n^{\lambda n} \cdot (k + 1)^{\lambda k + 1} \cdot (k - 1)^{\lambda k - 1} \cdots 2^{\lambda 2}
\]

where \( \lambda = \frac{q^k - 1}{q - 1} \) and let \( B = (A_+ \times C) \cup A_1 \).

**Proof.** Let us identify \( V \) with the field \( \mathbb{F}_q^n \) and let \( W \) be a subspace of \( V \) of dimension \( k \). Define \( A_1 = \{a \in A: \dim \alpha(a) = 1\} \) and \( A_+ = A \setminus A_1 \). Furthermore, let \((C, \gamma)\) be a 1-partition of \( W \) of type \( 1^\lambda \) where \( \lambda = \frac{q^k - 1}{q - 1} \) and let \( B = (A_+ \times C) \cup A_1 \).

Then we can define a function \( \beta : B \to 2^V \) as follows. If \( y = (a, c) \in A_+ \times C \), define \( \beta(y) = \beta(a, c) = \{x \cdot w : x \in \alpha(a), w \in \gamma(c)\} \). If \( y \in A_1 \), define \( \beta(y) = \{x \cdot w : x \in \alpha(y), w \in W\} \). We claim the pair \((B, \beta)\) is a \( \lambda \)-partition of \( V \). Indeed, if \( y = (a, c) \in A_+ \times C \), for any nonzero \( v_1, v_2 \in \beta(y) \) there exist \( x_1, x_2 \in \alpha(a), w_1, w_2 \in \gamma(c) \) such that \( v_1 = x_1 w_1 \) and \( v_2 = x_2 w_2 \). Since \( \gamma(c) \) is one-dimensional, there exists \( d \in \mathbb{F}_q \setminus \{0\} \) such that \( w_2 = dw_1 \), so \( v_2 = (dx_2)w_1 \). Hence, for any \( d' \in \mathbb{F}_q \setminus \{0\} \), we have \( v_1 + d'v_2 = x_1 w_1 + d'x_2 w_1 = (x_1 + d'x_2)w_1 \in \beta(y) \). Therefore \( \beta(y) \) is a subspace of \( V \). The proof that \( \beta(y) \) is a subspace of \( V \) when \( y \in A_1 \) is similar.

Note that for any \( x \in \mathbb{F}_q^n \) the function \( \phi_x : V \to V \) defined by \( \phi_x(v) = xv \) is a vector space automorphism. If \( y = (a, c) \in A_+ \times C \), then \( \gamma(c) \) is one-dimensional so for any nonzero \( w \in \gamma(c) \) we have \( \phi_w(\alpha(a)) = \{xw : x \in \alpha(a)\} = \{xw' : x \in \alpha(a), w' \in \gamma(c)\} = \beta(y) \). Hence \( \dim \beta(y) = \dim \alpha(a) \). Also, if \( y \in A_1 \), then \( \alpha(y) \) is one-dimensional so for any nonzero \( x \in \alpha(y) \) we have \( \phi_x(W) = \{xw : w \in W\} = \{(x'w) : w \in W, x' \in \alpha(y)\} = \beta(y) \). Therefore, \( \dim(\beta(y)) = \dim(W) = k \).

Next, we need to show that for any \( 0 \neq v \in V \) we have \( \{|x \in B : x \in \beta(y)\}| = \lambda \). But if \( y = (a, c) \in A_+ \times C \), we have \( v \in \beta(y) \Leftrightarrow \mathbb{F}_q w^{-1}v \subseteq \alpha(a) \) for some \( 0 \neq w \in \gamma(c) \). If \( y \in A_1 \), then \( v \in \beta(y) \Leftrightarrow \mathbb{F}_q w^{-1}v \subseteq \alpha(y) \) for some \( 0 \neq w \in \gamma(c) \). Therefore, since \((A, \alpha)\) is a 1-partition, \( \{|x \in B : x \in \beta(y)\}| = \{|\mathbb{F}_q w^{-1}v : 0 \neq w \in W\}| = \lambda \) since \( \dim(W) = k \).

Next, we use Theorem 10 to make an observation about the existence of a \( \lambda \)-partition of type \([(t_2, s), (t_1, r)]\) where \( r \) and \( s \) are distinct.

**Corollary 11** Let \( 1 < r \leq n \), \( 1 \leq s \leq n \) where \( r \neq s \). Then there exists a \( \left(\frac{q^a - 1}{q - 1}, s\right) \)-partition of type \( \left[\left(\frac{q^a - 1}{q - 1}, r\right), \left(\frac{q^n - q^r}{q - 1}, s\right)\right] \).

**Proof.** Let \( U \) be an \( r \)-dimensional subspace of \( V \). Let \( \mathcal{P} \) be a 1-partition consisting of \( U \) and all the one-dimensional subspaces not contained in \( U \). Then \( \mathcal{P} \) is a 1-partition of type \( r^1 q^1 \), where
\[ t = \frac{q^n - q^r}{q - 1}. \] Now we can apply Theorem 10 to this 1-partition to get a \( \left( \frac{q^n - 1}{q - 1} \right) \)-partition of \( V \) of type \( \left[ \left( \frac{q^n - 1}{q - 1}, r \right), \left( \frac{q^n - q^r}{q - 1}, s \right) \right] \).

