COUNTING THE GAUSSIAN PARTITIONS OF A FINITE VECTOR SPACE

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ABSTRACT. A subspace partition Π of a finite vector space $V = V(n, q)$ of dimension n over $\mathrm{GF}(q)$ is a collection of subspaces of V such that the union of the subspaces in Π is equal to V, and the intersection of any two subspaces is the zero vector. The multiset T_Π of dimensions of subspaces in Π is called the type of Π, or, a Gaussian partition of V. Previously, we showed that subspace partitions and their types are natural, combinatorial q-analogues of the set partitions of $\{1, \ldots, n\}$ and integer partitions of n respectively. In this paper, we connect all four types of partitions through the concept of "canonical" set, subspace, and Gaussian partitions, which are in one-to-one correspondence with the integer partitions of n. In particular, we use the most natural construction (due to Beutelspacher) of subspace partitions to derive a set G of Gaussian partitions of V starting from the canonical Gaussian partitions. We then show that the cardinality of G is a rational polynomial $R(q)$ in q, with $R(1) = \mathrm{p}(n)$, where p is the integer partition function.

1. INTRODUCTION

Let $V = V(n, q)$ be the n-dimensional vector space over $\mathrm{GF}(q)$, where n is a positive integer and q is a prime power. A *subspace partition*[1] of V is a collection Π of subspaces of V such that the union of the subspaces in Π is equal to V, and the intersection of any two subspaces is the zero vector (e.g., see the recent survey by Heden [14]). Subspace partitions are used to construct translation planes and nets [3, 4, 8], error-correcting codes [15, 20, 21, 22], orthogonal arrays [12], and designs [13, 25]). The origins of subspace partitions can be traced back to the general problem of partitioning a finite group into subgroups intersecting only at the identity element (e.g., see [19, 23] and the survey by Zappa [26]).

Let Π be a subspace partition of $V = V(n, q)$. Suppose that Π consists of x_i subspaces of dimension d_i for $1 \le i \le k$. The multiset $T_\Pi = d_1^{x_1} \ldots d_k^{x_k}$ of dimensions is then called a *partition type* of V. Clearly, not every multiset T that contains plausible dimensions is a partition type of V. However, if T *is* a partition type, then it must satisfy certain necessary conditions. One such condition, called the *packing condition*, is obtained by counting the nonzero vectors of V in two ways:

$$(1) \qquad \sum_{i=1}^{k} x_i(q^{d_i} - 1) = (q^n - 1).$$

[1] A *subspace partition* is also known as *vector space partition* in the literature.

A second necessary condition comes from dimension considerations. If U and W are sub-spaces of $V(n, q)$, then it is well known that the subspace spanned by $U \cup W$ has dimension $\dim(U) + \dim(W) - \dim(U \cap W)$. Therefore, if T is a partition type, then it must satisfy

$$(2) \qquad \begin{cases} 2d_i \leq n, & \text{if } x_i \geq 2, \\ d_i + d_j \leq n, & \text{if } i \neq j. \end{cases}$$

The necessary conditions (1) and (2) are not sufficient in general. For instance, $2^{10}1^1$ is not a partition type of $V(5, 2)$. There are several other nontrivial necessary conditions (e.g., see [16], [17], and [18]).

In our papers [1, 2], we studied the lattice of subspace partitions of $V = V(n, q)$ and the poset of partition types of V (which we called the *Gaussian partitions* of V). We proved several results, revealing these two objects as natural, combinatorial q-analogues of the set partitions of $\mathbf{n} = \{1, \ldots, n\}$ and the integer partitions of n respectively. In particular, we showed that the number of all subspace partitions of V is congruent to the number of set partitions of $\mathbf{n}$ modulo $q - 1$. In this paper, we connect all four types of partitions through the concept of "canonical" set, subspace, and Gaussian partitions, which are in one-to-one correspondence with the integer partitions of n. In particular, we use the most natural construction (due to Beutelspacher [6]) of subspace partitions to derive a set G of Gaussian partitions of V starting from the canonical Gaussian partitions. Henceforth, the phrases "subspace partition" and "Gaussian partition" will only refer to the subspace partitions of $V(n, q)$ and their types constructed in the manner we will describe, as opposed to the full collections of subspace partitions discussed in [1, 2]. Our main result is as follows.

