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On deformation of associative algebras and graph homology

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Abstract

Deformation theory of associative algebras and in particular of Poisson algebras is reviewed. The role of an "almost contraction" leading to a canonical solution of the corresponding Maurer–Cartan equation is noted. This role is reminiscent of the Homotopical Perturbation Lemma, with the infinitesimal deformation cocycle as "initiator."

Applied to star-products, we show how Moyal's formula can be obtained using such an almost contraction and conjecture that the "merger operation" provides a canonical solution at least in the case of linear Poisson structures.

Keywords: Deformation theory; Star-product; Graph homology

Contents

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1. Introduction

The aim of this article is to apply perturbation techniques to the case of the differential graded Lie algebras (DGLA) of graphs [4] which controls the deformation theory of associative algebras ([1,2], etc.).

Specifically, we investigate the Maurer–Cartan equation in the case of a differential Lie algebra in the presence of an "almost contraction" which leads to a "canonical solution." The role of the "merger operation" of [6] is unveiled, as providing such a mapping in the well-known case of Moyal formula, which provides a star-product in the case of a constant Poisson structure. It is conjectured that a similar merger operation exists in the general case (Conjecture 14), where the suitable combinatorial factors are still to be determined in a subsequent article [11]. The similarity with the homotopy perturbation lemma [12] is mentioned, to be exploited in the future work.

As a second "improvement" over the classical approach [1,2], we reduce the Maurer–Cartan equation to a Lie algebra equation, and point out, in a special case, the role of symmetry which seems to be the key for finding such a solution (Definition 8), a role also noted informally in the "correction analysis" of [6, p. 15].

The paper is organized as follows. We start with a brief review of Gerstenhaber theory of deformations of algebras [1], phrased in the context of differential graded Lie algebras, avoiding the Gerstenhaber pre-Lie operation. An "almost contraction" (2) is defined and the corresponding solution is constructed.

Section 3 applies the above technique to the generic case of the DGLA of graphs. In the constant Poisson structure case the Moyal formula is obtained in a way which gives us hope for the general case: Conjecture 14.

On the other hand, since the DGLA of graphs is a differential graded Lie algebra with differential $\partial = ad_m$, the bracket with a degree one element, a direct proof for the associativity of the Moyal formula at the level of Lie algebras is provided. It unravels a symmetry which will be studied in the general case, as part of the future work sketched in the concluding section.

2. Deformation theory of associative algebras

Given an associative algebra (A, m), a *star product* (deformation of m) is an associative $k[\![\hbar]\!]$-bilinear operation on $A_\hbar = A[\![\hbar]\!]$ [2, p. 5]. It is determined by the its values on $u, v \in A$:

$$u \star v = m(u, v) + \hbar m_1(u, v) + \hbar^2 m_2(u, v) + \cdots.$$

We will recall the constraints on the coefficients imposed by the associativity requirement.

2.1. Maurer–Cartan equation

Associativity of $m = m_0$ as well as of the star-product can be expressed conveniently using Gerstenhaber composition: $m \circ m = 0$ ([7, p. 9]; [1]). Let $\partial = [m, \cdot]$ be "bracketing with m," a square-zero differential, where $[\,,\,]$ denotes Gerstenhaber graded Lie bracket associated to the pre-Lie operation $\circ$, where the grading is the usual shifted degree of Hochschild DGLA $g = C^\bullet(A; A)$, so that $\deg(m_i) = 1$, $m_i : A \otimes A \to A$.