Next, we note that if we are given a \( \lambda \)-partition \((A, \alpha)\), we can also take “multiples” of \((A, \alpha)\) as follows. For each positive integer \( k \), let \( kA \) be the set \( A \times \{1, 2, \ldots, k\} \) and define the function \( k\alpha : kA \to 2^V \) by \( (k\alpha)(x, i) = \alpha(x) \) for all \( x \in A \) and \( 1 \leq i \leq k \). Then \((kA, k\alpha)\) is a \( k\lambda \)-partition of \( V \). If \( \mathcal{P} = (A, \alpha) \), the we write \( k\mathcal{P} \) to indicate \((kA, k\alpha)\). Note that if \( \mathcal{P} = (A, \alpha) \) is of type \( n_1^{k_1} n_2^{k_2} \cdots n_s^{k_s} \), then \( k\mathcal{P} = (kA, k\alpha) \) is of type \( n_1^{kt_1} n_2^{kt_2} \cdots n_s^{kt_s} \).

In some sense, we can reverse the above process using the concept of multiplicity. We define the \textit{multiplicity} of the \( \lambda \)-partition \( \mathcal{P} = (A, \alpha) \) as the greatest common divisor of the set \( \{ |\alpha^{-1}(\alpha(a))| : a \in A \} \).

**Lemma 12** Let \((A, \alpha)\) be a \( \lambda \)-partition of multiplicity \( m > 1 \). Then there exists a \( (\lambda/m) \)-partition \((B, \beta)\) such that \((A, \alpha)\) is equivalent to \((mB, m\beta)\).

**Proof.** Let \((A, \alpha)\) be a \( \lambda \)-partition of \( V \) of multiplicity \( m \). Therefore, for every subspace \( W \in \{ \alpha(a) : a \in A \} \) there exists a positive integer \( k_W \) such that \( W \) occurs \( k_W m \) times in the multiset image of \( \alpha \). Now let \((B, \beta)\) be the \( (\lambda/m) \)-partition corresponding to the multiset where every \( W \in \{ \alpha(a) : a \in A \} \) occurs \( k_W \) times. Then it is straightforward to check \((A, \alpha)\) is equivalent to \((mB, m\beta)\) since they have the same multiset image.

**3 Dual \( \lambda \)-Partitions**

In this section, we use vector space duals to define the dual of a \( \lambda \)-partition. This is slightly more complicated than taking the dual of each subspace in a \( \lambda \)-partition since we can increase multiplicities when doing this. Therefore, to get the dual of a \( \lambda \)-partition, we take the vector space duals of each subspace and then adjust the multiplicity of the resulting \( \lambda' \)-partition to match that of the original \( \lambda \)-partition. In the lemma below, we state some basic results about vector spaces and their duals using non-degenerate symmetric bilinear forms. Refer to [1, Chapter 3] or [6, Chapter 8, §27] for proofs of these results.

Let \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}_q \) be a non-degenerate symmetric bilinear form. For example, we could use the standard dot product when \( V = \mathbb{F}_q^n \). Then \( \langle \cdot, \cdot \rangle \) induces an isomorphism between \( V \) and its dual, \( V^* = \text{Hom}(V, \mathbb{F}_q) \). For any subset \( S \subseteq V \), we define \( S^\perp = \{ v \in V : \langle v, x \rangle = 0 \text{ for every } x \in S \} \).

When \( x \in V \), we denote \( \{ x \}^\perp \) by writing \( x^\perp \).

**Lemma 13** Let \( S, T \) be subsets of a finite-dimensional vector space \( V \) over \( F \) and let \( \langle \cdot, \cdot \rangle : V \times V \to F \) be a symmetric non-degenerate bilinear form on \( V \). Then we have the following properties:

1. \( S^\perp \) is a subspace of \( V \).
2. \( S \subseteq T \Rightarrow T^\perp \subseteq S^\perp \).
3. \( S^\perp = \text{span}(S)^\perp \).
4. $\dim(S^\perp) = n - \dim(\text{span}(S))$.

5. $(S^\perp)^\perp = \text{span}(S)$.

6. $(S \cup T)^\perp = S^\perp \cap T^\perp$.

7. $(\text{span}(S) \cap \text{span}(T))^\perp = S^\perp + T^\perp$.

In the proofs below, we will use some of these standard properties of $S^\perp$. We start with an important example that we will use to build dual $\lambda$-partitions.

**Example 3**

Let $J \subseteq V$ be a set of nonzero vectors representing the one-dimensional subspaces of $V$. So if $J = \{x_1, x_2, \ldots, x_k\}$, we have the following properties:

1. $\bigcup_{i=1}^{k} F_q x_i = V$,

2. for any $x, y \in J$, we have $F_q x \cap F_q y \neq \{0\} \Rightarrow x = y$.

Note here that $k = |J| = \frac{q^n - 1}{q - 1}$.