Theorem 1. *Let q be a prime power and n be a positive integer. The number of Gaussian partitions of $V(n, q)$ is a rational polynomial $R(q)$ in q. Moreover, we have $R(1) = \mathrm{p}(n)$, where p is the integer partition function.*

2. Set, Integer, and Subspace Partitions

2.1. Set Partitions.

Definition 1 (Split of subset). *A split of a subset D of $\mathbf{n} = \{1, \ldots, n\}$ with $d = |D| \geq 2$ is an operation denoted by (a, b), where $a + b = d$ and $a \geq b \geq 1$, that results in partitioning D into two disjoint subsets A and B of cardinalities a and b respectively.*

The partition $\{A, B\}$ of D is not unique as defined. However, we can make it unique as follows:

Definition 2 (Ordering split of subset). *An ordering split of a subset D of $\mathbf{n}$ is a split (a, b) as in the above definition such that any element of A is strictly less than any element of B.*

Lemma 3. *Any set partition of $\mathbf{n}$ can be obtained by applying a sequence of splits to $\mathbf{n}$ (after the first split, we understand that each subsequent split is applied to a smaller subset generated previously). The empty sequence corresponds to the partition $\{\mathbf{n}\}$ with one part.*

Definition 4 (Canonical set partition). *A set partition of $\mathbf{n} = \{1, \ldots, n\}$ with k parts will be called canonical if its parts can be labeled $D_1, \ldots, D_k$, with cardinalities $d_1, \ldots, d_k$ respectively, in such a way that*

    (i) $d_1 \geq \cdots \geq d_k \geq 1$, *and*

(ii) *For all i with $1 \leq i \leq k$, the collection $D_1, \ldots, D_i$ is a set partition of $\mathbf{d} = \{1, \cdots, d\}$, where $d = d_1 + \cdots + d_i$.*

For the next two lemmas, let us adopt the notation of Definition 4.

Lemma 5. *The set of canonical set partitions of $\mathbf{n}$ is in one-to-one correspondence with the integer partitions of n, via*

$$\{D_1, \ldots, D_k\} \leftrightarrow d_1 \ldots d_k.$$

Lemma 6. *Any canonical set partition of $\mathbf{n}$ with parts described as in Definition 4 can be obtained by applying the sequence*

$$(d_1 + \cdots + d_{k-1}, d_k), (d_1 + \cdots + d_{k-2}, d_{k-1}), \ldots, (d_1, d_2)$$

of ordering splits to $\mathbf{n}$. By definition, the empty sequence corresponds to $\{\mathbf{n}\}$.

2.2. Subspace Partitions.

2.2.1. *Splits of subspaces.* The study of all possible subspace partitions and their types considered in [1, 2] is hampered by the fact that even in small dimensions, the maximal subspace partitions of $V(n, q)$ have not been enumerated for all q, and even their types remain a mystery. Examples are the number of 2-spreads of $V(4, q)$ and the types of the exceptional partitions of $V(6, q)$ that we mentioned in [1]. As a matter of fact, when we put aside the dozens of special cases of partition constructions of novel types (e.g., see [7, 17, 24] and the references therein), there have been only two basic existence theorems in the literature that are used consistently:

(A) If d divides n, then André [3] proved that $V(n, q)$ has a refinement of type $d^{\frac{q^n - 1}{q^d - 1}}$, which is better known as a *d-spread* of $V(n, q)$.
(B) If $1 \leq d < n/2$, then it was proved by Beutelspacher [6], and independently by Bu [7], that $V(n, q)$ has a refinement of type $(n - d)^1 d^{q^{n-d}}$.