Grouping together coefficients of the powers of $\hbar$, we obtain the associativity conditions

$$\begin{aligned}
&m_0 \circ m_0 = 0,\\
&[m_0, m_1] = \partial m_1 = 0,\\
&[m_0, m_2] + m_1 \circ m_1 = \partial m_2 + m_1 \circ m_1 = 0,\\
&[m_0, m_3] + [m_1, m_2] = \partial m_3 + [m_1, m_2] = 0,\\
&\quad\vdots\\
&m_0 \circ m_n + m_1 \circ m_{n-1} + \cdots + m_{n-1} \circ m_1 + m_n \circ m_0\\
&\quad = \partial m_n + \sum_{j,k \geqslant 1,\ j+k=n} m_j \circ m_k = 0,\\
&\quad\ddots
\end{aligned} \tag{1}$$

The equations are equivalent to the Maurer–Cartan equation satisfied by the perturbation $\gamma = \star - m$ of m:

$$\partial\gamma + \frac{1}{2}[\gamma, \gamma] = 0.$$

Define trilinear maps

$$D_n = -\sum_{j,k \geqslant 1,\ j+k=n} m_j \circ m_k, \quad n \geqslant 1,$$

where the empty sum is zero. Note that by doubling terms and using the fact $[m_j, m_k] = [m_k, m_j]$ (all m_is are odd elements), we may rewrite

$$D_n = -\frac{1}{2} \sum_{j,k \geqslant 1,\ j+k=n} [m_j, m_k],$$

which has the advantage of involving the Lie algebra structure only, without making explicit use of the non-associative pre-Lie operation.

Lemma 1. *The following are equivalent*:

(i) *the product $\star$ is associative,*
(ii) $D_n = \partial m_n$, $n \geqslant 1$,
(iii) $[\star, \star] = 0$.

Proof. Regarding the equivalence between (i) and (ii), we only need to note that

$$[\star,\star]_n = \sum_{i,j\geqslant 0,\ i+j=n} [m_i, m_j] = 2(\partial m_n - D_n). \qquad \square$$

If the equations are satisfied up to order r we say $\star$ is an *rth order deformation of* m_0. Then the D_n satisfy the above equation up to order r, i.e. D_n are boundaries for $1 \leqslant n \leqslant r$.

As a consequence the following folklore fact is obtained ([1]; the "simple computation" of [2, p. 6]).

Lemma 2. *Let $m_1, \ldots, m_n$ be bilinear maps with $\partial m_1 = 0$. If $D_r = \partial m_r$ are boundaries for $2 \leqslant r \leqslant n$, then D_{n+1} is a cocycle*: $\partial D_{n+1} = 0$.

Proof. The key point is that $\star$ is a homogeneous element of degree one (after shifting), so that by the graded Jacobi identity

$$\big[[\star,\star],\star\big] = 0$$

the $(r+1)$-component vanishes

$$\sum_{i=1}^{r+1} \big[[\star,\star]_i, m_{r+1-i}\big] = 0.$$

The first r terms vanish anyway, since the assumption $D_i = \partial m_i$ is equivalent (after the "doubling trick") to $[\star,\star]_i = 0$ (see (iii) from Lemma 1). Therefore

$$\big[[\star,\star]_{r+1}, m_0\big] = 0,$$

i.e. $[\star,\star]_{r+1} = 2(\partial m_{r+1} - D_{r+1})$ is a cocycle. Then, since ∂m_{r+1} is a boundary, D_{r+1} is also a cocycle, concluding the proof. $\square$

2.2. Obstructions

We now review the problem of extending r-order deformations to $(r+1)$-order deformations for given initial conditions:

$$\star(0) = m, \qquad \frac{d\star}{d\hbar}(0) = m_1.$$

The first extension is possible if the homology class of $D_2 = -[m_1, m_1]$ is trivial. There are no possible "obstructions" if $H^3(C, \partial) = Z^3/B^3$ vanishes, where $C^m = \mathrm{Hom}(A^m, A)$, $Z^3 = \ker \partial_3$ and $B_3 = \mathrm{Im}\,\partial_2$:

$$0 \longrightarrow C_1 \xrightarrow{\partial_1} C_2 \xrightarrow{\partial_2} C_3 \xrightarrow{\partial_3} \cdots.$$

On the other hand, the deformation is equivalent to the trivial deformation $\star = m$ if $H^2(C, \partial) = 0$.