Next, define a function $\alpha : J \rightarrow 2^V$ by $\alpha(x) = x^\perp$ for all $x \in J$. We claim that $(J, \alpha)$ forms a $\left(\frac{q^n-1}{q-1}\right)$-partition of $V$. It is clear that $\alpha(x) = x^\perp$ is a subspace for every $x \in J$. Also, for any $0 \neq v \in V$, we have $v \in x^\perp = \alpha(x) \iff x \in v^\perp$. So, since $\dim v^\perp = n - 1$, there are exactly $\left(\frac{q^n-1}{q-1}\right)$ elements $x \in J$ such that $v \in \alpha(x)$. Hence $(J, \alpha)$ is the claimed $\left(\frac{q^n-1}{q-1}\right)$-partition of $V$ of type $\left[\left(\frac{q^n-1}{q-1}\right), n - 1\right]$. Indeed, $(J, \alpha)$ is just the $q$-Grassmanian $G(n, n - 1)$ mentioned in our introduction.

Given a $\lambda'$-partition of $V$, we use Proposition 14 as a first step in accomplishing our goal of defining a $\lambda$-partition that is dual to the initial $\lambda'$-partition. We will then create such a dual through a series of reductions starting from the above example.

**Proposition 14** Let $U \subseteq V = V_n(q)$ be a subspace of dimension $r$. Let $Q \subseteq U$ consist of one nonzero vector representative for each one-dimensional subspace of $U$. (So for each $0 \neq u \in U$ there exists $x \in Q$ such that $F_q u = F_q x$; and for any $x, y \in Q$, if $F_q x = F_q y$, then $x = y$.) Then the following hold:

1. If $r = \dim(U) \geq 2$, then $\bigcup_{x \in Q} x^\perp = V$.

2. If $w \in U^\perp$, then the set $\{x \in Q : w \in x^\perp\}$ has order $\frac{q^r-1}{q-1}$.

3. If $w \not\in U^\perp$, then the set $\{x \in Q : w \in x^\perp\}$ has order $\frac{q^{r-1}-1}{q-1}$.
Proof. Choose \( x_1, \ldots, x_r \in Q \) so that \( \{x_1, \ldots, x_r\} \) is a basis of \( U \). Let \( 0 \neq v \in V \) and for each \( 1 \leq i \leq r \) define \( \gamma_i = \langle x_i, v \rangle \). If \( \gamma_j = 0 \) for any \( j \), then \( v \in x_j^\perp \subseteq \bigcup_{i=1}^r x_i^\perp \). If \( \gamma_j \neq 0 \) for all \( j \), then the vector 

\[
y = \left( \sum_{i=2}^r \gamma_i \right) x_1 - \gamma_1 \left( \sum_{i=2}^r x_i \right) \in U \setminus \{0\}
\]

satisfies 

\[
\langle y, v \rangle = \left( \sum_{i=2}^r \gamma_i \right) \langle x_1, v \rangle - \gamma_1 \left( \sum_{i=2}^r \langle x_i, v \rangle \right) = \left( \sum_{i=2}^r \gamma_i \right) \gamma_1 - \gamma_1 \left( \sum_{i=2}^r \gamma_i \right) = 0.
\]

So \( v \in y^\perp \). Since \( y \neq 0 \), there exists \( z \in Q \) such that \( \mathbb{F}_q y = \mathbb{F}_q z \). Therefore, \( v \in z^\perp \subseteq \bigcup_{x \in Q} x^\perp \). So we have established that \( \bigcup_{x \in Q} x^\perp = V \).

Next, since \( Q \subseteq U \), for every \( x \in Q \) we have \( U^\perp \subseteq x^\perp \); so for any \( w \in U^\perp \), the set \( \{x \in Q : w \in x^\perp\} = Q \), hence has order \( \frac{q^r - 1}{q - 1} \) as claimed.

Finally, if \( w \not\in U^\perp \), then for any \( x \in Q \subseteq U \) we have \( w \in x^\perp \iff x \in w^\perp \cap U \). But \( \dim(w^\perp \cap U) = r - 1 \) since \( \dim w^\perp = n - 1 \) and \( U \not\subseteq w^\perp \). Hence, there are exactly \( \frac{q^r - 1}{q - 1} \) one-dimensional subspaces of \( w^\perp \cap U \). So it follows that the order of the set \( \{x \in Q : w \in x^\perp\} \) is \( \frac{q^r - 1}{q - 1} \). 

We can use the above observations to make a “reduction” in the \( \lambda \)-partition \( P \) given in Example 3. In particular, based on the above proposition, if we are given an \( r \)-dimensional subspace \( U \subseteq V \), we can reduce \( \lambda \) by \( \frac{q^r - 1}{q - 1} \) by eliminating \( \frac{q^r - 1}{q - 1} \) subspaces of dimension \( n - 1 \) (corresponding to the \( x \in J \cap U \), where \( J \) is the set defined in Example 3) and replacing them with \( \lambda' \). Using the technique described above, given a \( \lambda' \)-partition \( (A, \alpha) \) of \( V \), if we naively try to define \( \alpha' : A \to 2^V \) by \( \alpha'(a) = (\alpha(a))^\perp \) for all \( a \in A \), we will not in general get a \( \lambda'' \)-partition for some \( \lambda'' \). Proposition 14 suggests a minor modification to this strategy to create such a \( \lambda'' \)-partition. We first demonstrate this technique through an example.