The case $d = n/2$ is covered by (A). If d divides n but is not equal to $n/2$, then finitely many applications of move (B) will give us a spread as in (A). Thus, these two refinements can be combined into a single one:

(C) If $1 \leq d \leq n/2$, then $V(n, q)$ has a refinement of type $(n - d)^1 d^{q^{n-d}}$.

Definition 7 (Split of subspace)**.** *A split is a refinement of the form (C) on any one subspace in a subspace partition. We will let (a, b) (for $a \geq b \geq 1$) denote a subspace split that produces the refinement $(a + b)^1 \rightarrow a^1 b^{q^a}$ of the type $(a + b)^1$.*

Note that a split only shows the type of the move and not the subspace it is applied to. It is possible to obtain many different refined subspace partitions (of the same type) by applying a split to a specific subspace partition, just as in the case of set partitions.

2.2.2. *The Mechanism.* We will use the construction of Beutelspacher [6] (and Bu [7]) that yields the partitions in the statement (B) discussed earlier. This construction starts from a given direct sum decomposition $W \oplus U$ of $V(n, q)$ and a partition of U to give us a partition of $V(n, q)$ that includes U and W. Moreover, the new subspaces in the partition reproduce the dimension of U.

Theorem 2 (Beutelpacher [6]). *Let $V = V(n,q)$, U and W be subspaces of V such that $V = W \oplus U$, and $s = \dim(W) \geq \dim(U) = t$. Let $\{w_1, \ldots, w_s\}$ be a basis of W, and $\{u_1, \ldots, u_t\}$ be a basis of U. Moreover, we identify W with the field $GF(q^s)$. For every element $\gamma \in W \setminus \{0\}$, define a subspace U_γ of V by*

$$U_\gamma = \text{span}(\{u_1 + \gamma w_1, \ldots, u_t + \gamma w_t\}).$$

Then $\dim(U_\gamma) = t$, $U_\gamma \cap U_{\gamma'} = \{0\}$ for $\gamma \neq \gamma'$, and the collection

$$\{U, W\} \cup \{U_\gamma : \gamma \in W \setminus \{0\}\}$$

of subspaces forms a partition of V.

Theorem 2 can be used to accomplish refinements described in (C):

Corollary 8. *Choosing $\dim(W) = n - d$ and $\dim(U) = d$ (where $d \leq n/2$) in Theorem 2, we obtain a subspace partition of $V(n,q)$ of type $(n-d)^1 d^{q^{n-d}}$.*

We will informally designate the new subspaces U_γ created in the above corollary as "copies" of U.

2.2.3. *Canonical subspace partitions.* Let us fix a basis $S = \{e_1, \ldots, e_n\}$ of $V(n,q)$, and identify it with **n** via the subscripts.

Definition 9 (Ordering split of special subspace). *Let D be a nonempty subset of S. An ordering split of type (a,b) of $\langle D \rangle$, the subspace of $V(n,q)$ generated by D, is one that is obtained by applying the ordering split (a,b) to the set D to obtain a partition $\{A, B\}$ of D, then applying the construction in Corollary 8 to $\langle D \rangle = \langle A \rangle \oplus \langle B \rangle$, with $W = \langle A \rangle$ and $U = \langle B \rangle$.*

Definition 10 (Canonical Gaussian partition). *We call a subspace partition Π of $V(n,q)$ canonical if it can be obtained by applying a sequence of ordering splits to $V = \langle S \rangle$ that would have resulted in the corresponding canonical set partition of S.*

Proposition 11. *Let Π be a canonical subspace partition as described above, and let $D_1, \ldots, D_k$ be the corresponding canonical set partition of the basis S of $V(n,q)$. Then Π contains the subspaces $\langle D_1 \rangle, \ldots, \langle D_k \rangle$ of $V(n,q)$.*

2.2.4. *The construction of subspace partitions.* From this point on, we will only consider subspace partitions of $V(n,q)$ that are either canonical or are obtained from a canonical one by finitely many splits via the mechanism described in Corollary 8 and the three basic rules that we will outline below. This convention is akin to leaving out the exceptional groups in the classification of finite simple groups, whose existence and structures require the use of more customized techniques.