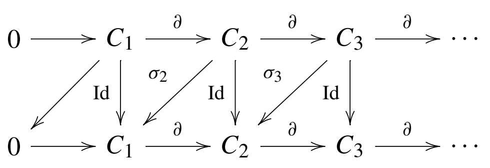
Assume a choice of m_2 such that $\partial m_2 = D_2$ has been made. Then the next obstruction is the homology class of D_3, and so on.

Even if H^3 is not zero, an inductively defined deformation exists if there is an *almost contraction* in degree three, i.e. a mapping σ satisfying the equation

$$\sigma : \mathcal{D} \subset Z_3 \to X_2, \qquad \partial\sigma + \sigma\partial = 1_{\mathcal{D}}, \tag{2}$$

where $\mathcal{D}$ is a subspace of cocycles containing D_n corresponding to the inductively defined m_n for all n.

Recall that if a contracting homotopy exist globally (for $n \geqslant 1$):



then the cohomology of the complex must be trivial $H(C^\bullet, \partial) = 0$.

2.3. *Almost contractions and homotopy perturbation theory*

Even if there is no contracting homotopy in degree 3, we still have a canonical solution if there are maps σ_3 and σ_4 acting as an *almost contraction*:

$$\partial\sigma_3 D_n + \sigma_4\partial D_n = D_n,$$

which continue to satisfy this identity as each D_n is computed out of the inductively defined m_n.

Indeed, if $m_n = \sigma D_n$, then $\partial m_n = D_n$ is equivalent to the above condition, since D_n are cocycles anyway. In lower degrees this yields

$$D_2 = -\frac{1}{2}[m_1, m_1], \tag{3}$$

$$m_2 = \sigma D_2 = -\frac{1}{2}\sigma\big([m_1, m_1]\big), \tag{4}$$

$$D_3 = -\frac{1}{2}\big([m_2, m_1] + [m_1, m_2]\big) = \frac{1}{2}\big[m_1, \sigma[m_1, m_1]\big], \tag{5}$$

$$m_3 = \sigma D_3 = \frac{1}{2}\sigma\big[m_1, \sigma[m_1, m_1]\big]. \tag{6}$$

Define $t = \sigma \circ ad_{m_1}$ and $\hat{m}_1^{n+1} = t^n(m_1)$, $n \geqslant 0$. Then we have

$$D_4 = -\frac{1}{2}\big([m_3, m_1] + [m_2, m_2] + [m_1, m_3]\big), \tag{7}$$

$$m_4 = \sigma D_4 = \hat{m}_1^4 - \frac{1}{2}\sigma\big(\big[\hat{m}_1^2, \hat{m}_1^2\big]\big). \tag{8}$$

It is natural to investigate the conditions under which such a "minimal procedure" with "initiator" t and cocycle m_1 exists. Its interpretation from the perspective of the Homotopical Perturbation Lemma [3, p. 10] will be considered elsewhere.

A case when such a procedure is successful is the one of the Moyal star-product

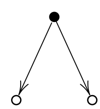
$$\star = \exp(\hbar m_1),$$

as it will be explained next, at the level of graphs.

3. Application to graphs

Let $\mathcal{G}_{n,m}$ be the set of *orientation classes* of *Lie admissible edge labeled graphs* of [9, p. 3], corresponding to **linear Poisson structures** (see also [4]). An element $\Gamma \in \mathcal{G}_{n,m}$ is a directed graph with n internal vertices, m labeled boundary vertices $1, 2, \dots, m$ (left to right in figures), such that each internal vertex is trivalent with exactly two descendants. The corresponding two outgoing arrows will be labeled left/right, defining the *orientation class* of the graph Γ up to a "negation" of the edge labeling in any two internal vertices [9]. The orientation class of a graph embedded in the plane will be determined by the positive orientation of the plane. The corresponding (graded) space is denoted by $\mathcal{G} = \bigcup \mathcal{G}_m$, where $\mathcal{G}_m = \bigcup_{n\in\mathbb{N}} \mathcal{G}_{n,m}$. Let C be the quotient of the DGLA of graphs $k\mathcal{G}$, with pre-Lie composition $\circ$ and differential $\partial = [b_0, \cdot]$ of [4], where $b_0 \in \mathcal{G}_{0,2}$, by the ideal generated by the Jacobi identity (9) [9].