Example 4

Let \( V = V_6(2) \). For convenience, we can view the vectors of \( V_6(2) \) as a binary representation of an integer and then convert this to decimal form to represent this vector. Hence we use decimal notation to represent the nonzero vectors in \( V_6(2) \) in this example. For example, the vector \( (1, 1, 0, 1, 0, 1) \) would be represented by \( 1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 53 \).

Now consider the following subspaces of \( V_6(2) \), where we give only the nonzero vectors in each subspace: 
\[
U_1 = \{1, 2, 3, 4, 5, 6, 7\}, \ U_2 = \{8, 16, 24, 32, 40, 48, 56\},
\]
\[ U_3 = \{9, 18, 36, 27, 54, 63, 45\}, \quad U_4 = \{20, 35, 30, 55, 61, 41, 10\}, \]
\[ U_5 = \{38, 31, 53, 57, 42, 12, 19\}, \quad U_6 = \{29, 49, 58, 44, 11, 22, 39\}, \]
\[ U_7 = \{28, 46, 50\}, \quad U_8 = \{15, 51, 60\}, \quad U_9 = \{21, 43, 62\}, \quad U_{10} = \{14, 33, 47\}, \]
\[ U_{11} = \{13, 23, 26\}, \quad U_{12} = \{17, 37, 52\}, \quad U_{13} = \{25, 34, 59\}. \]

Then \( \{U_1, U_2, \ldots, U_{13}\} \) is a 1-partition of \( V_6(2) \) of type \([6, 3), (7, 2)\].

Next, we consider the following subspaces (again we only indicate the nonzero vectors in each subspace), where we use the standard dot product to define \( S^\perp \) for any subset \( S \subseteq V_6(2) \):
\[ U_1^\perp = \{8, 16, 24, 32, 40, 48, 56\}, \quad U_2^\perp = \{1, 2, 3, 4, 5, 6, 7\}, \]
\[ U_3^\perp = \{9, 18, 36, 27, 54, 63, 45\}, \quad U_4^\perp = \{11, 20, 31, 33, 42, 53, 62\}, \]
\[ U_5^\perp = \{15, 17, 30, 35, 44, 50, 61\}, \quad U_6^\perp = \{14, 19, 29, 39, 41, 52, 58\}, \]
\[ U_7^\perp = \{1, 12, 13, 22, 23, 26, 27, 34, 35, 46, 47, 52, 53, 56, 57\}, \]
\[ U_8^\perp = \{3, 12, 15, 21, 22, 25, 26, 37, 38, 41, 42, 48, 51, 60, 63\}, \]
\[ U_9^\perp = \{7, 10, 13, 19, 20, 25, 30, 34, 37, 40, 47, 49, 54, 59, 60\}, \]
\[ U_{10}^\perp = \{6, 10, 12, 16, 22, 26, 28, 33, 39, 43, 45, 49, 55, 59, 61\}, \]
\[ U_{11}^\perp = \{5, 11, 14, 18, 23, 25, 28, 32, 37, 43, 46, 50, 55, 57, 60\}, \]
\[ U_{12}^\perp = \{2, 8, 10, 21, 23, 29, 31, 36, 38, 44, 46, 49, 51, 57, 59\}, \]
\[ U_{13}^\perp = \{4, 9, 13, 17, 21, 24, 28, 34, 38, 43, 47, 51, 55, 58, 62\}. \]

It is straightforward to check that \( \{U_7, U_8, \ldots, U_{13}, 2U_1, 2U_2, \ldots, 2U_6\} \) is a 3-partition of \( V_6(2) \) of type \([7, 4), (12, 3)\), where we use \( 2U_j^\perp \) to denote two copies of \( U_j^\perp \). Note that here we needed two copies of the \( U_j^\perp \) of smallest dimension in order to make this a 3-partition.

Moreover, if we repeat this procedure for this new 3-partition (doubling \( U_i = (U_i^\perp)^\perp \) for \( 7 \leq i \leq 13 \), we get a 2-partition of type \([12, 3), (14, 2)\], which consists of two copies of the original 1-partition \( \{U_1, U_2, \ldots, U_{13}\} \), hence it has multiplicity 2.

Theorem 15 takes into account the multiplicities that can occur and uses Lemma 12 to give us a range of possible candidates for a dual partition. We then identify the candidate with the same multiplicity as the original \( \lambda \)-partition to be the dual partition.

Before stating Theorem 15, we will need to introduce the concept of d-multiplicity. Given a \( \lambda \)-partition \( \mathcal{P} = (Y, \omega) \) of \( V \), let \( D = \{\dim \omega(y) : y \in Y\} \). For each \( d \in D \) define the d-multiplicity \( \mu_d \) of \( \mathcal{P} \) to be the greatest common divisor of the set \( \{\omega^{-1}(\omega(y)) : y \in Y \text{ and } \dim \omega(y) = d\} \). (If \( d \notin D \), we can define \( \mu_d \) to be 0.) It follows from the definitions that the multiplicity of \( \mathcal{P} \) is the greatest common divisor of \( \{\mu_d : d \in D\} \).