1. The unique ancestor rule: During the construction of a specific subspace partition Γ starting from a canonical partition Π as given in Definition 10 and Proposition 11, the subspaces $\langle D_i \rangle$ will be left intact, reflecting the corresponding partitioning of the basis S as a set. This way, we can trace every subspace partition back to a unique canonical partition.

2. The dimension rule: Let $f_1 \cdots f_s$ be the Gaussian partition describing the nonincreasing dimensions of the subspaces that exist at any stage of the construction, before we apply a split (a,b) to a subspace of dimension $f_i = a + b$. This rule dictates that the parts a and b of the split cannot be strictly smaller than the dimensions to the right of f_i, if any: we require that $f_i > a \geq b \geq f_{i+1}$. The *same* split (a,b) may be applied to several subspaces of

dimension f_i in a cluster. When applying any split to the last dimension f_s, the dimension rule is automatically satisfied.

3. The left-right rule: We will mark the subspace of dimension a that results from a split of type (a, b) as "left" and the q^a subspaces of dimension b that are produced in the same split as "right", since b will appear to the right of a in the new Gaussian partition. This rule dictates that we are allowed to split only *copies* of $\langle D_i \rangle$ (see Rule 1) and subsequently, only left subspaces. This rule will hold even in the case $a = b$; there is exactly one left subspace produced by a split, regardless of dimension.

The effects of this construction on the shape of the Gaussian partitions will be discussed further while enumerating these partitions.

3. Gaussian Partitions Extending Integer Partitions

3.1. **Gaussian Partitions.** The definition and notation of splits can be applied to the main object in this paper, namely, the Gaussian partitions associated with $V(n, q)$. We will denote a Gaussian partition that is the type of some subspace partition Γ by T_Γ. If c is a dimension appearing in T_Γ, then a split (a, b) with $a + b = c$ that does not violate the dimension rule will result in the insertion of the symbols $a^1 b^{q^a}$ between c and the adjacent dimension on the right, while reducing the exponent of c by 1. Similarly, j repeated applications of (a, b) will reduce the exponent of c by j and result in the insertion of $a^j b^{jq^a}$.

3.2. **Canonical Gaussian Partitions.**

Definition 12 (Canonical Gaussian partition)**.** *A Gaussian partition of $V(n, q)$ is called* canonical *if it is the type of a canonical subspace partition of $V(n, q)$.*

Proposition 13. *The canonical subspace partitions of $V(n, q)$, the canonical Gaussian partitions of $V(n, q)$, and the canonical set partitions of $\mathbf{n}$ are in one-to-one correspondence with the integer partitions of n.*

Example 14. *The integer partition $5^2 3^1 1^4$ of $n = 17$ is represented by the canonical Gaussian partition*
$$5^1 5^{q^5} 3^{q^{10}} 1^{q^{13}} 1^{q^{14}} 1^{q^{15}} 1^{q^{16}} = 5^{1+q^5} 3^{q^{10}} 1^{q^{13}+q^{14}+q^{15}+q^{16}}$$
of $V(17, q)$. Note that the exponent of 5^{q^5} tells us that the sum of the dimensions that come before (equivalently, the parts of the corresponding integer partition) is 5, the exponent of $3^{q^{10}}$ tells us that the previous sum is 10, and the exponent of $1^{q^{13}}$ tells us that the previous dimensions add up to 13, etc.

Let us state our observations about the shape of a canonical Gaussian partition formally.

