On the existence of a $(2,3)$-spread in $V(7,2)$

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Abstract

An (s,t)-spread in a finite vector space $V = V(n,q)$ is a collection $\mathcal{F}$ of t-dimensional subspaces of V with the property that every s-dimensional subspace of V is contained in exactly one member of $\mathcal{F}$. It is remarkable that no (s,t)-spreads has been found yet, except in the case $s = 1$.

In this note, the concept α-point to a $(2,3)$-spread $\mathcal{F}$ in $V = V(7,2)$ is introduced. A classical result of Thomas, applied to the vector space V, states that all points of V cannot be α-points to a given $(2,3)$-spread $\mathcal{F}$ in V. In this note, we strengthened this result by proving that every 6-dimensional subspace of V must contain at least one point that is not an α-point to a given $(2,3)$-spread of V.

1 Introduction

An (s,t)-*spread* in the finite vector space $V = V(n,q)$ over $\mathrm{GF}(q)$ is a collection $\mathcal{F}$ of t-dimensional subspaces of V with the property that every s-dimensional subspace of V is contained in exactly one member of $\mathcal{F}$. So far no (s,t)-spread, with $s > 1$, has been found, and it was conjectured by Metsch that none exists, see [1] for a survey.

If there exists an (s,t)-spread $\mathcal{F}$ in V then for any point P in V, the members of $\mathcal{F}$ that contain P induce an $(s-1,t-1)$-spread $\mathcal{F}_P$ in the quotient space V/P. A $(1,t)$-spread, or for short *spread*, $\mathcal{S}$ of V is called *geometric* if for any three members S_1, S_2 and S_3 of $\mathcal{S}$ such that $S_3 \cap \langle S_1 \cup S_2 \rangle \neq \{0\}$, we have $S_3 \subseteq \langle S_1 \cup S_2 \rangle$.

Thomas [2] proved the following theorem.

Theorem 1 *Given a $(2,t)$-spread $\mathcal{F}$ of $V = V(n,q)$, there exists a point P in V such that the derived $(1,t-1)$-spread $\mathcal{F}_P$ is not geometric.*

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It must be remarked that geometric spreads are the spreads that are most natural and "easiest" to construct, although most of the spreads are not geometric.

The existence of $(2,3)$-spreads in $V(7,2)$ is the "first" open case for this conjecture. In this note, we give a property of $(2,3)$-spreads in $V(7,2)$, which, in this particular case, yields the result of Thomas as a corollary.

Assume that $\mathcal{F}$ is a $(2,3)$-spread in $V = V(7,2)$. As every spread in a 6-dimensional subspace U of V is of size 21, we get that every 1-dimensional subspace P, or *point*, of V is contained in 21 members of $\mathcal{F}$. As each of these 21 members of $\mathcal{F}$ contains 7 points, of which three belongs to U, it follows that U contains 45 members of $\mathcal{F}$. Similarly, we may derive that every point P in U is contained in exactly 5 of these 45 members of $\mathcal{F}$ and that every 5-dimensional subspace T of U contains exactly five members of $\mathcal{F}$.

We will say that a point P is an *α-point* to $\mathcal{F}$ if every 5-dimensional subspace T of V that contains two of the members of $\mathcal{F}$ that meet at P, has the property that all its five members from $\mathcal{F}$ will meet at the point P. From the definition of a geometric spread, it follows that in the case of $(2,3)$-spreads in $V = V(7,2)$, Theorem 1 of Thomas states that at least one point of V is not an α-point to $\mathcal{F}$.

We will show the following Theorem.

Theorem 2 *Assume that $\mathcal{F}$ is a $(2,3)$-spread in $V = V(7,2)$. Every 6-dimensional subspace of V contains at least one point which is not an α-point to $\mathcal{F}$.*

2 Proof of Theorem 2

Assume that $\mathcal{F}$ is a $(2,3)$-spread in $V = V(7,2)$. Let U be any 6-dimensional subspace of V. Assume that all points in U are α-points to $\mathcal{F}$. Then every 5-dimensional subspace T of U will contain a point P where all its five members of $\mathcal{F}$ meet. This point P will be called the *α-point of T*. Moreover, each point P of U is contained in exactly five of the members of $\mathcal{F}$ that belong to U, and hence these five members of $\mathcal{F}$ that meet the point P will all belong to the same 5-dimensional subspace T of U.

We claim that there is a 4-dimensional subspace W of U that does not contain any member of $\mathcal{F}$. To see this, just observe that every 3-dimensional subspace of a 5-dimensional subspace T of U is contained in exactly three 4-dimensional subspaces of T, and as T contains exactly five members of $\mathcal{F}$, there will be at least 16 subspaces W of dimension 4 of T that do not contain any member of $\mathcal{F}$. Such a 4-dimensional subspace W of U will be called a *poor space*.