**Theorem 15** Let \( \mathcal{P} = (Y, \omega) \) be a \( \lambda \)-partition of \( V = V_n(q) \) of type \([a_k, k), (a_{k-1}, k-1), \ldots, (a_{s+1}, s+1), (a_s, s)\] where \( a_k a_s \neq 0 \). For each \( s \leq d \leq k \), let \( \mu_d \) denote the d-multiplicity of \( \mathcal{P} \). Then for every \( \ell \geq 1 \) such that \( \ell \) is a common divisor of the set \( \{\mu_k q^k, \mu_{k-1} q^{k-1}, \ldots, \mu_s q^s\} \), there exists a \( \lambda_\ell \)-partition \( \mathcal{P}(\ell) = (C_\ell, \gamma_\ell) \) of \( V \) such that:

1. \( \lambda_\ell = \frac{1}{\ell} \left( \sum_{i=s}^{k} a_i \right) - \lambda = \frac{1}{\ell} (|Y| - \lambda) \).

2. \( \mathcal{P}(\ell) \) is of type

\[
\left( \left( \frac{a_s q^s}{\ell}, n-s \right), \left( \frac{a_{s+1} q^{s+1}}{\ell}, n-s-1 \right), \ldots, \left( \frac{a_k q^k}{\ell}, n-k \right) \right).
\]
3. \( \{ \gamma_{\ell} : c \in C_{\ell} \} = \{ \omega(y)^{\perp} : y \in Y \} \) as sets.

4. \( \gamma_{\ell}^{-1} \left( \omega(y)^{\perp} \right) = \frac{q_{\ell}^{r_{y}}}{\ell} \left| \omega^{-1}(\omega(y)) \right| \) where \( r_{y} = \dim \omega(y) \).

Proof. Let \((J, \alpha)\) be the \( \left( \frac{q^{n-1} - 1}{q - 1} \right) \)-partition of \((n-1)\)-dimensional subspaces of \(V\) defined in Example 3, where \( \alpha(x) = x^{\perp} \) for all \( x \in J \). Let \((Y, \omega)\) be a \( \lambda \)-partition of \(V\) of type \([(a_{k}, k), \ldots, (a_{s}, s)]\), where \( a_{k}a_{s} \neq 0 \) and \( m = \sum_{i=s}^{k} a_{i} \) is the size of \((Y, \omega)\). For each \( y \in Y \), let \( r_{y} = \dim(\omega(y)) \).

Next, consider the Cartesian product \( J \times Y \) and the canonical projection \( \pi : J \times Y \to J \) onto \( J \) defined by \( \pi(x, y) = x \) for all \((x, y) \in J \times Y \). Define

\[ A = \{ (x, y) \mid y \in Y, x \in \omega(y) \} \subseteq J \times Y. \]

We claim that \((A, \alpha\pi)\) is a \( \lambda \left( \frac{q^{n-1} - 1}{q - 1} \right) \)-partition of \(V \). Clearly \( \alpha\pi(x, y) = \alpha(x) = x^{\perp} \) is a subspace for all \((x, y) \in A \). Let \( 0 \neq v \in V \). Then

\[ v \in \alpha\pi(x, y) \iff v \in x^{\perp} \text{ and } x \in \omega(y) \iff x \in v^{\perp} \cap \omega(y). \]

So

\[ \left| \{ (x, y) \in A : v \in \alpha\pi(x, y) \} \right| = \sum_{y \in Y} \frac{1}{q - 1} \left| v^{\perp} \cap \omega(y) \right| = \lambda \left( \frac{q^{n-1} - 1}{q - 1} \right), \]

where the last equality follows because \((Y_{W}, \omega_{W})\) is a \( \lambda \)-partition of \( W = v^{\perp} \) by Lemma 1.

Now, for each \( y \in Y \), let \( A_{y} = \{ (x, y) \in A : x \in \omega(y) \} \), and define \( \alpha_{y} : A_{y} \to 2^{V} \) to be the restriction of \( \alpha\pi \) to \( A_{y} \). Then \((A, \alpha\pi) = \left( \bigcup_{y \in Y} A_{y}, \bigcup_{y \in Y} \alpha_{y} \right) \). For each \( y \in Y \), choose a subset \( B_{y} \subseteq A_{y} \) of cardinality \( q^{r_{y} - 1} \), let \( B = \bigcup_{y \in Y} B_{y} \), and define a function \( \beta : A \to 2^{V} \) by

\[ \beta(x, y) = \begin{cases} \omega(y)^{\perp} & \text{if } (x, y) \in B \\ V & \text{if } (x, y) \in A \setminus B \end{cases} \]

for all \((x, y) \in A \).

We claim that \((A, \beta)\) is a \( \lambda \left( \frac{q^{n-1} - 1}{q - 1} \right) \)-partition of \(V \).