Proposition 15. *The canonical Gaussian partition for $V(n, q)$ that corresponds to the integer partition $d_1 \cdots d_k$ of n, with $d_1 \geq \cdots \geq d_k$, is given by*
$$T = d_1^1 d_2^{q^{d_1}} d_3^{q^{d_1+d_2}} \cdots d_k^{q^{d_1+\cdots+d_{k-1}}}.$$
Conversely, a partition of type T, where $d_1 \geq \cdots \geq d_k$ and $d_1 + \cdots + d_k = n$, is canonical.

Addition Property 16. *For a Gaussian partition written as in Proposition 15, the exponent q^t of any dimension d_i reflects the sum $t = d_1 + \cdots + d_{i-1}$ of the parts of the corresponding integer partition that come before d_i (the empty sum is zero).*

Remark 17. *If we require the parts d_i in Proposition 15 to be distinct, then the canonical Gaussian partition corresponding to the integer partition $d_1^{n_1} \cdots d_k^{n_k}$ of n, with $d_1 > \cdots > d_k$, is given by*

$$
T = d_1^{1 + q^{d_1} + \cdots + q^{(n_1-1)d_1}} d_2^{q^{n_1 d_1} + q^{n_1 d_1 + d_2} + \cdots + q^{n_1 d_1 + (n_2-1)d_2}} \cdots
$$
$$
d_k^{q^{(n_1 d_1 + \cdots + n_{k-1} d_{k-1})} + \cdots + q^{(n_1 d_1 + \cdots + (n_k-1)d_k)}}.
$$

Moreover, the uniqueness of the exponents in T as a polynomial in q with integer coefficients 0 or 1 follows from the uniqueness of digits in the base-q representation of positive integers.

The two depictions of T in Proposition 15 and Remark 17 correspond to the left- and right-hand sides of the equation in Example 14 respectively.

3.3. **The structure of Gaussian partitions.** It is possible to describe the genesis of Gaussian partitions of $V = V(n, q)$ without any reference to the process outlined in Section 2.2.4. Consider a canonical subspace partition Π of V given as in Definition 10 and Proposition 11. Let T_Π be the corresponding Gaussian partition, written as in Proposition 15. The construction of a subspace partition Γ from Π results in the following properties for the Gaussian partition T_Γ starting from T_Π:

(1) Let d_i denote $\dim(\langle D_i \rangle)$ where D_i is a subset of the basis S as before. *Splits (a, b) may be applied to the dimension d_i in T_Π only if $i = k$ or $d_i > d_{i+1}$ (and $i \neq 1$).* Clearly, strictly smaller dimensions a and b cannot be placed in between the same two integers d_i and d_i in T_Π by the dimension rule, and splitting a unique maximal dimension is prohibited by the unique ancestor rule. Hence, *only the largest power $q^u \neq 1$ of q in the exponent of any one dimension d_i of T_Π may be dissolved* (when like bases are combined, as in Remark 17). In fact, *at most $q^u - 1$ splits may be applied to powers of d_i*, as $\langle D_i \rangle$ in Π must remain intact by the unique ancestor rule.

(2) *Only one kind of split (a, b) may be applied to a dimension d_i.* If a different one, say (r, s) with $a > r$ and $b < s$ is attempted before or after (a, b), then the resulting order of dimensions would be d_i, r, s, a, b, d_{i+1} or d_i, a, b, r, s, d_{i+1}, violating the dimension rule.

(3) Let $N = q^u - 1$. Then a split (a, b) applied to the dimension d_i in T_Π several times in a sequence would result in one of the N possible Gaussian partitions, with the powers $a^1, \ldots, a^N$ of the dimensions a of the left subspaces. We will call this linearly ordered collection of Gaussian partitions the **spine** corresponding to the split (a, b). At any power a^k, we are allowed to branch off into a split (r, s) applied to some copies of a (such that the dimension rule is not violated), where we may dissolve all k powers if we wish to, obtaining a **spinelet** of new possible partitions. By the same reasoning as above, *at most one kind of split of a is permitted at this stage*. The various spines and spinelets emanating from T_Π help us visualize the universe of possibilities for Gaussian partitions T_Γ constructed from T_Π.