The initial conditions of the "universal" deformation problem are $m_0 = b_0$ and $m_1 = b_1$, where

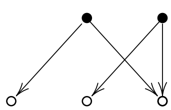
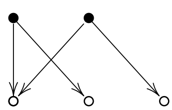
$b_0 =$, $b_1 =$  .

Recall that $b_0 \circ b_0 = 0$ and $[b_0, b_1] = 0$ [4, p. 13].

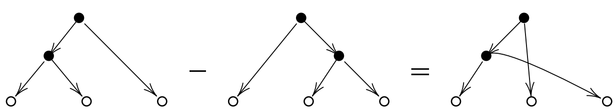
The first possible obstruction is the homology class of

$$D_2 = -b_1 \circ b_1 = -\left(t_2^R - t_2^L + c_2^L - c_2^R\right)$$

where

$c_2^R =$  and $c_2^L =$ 

and the graphs t_2^R, t_2^L are depicted in the LHS of the following diagram representing the Jacobi identity $t_2^R - t_2^L = c_2$

 (9)

Using this identity, D_2 simplifies to $D_2 = c_2^R - c_2^L - c_2$ (for additional details, see [4, p. 16]; [5, p. 20]).

3.1. Candidates for almost contractions

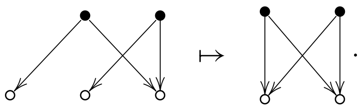
We claim that an almost contraction as needed earlier is the "merger operation" ([4, p. 10]; see also [6, p. 17]):

$$\sigma_i(\Gamma) = \Gamma/\big(i(i+1)\big), \quad \Gamma \in \mathcal{G}_{n,m}, \tag{10}$$

$$\sigma(\Gamma) = \frac{1}{2(2^n-2)} \sum_{i=1}^{m-1} (-1)^{i-1} \sigma_i(\Gamma), \tag{11}$$

where the quotient graph from the RHS of (10) is obtained by merging the ith and $(i+1)$st boundary points. If a non-admissible graph emerges after the merger, the result is considered to be zero.

For example, we have $\sigma(c_2^R) = \sigma_1(c_2^R) = \frac{1}{4}b_1^2$ (similarly $\sigma(c_2^L) = -\frac{1}{2}b_1^2$):



We will investigate the above claims in the special cases of constant and linear Poisson structures.

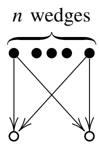
3.2. Constant Poisson structures

As an example we derive Moyal's formula along the previous lines using the "merger of legs" as an almost contracting operation.

The benefit of having a Poisson structure with constant coefficients is that a graph with an arrow landing on an internal vertex evaluates to zero under Kontsevich rule $B(\Gamma) = \mathcal{U}_\Gamma(\alpha^{\wedge n})$ where $\Gamma \in \mathcal{G}_{n,m}$ [7, pp. 23, 28].

Therefore

$\Gamma = b_1^n =$



is the unique graph in $\mathcal{G}_{n,2}$ not in the kernel of B.

In particular, the Jacobi identity (9) is automatically satisfied, since all the terms evaluate to zero under Kontsevich rule

$$B\big(t_2^R\big) = B\big(t_2^L\big) = B(c_2) = 0.$$

Lemma 3. *For any $i, j \geqslant 0$ we have*

$$\sigma\big(\big[b_1^i, b_1^j\big]\big) = -\frac{1}{2^{i+j-1}-1} b_1^{i+j},$$

where $b_1^n \in \mathcal{G}_{n,2}$, $n \geqslant 1$, with the natural orientation.