Proof of Claim: It is clear that \( \beta(x, y) \) is a subspace of \(V\) for all \((x, y) \in A \). Next, for any \( 0 \neq v \in V \), we let \( S_{v} = \{ (x, y) \in A : v \in \alpha\pi(x, y) \} \) and \( T_{v} = \{ (x, y) \in A : v \in \beta(x, y) \} \). We prove that \( |T_{v}| = |S_{v}| \) and we know \( |S_{v}| \) has the required cardinality since \((A, \alpha\pi)\) is a \( \lambda \left( \frac{q^{n-1} - 1}{q - 1} \right) \)-partition of \(V \).

Note that since \( A \) is the disjoint union of the \( A_{y} \) for \( y \in Y \), it suffices to show that \( |T_{v} \cap A_{y}| = |S_{v} \cap A_{y}| \) for all \( y \in Y \). So fix \( y \in Y \). If \( v \in \omega(y)^{\perp} \), then \( A_{y} \cap T_{v} = A_{y} = A_{y} \cap S_{v} \), where the last equality follows from Proposition 14(2). If \( v \notin \omega(y)^{\perp} \), then \( |A_{y} \cap T_{v}| = |A_{y}| - |B_{y}| = \frac{q^{r_{y} - 1} - 1}{q - 1} \) and, it follows from Proposition 14(3) that \( |A_{y} \cap T_{v}| = |A_{y} \cap S_{v}| \). Therefore, our claim is established.
Now consider the pair \((B, \beta_0)\), where \(\beta_0\) is the restriction of \(\beta\) to \(B\). By definition, it follows that \(\{\beta_0(x, y) : (x, y) \in B\} = \{\omega(y)^{\perp} : y \in Y\}\) as sets. Furthermore, \((B, \beta_0)\) is also a \(\lambda_0\)-partition of \(V\) for some \(\lambda_0\) since for all \((x, y) \in A \setminus B, \beta(x, y) = V\). We can compute \(\lambda_0\) as follows.

\[
\lambda_0 = \lambda \left( \frac{q^{n-1} - 1}{q - 1} \right) - \sum_{y \in Y} \left( \frac{q^{s-1} - 1}{q - 1} \right) = \lambda \left( \frac{q^{n-1} - 1}{q - 1} \right) - \sum_{k=0}^{n} a_i \left( \frac{q^{i-1} - 1}{q - 1} \right).
\]

But, since \((Y, \omega)\) is a \(\lambda\)-partition, we know

\[
\sum_{i=0}^{n} a_i (q^i - 1) = \lambda (q^n - 1) \Rightarrow \lambda q^{n-1} - \left( \sum_{i=0}^{n} a_i q^{i-1} \right) = \frac{1}{q} \left( \lambda - \left( \sum_{i=0}^{n} a_i \right) \right).
\]

Hence we see that

\[
\lambda_0 = \frac{1}{q - 1} \left[ \left( \lambda q^{n-1} - \sum_{i=0}^{n} a_i q^{i-1} \right) - \left( \lambda - \sum_{i=0}^{n} a_i \right) \right] = \frac{1}{q - 1} \left[ \left( \frac{1}{q} \left( \lambda - \sum_{i=0}^{n} a_i \right) \right) - \left( \lambda - \sum_{i=0}^{n} a_i \right) \right] = \frac{1}{q - 1} \left( \frac{1-q}{q} \right) \left( \lambda - \sum_{i=0}^{n} a_i \right) = \frac{1}{q} \left( \sum_{i=0}^{n} a_i \right) - \lambda = \frac{1}{q} (|Y| - \lambda).
\]

Furthermore, \((B, \beta_0)\) is of type

\[
\left[ (a_s q^{s-1}, n-s), (a_{s+1} q^{s}, n-s-1), \ldots, (a_{k} q^{k-1}, n-k) \right].
\]

Because \(\beta_0\) is constant when restricted to \(B_y = A_y \cap B\), in \((B, \beta_0)\) we have \(|\beta^{-1}_0(\omega(y)^{\perp})| = |\beta^{-1}_0(\beta_0(x, y))| = |B_y|\omega^{-1}(\omega(y)) = q^{s-1}|\omega^{-1}(\omega(y))|\), where \((x, y) \in B\). Therefore, for any \(s \leq d \leq k\), the \((n-d)\)-multiplicity of \((B, \beta_0)\) is \(\mu_d q^{d-1}\). Hence the multiplicity of \((B, \beta_0)\) is the greatest common divisor \(g\) of the set \(\{\mu_s q^{s-1}, \mu_{s+1} q^{s-1}, \ldots, \mu_k q^{k-1}\}\). So by Lemma 12, there exists a \(\lambda'-\)subpartition \((C, \gamma)\) of \((B, \beta_0)\) of multiplicity 1 of type

\[
\left[ \left( \frac{a_s q^{s}}{qg}, n-s \right), \left( \frac{a_{s+1} q^{s+1}}{qg}, n-s-1 \right), \ldots, \left( \frac{a_k q^{k}}{qg}, n-k \right) \right],
\]

where

\[
\lambda' = \frac{\lambda_0}{g} = \frac{1}{qg} (|Y| - \lambda).
\]

Furthermore, for every \((x, y) \in B\), there exists a \(c \in C\) such that \(\gamma(c) = \beta_0(x, y) = \omega(y)^{\perp}\).