(4) *We may apply the same split (a, b) to a Gaussian partition only back-to-back and may never revisit it while working on the same spine*, lest we violate the dimension rule.

(5) *None of the intermediate splittings between dimensions d_i and d_{i+1} results in another canonical Gaussian partition.* However, if we were to split the last dimension d_i (corresponding to the subspace $\langle D_i \rangle$) using an $(N+1)$st split (a, b), then we would arrive at another canonical Gaussian partition. This is stated as Proposition 19.

(6) *The construction process of any Gaussian partition is unique up to order.* That is, it can be traced back to a unique canonical Gaussian partition, and there is only one possible set of splits that results in that partition (splits starting out of different places in T_Π commute). This is stated as Proposition 20.

Example 18 (The Gaussian partitions of $V(6, q)$).

The eleven canonical Gaussian partitions of $V(6, q)$ are 6^1, $5^1 1^{q^5}$, $4^1 1^{q^4 + q^5}$, $3^1 1^{q^3 + q^4 + q^5}$, $2^1 1^{q^2 + q^3 + q^4 + q^5}$, $1^{1 + q + q^2 + q^3 + q^4 + q^5}$, $4^1 2^{q^4}$, 3^{1+q^3}, $3^1 2^{q^3} 1^{q^5}$, $2^{1 + q^2 + q^4}$, and $2^{1 + q^2} 1^{q^4 + q^5}$. The first five cannot be split further without running into another canonical partition, and the sixth one is already minimal. The non-canonical Gaussian partitions obtained from the remaining five are described in the following table:

Canonical	Splits	Non-canonicals	Range
$4^1 2^{q^4}$	$(1, 1)$	$4^1 2^{q^4 - i} 1^{i(q+1)}$	$1 \le i \le q^4 - 1$
3^{1+q^3}	$(2, 1)$	$3^{1 + q^3 - i} 2^i 1^{iq^2}$	$1 \le i \le q^3 - 1$
3^{1+q^3}	$(2, 1) \text{ and } (1, 1)$	$3^{1 + q^3 - i} 2^{i-j} 1^{iq^2 + j(1+q)}$	$1 \le i \le q^3 - 1; \ 1 \le j \le i$
$3^1 2^{q^3} 1^{q^5}$	$(1, 1)$	$3^1 2^{q^3 - i} 1^{i(q+1) + q^5}$	$1 \le i \le q^3 - 1$
$2^{1 + q^2 + q^4}$	$(1, 1)$	$2^{1 + q^2 + q^4 - i} 1^{i(q+1)}$	$1 \le i \le q^4 - 1$
$2^{1 + q^2} 1^{q^4 + q^5}$	$(1, 1)$	$2^{1 + q^2 - i} 1^{i(q+1) + q^4 + q^5}$	$1 \le i \le q^2 - 1$

Consider the number of non-canonical Gaussian partitions of $V(6, q)$: there are

$$s(q) = \sum_{i=1}^{q^3 - 1} \sum_{j=0}^{i} 1 = \sum_{i=1}^{q^3 - 1} (i + 1) = \frac{(q^3 - 1) q^3}{2} + (q^3 - 1)$$

of them that are obtained from 3^{1+q^3}, and the total is

$$(q^4 - 1) + s(q) + (q^3 - 1) + (q^4 - 1) + (q^2 - 1),$$

a rational polynomial in q with root $q = 1$.