Proof. It is enough to note that the only term of $b_1^i \circ_1 b_1^j$ not vanishing after the application of σ, is the one for which all i of the left legs of b_1^i land on the left boundary point of b_1^j, since otherwise all consecutive boundary points are "bridged" by some b_1, and therefore the term vanishes under the merger operation

$$\sigma\left(b_1^i \circ_1 b_1^j\right) = -\frac{1}{2(2^{i+j}-2)} b_1^{i+j}. \qquad \square$$

It follows that $m_2 = \sigma D_2 = b_1^2/2$ and in general, we have

Lemma 4. *If $m_0 = b_0$, $m_1 = b_1$ and $m_n = \sigma D_n$, $n \geqslant 2$, then $\forall n$, $m_n = b_1^n/n!$.*

Proof. Assuming inductively that $m_k = b_1^k/k!$ for $1 \leqslant k \leqslant n-1$, then

$$m_n = \sigma D_n = -\frac{1}{2} \sum_{i+j=n,\ i,j\geqslant 1} \sigma\left(\left[\frac{b_1^i}{i!}, \frac{b_1^j}{j!}\right]\right) \tag{12}$$

$$= \left(-\frac{1}{2}\right)\left(-\frac{1}{2^{n-1}-1}\right) b_1^n \sum_{i+j=n,\ i,j\geqslant 1} \frac{1}{i!}\frac{1}{j!} = \frac{b_1^n}{n!}. \qquad \square \tag{13}$$

Now since the Moyal formula provides an associative product

$$* = e^{b_1 h}, \qquad [*,*] = 0,$$

$D_n = \partial m_n$ are boundaries and therefore, together with $m_n = \sigma D_n$, it implies that σ is an almost contraction for the inductively defined $m_n = \sigma D_n$, starting with the cocycle m_1:

$$\partial\sigma D_n + \sigma\partial D_n = D_n, \quad n \geqslant 2.$$

This, of course, amounts to $\partial\sigma D_n = D_n$, which in turn is equivalent to the original equation in degree n. Therefore we will give a direct proof that the above star-product is associative, in order to better understand the combinatorics involved. In contrast with the previous more general approach, we will take advantage of the fact that the differential ∂ is defined as a Lie bracket, and focus on the Lie algebra structure.

Proposition 5.

$$[*,*] = 0.$$

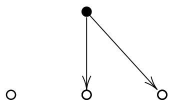
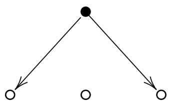
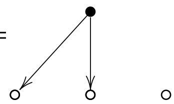
Proof. The nth homogeneous degree of the above equation is:

$$\sum_{i+j=n,\ i,j\geqslant 0} [m_i, m_j] = 0, \quad m_k = b_1^k/k!. \tag{14}$$

To prove it we will start by determining the structure coefficients of the Lie bracket. In order to isolate the combinatorial factors from the Lie algebra structure constants, it is better to adopt a basis with elements of the form $\Gamma/|\mathrm{Aut}(\Gamma)|$.

Consider $\{B_n = b_1^n/n!\}$ as a basis in $k\mathcal{G}_{\bullet,2}$. Incidentally, the solution of $[Z, Z] = 0$ is therefore the corresponding "integral" $* = \sum_n B_n$.

Consider the graphs $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{G}_{1,3}$, defined as follows:

$\Gamma_1 =$  , $\Gamma_2 =$  , and $\Gamma_3 =$  .