Finally, to get the partition \(P^{(s)} = (C, \gamma)\), we take the \((qg)/\ell\) multiple of \((C, \gamma)\) as described in Lemma 12 and the discussion immediately preceding it. Then \(P^{(s)}\) satisfies the conclusion of the
Given a $\lambda'$-partition $\mathcal{P}$ of $V$, in Theorem 15 there is a smallest partition $\mathcal{P}^{\min}$ of multiplicity 1 that occurs when $\ell$ is maximized.

**Definition 2** Let $\mathcal{P} = (Y, \omega)$ be a $\lambda'$-partition of a vector space $V$ of multiplicity $m$. The dual $\lambda$-partition $\mathcal{P}^*$ of $\mathcal{P}$ is the $\lambda$-partition of multiplicity $m$ given by $m\mathcal{P}^{\min}$.

It follows from the definition of $\mathcal{P}^*$ that $(m\mathcal{P})^* = m(\mathcal{P}^*)$ for any $m \geq 1$.

**Corollary 16** Let $\mathcal{P}$ be a $\lambda$-partition. Then $(\mathcal{P}^*)^* = \mathcal{P}$.

**Proof.** Note that since for any $\lambda$-partition we have $(m\mathcal{P})^* = m(\mathcal{P}^*)$, it suffices to assume the multiplicity of $\mathcal{P}$ is 1.

Let $\mathcal{P} = (Y, \omega)$ be a partition of multiplicity 1 of type $[(a_k, k), \ldots, (a_s, s)]$, where $a_k a_s \neq 0$. Let $\mu_y$ denote the $d$-multiplicity of $\mathcal{P}$ for all $s \leq d \leq k$. Furthermore, let $\mathcal{P}^* = (Z, \xi)$ and $(\mathcal{P}^*)^* = (Z, \xi)$. Then it follows from Theorem 15(3) that

$$
\{\xi(z) : z \in Z\} = \{\gamma(c) : c \in C\} = \left\{\left(\omega(y)\right)^{\perp} : y \in Y\right\} = \{\omega(y) : y \in Y\}.
$$

Let $y \in Y$ and $z \in Z$ such that $\xi(z) = \omega(y)$. It suffices to show $|\xi^{-1}(\xi(z))| = |\omega^{-1}(\omega(y))|$. Let $c \in C$ be such that $\gamma(c)^\perp = \omega(y) = \xi(z)$. By Theorem 15(4) it follows that the $d$-multiplicity of $\mathcal{P}^*$ is $(\mu_{n-d} q^{n-d})/g$ for $n-k \leq d \leq n-s$, so

$$
|\xi^{-1}(\xi(z))| = \frac{q^{n-r_y}}{g'} |\gamma^{-1}(\omega(y)^\perp)| = \frac{q^s q^{n-r_y}}{g' g} |\omega^{-1}(\omega(y))|
$$

where $r_y = \text{dim} \omega(y)$, $g$ is the gcd of $\{\mu_k q^k, \mu_{k-1} q^{k-1}, \ldots, \mu_s q^s\}$, and $g'$ is the gcd of the set

$$
\left\{\frac{\mu_s q^s}{g} q^{n-s}, \frac{\mu_{s-1} q^{s-1}}{g} q^{n-s+1}, \ldots, \frac{\mu_k q^k}{g} q^{n-k}\right\}.
$$

Therefore, $g' g$ is the gcd of the set $\{\mu_k q^k, \mu_{k-1} q^{k-1}, \ldots, \mu_s q^s\}$, hence $g' g = q^n$ since we assumed the multiplicity of $\mathcal{P}$ was 1. So it follows that $|\xi^{-1}(\xi(z))| = |\omega^{-1}(\omega(y))|$, hence $(\mathcal{P}^*)^* = \mathcal{P}$, as claimed.

Many of the $\lambda$-partition types that we have discussed above seem realizable to be duals of 1-partitions. An example of a minimal $\lambda$-partition that is not the dual of a 1-partition is the 7-partition of $V_8(2)$ of type $3^{255}$. In order for this to have been a dual partition of a 1-partition, we would need a 1-partition of $V_8(2)$ of type $5^{255}$, which is clearly impossible.

### 4 $\lambda$-partitions and Designs Over Finite Fields

A number of well-studied mathematical structures arise from certain partitions of finite vector spaces. For example, if $\mathcal{P}$ is the set of all subspaces of $V_n(q)$ (which is a $\lambda$-partition of $V_n(q)$),
then the set of all cosets of the elements of $\mathcal{P}$, denoted by $\text{AG}(n,q)$, is what is known as the affine geometry of dimension $n$ over $\mathbb{F}_q$ (see [2]). Similarly, the set of all subspaces of $V_{n+1}(q)$, denoted by $\text{PG}(n,q)$, is the projective geometry of dimension $n$ over $\mathbb{F}_q$. Other designs arise similarly either from taking cosets of subspaces in a partition or from taking the subspaces themselves as blocks in the design. We will first define these terms.