There are three 5-dimensional subspaces T_1, T_2 and T_3 of U such that

$$W = T_1 \cap T_2 = T_1 \cap T_3 = T_2 \cap T_3 \ , \qquad \text{and} \qquad U = T_1 \cup T_2 \cup T_3 \ . \quad (1)$$

For $1 \leq i \leq 3$, let P_i be the α-point in the space T_i.

We first note that none of the points P_1, P_2, or P_3 belongs to W.

To prove this fact, assume for instance that P_1 belongs to W. Since W is a poor 4-dimensional space, each of the five members of $\mathcal{F}$ that belongs to U and contains the point P_1 meet W in two points, besides the point P_1. This leads to a contradiction since W contains 15 points and every point $Q \neq P_1$ in T_1 (and thus in W) belongs to exactly one of the five members of $\mathcal{F}$ in U that meet the point P_1.

Since $\mathcal{F}$ is a $(2,3)$-spread and since the points P_i, $1 \leq i \leq 3$, do not belong to W and they are the α-points of the respective spaces T_i, we can conclude that the members of $\mathcal{F}$ that are subspaces of T_i will intersect W in a spread $\mathcal{S}_i$. Furthermore, since $\mathcal{F}$ is a $(2,3)$-spread, these three spreads are mutually disjoint.

Now, let Q be any point of W. Let T_Q denote the unique 5-dimensional subspace of U, that contains the two members of $\mathcal{F}$ that meet the point Q and belong to T_1 and T_2, respectively. We note from Equation (1) that $P_1 \notin T_2 \cup T_3$ and $P_2 \notin T_1 \cup T_3$. Hence, T_Q cannot be one of the spaces T_i, $1 \leq i \leq 3$. As these are the only 5-dimensional subspaces of U that contain W, it follows that

$$\dim(T_Q \cap W) \leq 3 \ .$$

Moreover, since all 5-dimensional subspaces of U have a unique point where all its members of $\mathcal{F}$ meet, and as there are two members of $\mathcal{F}$ in T_Q meeting Q, we conclude that Q is the α-point of the space T_Q. This implies that the member of $\mathcal{F}$ that is a subspace of T_3 and meets the point Q must also belong to T_Q. This space will be denoted by $Z_{Q,3}$; and we define $Z_{Q,1}$ and $Z_{Q,2}$ similarly. For $1 \leq i \leq 3$, the intersection of $Z_{Q,i}$ with W is a 2-dimensional subspace which we denote by $L_{Q,i}$.

Now, the space $Z_{Q,3}$ is completely contained in T_Q and intersects W in the 2-dimensional space $L_{Q,3}$, which thus also must be a subspace of T_Q, so,

$$L_{Q,3} \subseteq T_Q \cap W = \langle L_{Q,1}, L_{Q,2} \rangle \ . \quad (2)$$

The last step in our proof is to show that there is at least one point Q in W, for which the above relation does not hold.

Let us assume for a moment that

$$\mathcal{S}_1 = \{\ L_1, L_2, \ldots, L_5\ \} \qquad \text{and} \qquad \mathcal{S}_2 = \{\ L'_1, L'_2, \ldots, L'_5\ \} \ .$$

Every member, or *line*, of $\mathcal{S}_2$ intersects three members of $\mathcal{S}_1$. Without loss of generality, we may assume that the line L'_5 does not intersect the lines

L_1 and L_2. These two lines together contain 6 points. Each of these 6 points is contained in exactly one of the lines of $\mathcal{S}_2$. As a line contains 3 points we get that there must be two lines, say L'_1 and L'_2, of $\mathcal{S}_2$ that meet both L_1 and L_2.

Let $Q = L_1 \cap L'_1$, $Q' = L_2 \cap L'_2$, $R_1 = L_1 \cap L'_2$ and $R_2 = L_2 \cap L'_1$, i.e., with the original notation

$$L_{Q,1} \cap L_{Q',2} = R_1 \qquad \text{and} \qquad L_{Q,2} \cap L_{Q',1} = R_2 \ . \tag{3}$$

Then the line L, that meets the points R_1 and R_2, satisfies the following relation

$$L = \langle R_1, R_2 \rangle = (T_Q \cap W) \cap (T_{Q'} \cap W) \ .$$

If the relation (2) holds for all points Q of W, then L will meet both the spaces $L_{Q,3}$ and $L_{Q',3}$. Note that L contains just three points, the above defined two points R_1 and R_2, and a third point R_3. So from Equation (3), we can infer that both the spaces $L_{Q,3}$ and $L_{Q',3}$ must meet L at the point R_3. This contradicts the fact that $\mathcal{S}_3$ is a spread and the proof is complete.

References

[1] K. Metsch, Bose-Burton type theorems for finite projective, Affine and Polar spaces, *Surveys in Combinatorics*, ed. by Lamb and Preece, London Mathematical Society, Lecture Notes Series 267, 1999.

[2] S. Thomas, Designs and partial geometries over finite fields, *G. Dedicata* **63** (1996), 247–253.

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