Then

$$\left\{\Gamma_{rst} = \left(\Gamma_1^r/r!\right)\left(\Gamma_2^s/s!\right)\left(\Gamma_3^t/t!\right)\right\}_{r,s,t\geqslant 0}$$

is a basis in $k\mathcal{G}_{\bullet,3}$ and

$$\forall i, j \geqslant 0, \quad [B_i, B_j] = \sum_{r+s+t=i+j} C_{(i,j)}^{(r,s,t)} \Gamma_{rst}.$$

To compute the coefficients C_I^J of Γ_J, where $I = (i, j)$ and $J = (r, s, t)$, consider $b_1^i \circ_1 b_1^j$ first and note that when splitting the i-left legs of b_1^i to make them land on the first two boundary points of b_1^j, the only graphs $\gamma = \Gamma_1^r \Gamma_2^s \Gamma_3^t$ that are involved are those for which $r + s = i$, $t = j$.

(1) If $r + s = i$ and $t = j$ then $b_1^i \circ_1 b_1^j$ contributes $i!/(r!s!)$ to γ, thus $C_I^J = 1$.
(2) If $r = i$ and $s + t = j$ then $b_1^j \circ_2 b_1^i$ contributes $-j!/(s!t!)$ to γ, thus $C_I^J = -1$.
(3) If $r + s = j$ and $t = i$ then $b_1^j \circ_1 b_1^i$ contributes $j!(r!s!)$ to γ, thus $C_I^J = 1$.
(4) If $r = j$ and $s + t = i$ then $b_1^i \circ_2 b_1^j$ contributes $-i!/(s!t!)$ to γ, thus $C_I^J = -1$.
(5) If none of the above cases hold then γ is not present in $[B_i, B_j]$, thus $C_I^J = 0$.

In conclusion we have the following lemma.

Lemma 6.

$$\forall i, j \geqslant 0, \quad [B_i, B_j] = \sum_{r+s=i,\ t=j} \Gamma_{(r,s,t)} - \sum_{r=i,\ s+t=j} \Gamma_{(r,s,t)} + \sum_{r+s=j,\ t=i} \Gamma_{(r,s,t)} - \sum_{r=j,\ s+t=i} \Gamma_{(r,s,t)}. \tag{15}$$

To understand the algebraic reason for the cancellation better, define the following codifferential (dual to addition in some sense):

$$\delta(i, j) = \sum_{r+s=i,\ t=j} (r, s, t) - \sum_{r=i,\ s+t=j} (r, s, t).$$

Then the bracket in Lemma 6 is its symmetrization:

$$[B_i, B_j] = \left\langle \Gamma, \delta(i, j) + \delta(j, i)\right\rangle, \quad \Gamma(r, s, t) = W_{i,j}^{(r,s,t)} \Gamma_{(r,s,t)},$$

where Γ is the linear operator extending the function defined on the corresponding domain in the (r, s, t)-space. The $W(r, s, t) = 1$ are the "true coefficients" of the Lie bracket, without the grading sign built into $\circ$, which is independent of the particular case under consideration.

For a geometric viewpoint of the "integration domain," consider the 3-simplex $0 \leqslant r, s, t \leqslant n$, where $n = i + j$ is fixed. Then $\{(r, s, t) \mid r + s + t = i + j = n\}$ is the front face, $r + s = i, t = j$ defines a segment parallel to the rs-plane and $r = i$, $s + t = j$ defines a segment parallel to the st-plane, both contained in the front face and having $(i, 0, j)$ as common point.

When summing over (i, j), $i + j = n$, both segments swipe the front face

$$\{r + s = i, t = j, i + j = n\} = \{r + s + t = n\} = \{r = i, s + t = j, i + j = n\}.$$

Now, due to the opposite signs, there is an overall cancellation:

Lemma 7.

$$\sum_{i+j=n,\ i,j\geqslant 0} \delta(i, j) = 0.$$

As a corollary, (14) holds true, concluding the proof of the proposition. □

Note that the proof of the proposition does not depend on the values $W(r, s, t)$, but rather on a certain symmetry of the basis elements involved in the Lie bracket.