A design is a pair $(X, \mathcal{A})$, where $X$ is a set of elements called points, and $\mathcal{A}$ is a collection of nonempty subsets of $X$ called blocks. Suppose $v \geq 2, \lambda \geq 1$, and $L \subseteq \{n \in \mathbb{Z} : n \geq 2\}$. A $(v, L, \lambda)$-pairwise balanced design (abbreviated $(v, L, \lambda)$-PBD) is a design $(X, \mathcal{A})$ where: (1) $|X| = v$, (2) $|A| \in L$ for all $A \in \mathcal{A}$, and (3) every pair of distinct points is contained in exactly $\lambda$ blocks. It is easy to see that a $(v, L, \lambda)$-PBD is equivalent to a decomposition of the $\lambda$-fold complete multigraph $\lambda K_v$ into complete subgraphs with orders in $L$. A $(v, \{k\}, \lambda)$-PBD is better known as a balanced incomplete block design and is denoted by $(v, k, \lambda)$-BIBD.

Suppose $(X, \mathcal{A})$ is a $(v, L, \lambda)$-PBD. A parallel class in $(X, \mathcal{A})$ is a subset of disjoint blocks from $\mathcal{A}$ whose union is $X$. A partition of $\mathcal{A}$ into $r$ parallel classes is called a resolution, and $(X, \mathcal{A})$ is said to be a resolvable PBD if $\mathcal{A}$ has at least one resolution.

A parallel class in a $(v, L, \lambda)$-PBD is uniform if every block in the parallel class is of the same size. Let $L = \{\ell_1, \ell_2, \ldots, \ell_r\}$ be an ordered set of integers $\geq 2$ and let $R = \{t_1, t_2, \ldots, t_r\}$ be an ordered multiset of positive integers. A uniformly resolvable design, denoted $(v, L, \lambda, R)$-URD, is a resolvable $(v, L, \lambda)$-PBD with $t_i$ parallel classes with blocks of size $\ell_i$ for $1 \leq i \leq r$. It is easy to see that a $(v, \{\ell_1, \ldots, \ell_r\}, \lambda, \{t_1, \ldots, t_r\})$-URD is equivalent to a factorization of $\lambda K_v$ into $t_i$ $K_{\ell_i}$-factors for $1 \leq i \leq r$. For some of the necessary conditions for the existence of URDs, we direct the reader to [7] and the references therein.

If $W$ is a subset of $V_n(q)$, we denote the complete graph with vertices labeled with elements of $W$ by $K(W)$. If $W$ and $X$ are subsets of $V_n(q)$ with $0 \notin X$, we define $G(W, X)$ to be the subgraph of $K(V_n(q))$ with edge set $\{(w, w + x) : w \in W, x \in X\}$. It is easy to see that if $X$ is a subspace of $V_n(q)$ of dimension $n_i$, then $G(V_n(q), X \setminus \{0\})$ is a $K_{q^{n_i}}$-factor of $K_{q^n}$. Moreover, if $X_1$ and $X_2$ are disjoint subspaces, then the factors they induce are also disjoint. Thus a $\lambda$-partition $\mathcal{P}$ of $V_n(q)$ of type $[(t_1, n_1), \ldots, (t_k, n_k)]$ induces a factorization of $\lambda K_{q^n}$ into $t_i$ $K_{q^{n_i}}$-factors for $1 \leq i \leq k$. Equivalently, if we let $\mathcal{A}$ denote the subspaces in $\mathcal{P}$, along with all their cosets, then, $(V_n(q), \mathcal{A})$ is a $(q^n, \{q^{n_1}, \ldots, q^{n_k}\}, \lambda, \{t_1, \ldots, t_k\})$-URD. Thus we have the following result on URDs as a corollary to Corollary 11.

**Corollary 17** Let $1 < r \leq n$, $1 \leq s \leq n$ where $r \neq s$ and let $q$ be a prime power. Then there exists a $(q^n, \{q^r, q^s\}, \frac{q^r - 1}{q^s - 1}, \frac{q^s - 1}{q^r - 1}, \frac{q^r - q^s}{q^r - 1})$-URD.

Similarly, we have the following result on resolvable designs as a corollary to Proposition 7.

**Corollary 18** Let $q$ be a prime power and let $k, n$ be positive integers with $k \leq n$. Let $r = \gcd(k, n)$. Then there exists a resolvable $(q^n, q^k, \frac{q^k - 1}{q^n - 1})$-BIBD.

Another related area with potential applications for $\lambda$-partitions with additional properties is the area of designs over finite fields (see [4], for example). A $t$-$(n, k, \lambda; q)$ design is a collection $\mathcal{B}$ of $k$-dimensional subspaces of an $n$-dimensional vector space over $\mathbb{F}_q$ with the property that any $t$-dimensional subspace is contained in exactly $\lambda^t$ members of $\mathcal{B}$. It is also called a design over a finite field or a $q$-analog of $t$-$(n, k, \lambda)$ design. The collection $\mathcal{B}$ is necessarily a $\lambda$-partition of $V_n(q)$.
The first nontrivial example for \( t \geq 2 \) was given by S. Thomas [13]. Namely, he constructed a series of \( 2-(n, 3, 7; 2) \) designs for all \( n \geq 7 \) satisfying \( (n, 6) = 1 \).

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References


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