Definition 8. The *antipodal map* of the DGLA of graphs is [4]:

$$S(\Gamma) = (-1)^m \Gamma^t, \quad \Gamma \in \mathcal{G}_{n,m},$$

where Γ^t is the *transposed graph*, i.e. the graph obtained by reversing the order on the boundary points.

For example $S(b_1) = -b_1^t = b_1$, since they define the same orientation class.

Lemma 9. *The antipodal map is a pre-Lie morphism*:

$$S(\Gamma_1 \circ \Gamma_2) = S(\Gamma_1) \circ S(\Gamma_2),$$

and therefore an involution of the Lie algebra of graphs.

The role of the symmetrization of a star-product was already noted in [6] and [4].

Remark 10. If we define:

$$\delta(n) = \sum_{i+j=n,\ i,j\geqslant 0} (i, j)$$

then the previous lemma says that $\delta^2 = 0$, i.e. δ is indeed a codifferential.

Note also that δ is associated with the asymmetric operation:

$$\{\Gamma_1, \Gamma_2\} = \Gamma_1 \circ_1 \Gamma_2 - \Gamma_2 \circ_2 \Gamma_1, \quad \Gamma_i \in G_{\bullet,2}.$$

Its properties will be investigated elsewhere.

As a second example we will consider the case of linear Poisson structure.

3.3. Linear Poisson structures

Explicit star-products for linear Poisson structures (e.g. dual of a Lie algebra) were known to exist since [8–10].

In this case the graphs not in the kernel of the Kontsevich rule are products of tree-like graphs, since at most one arrow may land on internal vertex in order to have a non-zero contribution.

A candidate for an almost contraction is the "merger operation" (10).

Lemma 11. *σ is a homological differential,*

$$\sigma^2 = 0.$$

Proof. Indeed, if $j \geqslant i$ then $\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$ and the opposite sign of the two terms yields a pairwise cancellation as usual. □

Specializing to degrees two and three we obtain ($b_0 = (12)$):

$$\sigma_2(\Gamma) = -1/\left(2^{n-1} - 1\right)\Gamma/b_0, \quad \Gamma \in \mathcal{G}_{n,2},$$

$$\sigma_3(\theta) = -1/\left(2^{n-1} - 1\right)\left(\theta/(12) - \theta/(23)\right), \quad \theta \in \mathcal{G}_{n,3}.$$

At present the relation between the two differentials σ and ∂ (insertion and merger of boundary vertices), is not clear.

Some elementary facts are recorded next.

Lemma 12. *For any graph $\Gamma \in \mathcal{G}_{n,1}$ we have*

$$(\partial\Gamma)/b_0 = 2^{i-1}\Gamma,$$

where i is the number of edges landing on the unique boundary vertex of Γ.

For Bernoulli graphs b_n [4, p. 5], we have the following.

Lemma 13.

(i) $\partial\sigma_2(b_n^L) = 0$,
(ii) $\sigma_3\partial(b_n^L) = 2^{n-1}b_n^L - S_R(b_n^L)/b_0^R$ *where S_R (respectively S_L) splits in all non-trivial ways the arrows landing on L (R).*

Conjecture 14. *A canonical solution is defined inductively by* $Z_n = \sigma D_n$.

Although stated in the context of linear Poisson structures, we believe that the above conjecture holds in general, with the appropriate combinatorial coefficients for the merger operations σ_i, to be discussed elsewhere [11].

4. Conclusions

We showed that Maurer–Cartan equation can be solved provided that there is an almost contraction. This is reminiscent of the homotopy perturbation lemma with the infinitesimal cocycle as "initiator" [3,12]. As an application to star-products, the Moyal's formula was obtained in this way.

It is conjectured that the "merger operation," which is a homology differential, provides such an almost contraction at least in the case of linear Poisson structures, leading to a canonical star-product. Further investigations will be reported in a forthcoming article [11].

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