Proposition 19. *Let*

$$\begin{aligned} T \ = \ & d_1^{1 + q^{d_1} + \cdots + q^{(n_1 - 1) d_1}} d_2^{q^{n_1 d_1} + q^{n_1 d_1 + d_2} + \cdots + q^{n_1 d_1 + (n_2 - 1) d_2}} \cdots \\ & d_k^{q^{(n_1 d_1 + \cdots + n_{k-1} d_{k-1})} + \cdots + q^{(n_1 d_1 + \cdots + (n_k - 1) d_k)}}, \end{aligned}$$

with dimensions $d_1 > \cdots > d_k$, be a canonical Gaussian partition. If (a, b) is an allowed split with $a + b = d_i$, then neither the spine obtained by applying (a, b) to T back-to-back $q^{n_1 d_1 + \cdots + (n_i - 1) d_i} - 1$ times, nor the set of Gaussian partitions obtained from the spine, contains a canonical partition. However, with the next application of the split (a, b) to the spine, we obtain another canonical Gaussian partition T'.

Proof. As long as the largest power of q in the exponent of d_i is partially decomposed, the Gaussian partition cannot be canonical, because the exponent of d_i does not correctly reflect Addition Property 16. However, once the highest power of q in the exponent of d_i is dissolved, we do get a canonical partition: this partition is

$$\begin{aligned} T' \ = \ & d_1^{1 + q^{d_1} + \cdots + q^{(n_1 - 1) d_1}} \cdots d_i^{q^{(n_1 d_1 + \cdots + n_{i-1} d_{i-1})} + \cdots + q^{(n_1 d_1 + \cdots + (n_i - 2) d_i)}} \\ & a^{q^{(n_1 d_1 + \cdots + (n_i - 1) d_i)}} b^{q^{(n_1 d_1 + \cdots + (n_i - 1) d_i + a)}} d_{i+1}^{q^{(n_1 d_1 + \cdots + n_i d_i)} + \cdots} \cdots \end{aligned}$$

The exponents of a and b, as well as the expression $q^{n_1 d_1}$ in the exponent of d_{i+1}, conform to the Addition Property; note that $a + b = d_i$, and it is possible to have equalities in $a \geq b \geq d_{i+1}$. $\square$

Proposition 20. *A Gaussian partition is uniquely defined by its canonical ancestor and the set of splits applied to it.*

Proof. This proof depends implicitly on the uniqueness of the base-q representation of positive integers. Every non-canonical Gaussian partition T' starts from *some* canonical Gaussian partition T by definition. It turns out that T can be uniquely reconstructed due to the structure of canonicals described in Proposition 15: let

$$T = d_1^1 d_2^{q^{d_1}} d_3^{q^{d_1+d_2}} \cdots d_k^{q^{d_1+\cdots+d_{k-1}}}.$$

The original dimensions $d_1 \geq \cdots \geq d_k$ of T are all present in T'. In fact, if $k \geq 2$, then d_1 and d_2 must be the leftmost two numbers in T', when dimensions are written in nonincreasing order without exponents, because of the unique ancestor rule (see Section 2.2.4). If the total exponent of d_2 is already q^{d_1}, then we factor out this power, and the next number to the right has to be $d_3 < d_2$. If the total exponent exceeds q^{d_1}, then we understand that $d_3 = d_2$, separate $d_2^{q^{d_1}}$, and check if the total exponent of d_3 is $q^{d_1+d_2}$, etc. As soon as we hit an exponent of some d_i that falls short of Addition Property 16, the first split (a, b) applied to d_i can be identified by the first integer $a < d_i$ to the right of the d_i's. If a^α is the collection of all a's in T', then we can locate $(d_i - a)^{\alpha q^a} = b^{\alpha q^a}$ somewhere to the right (and the next integer, if any, must be d_{i+1}). All subsequent splits, if any, can be put together in this fashion from inside out in nested intervals owing to the dimension rule (see Section 2.2.4). Then we start working on d_{i+1}, and so on, until all splits are repaired backwards and T is re-created. $\square$

3.4. **Some Preliminary Counting.** We will use the following lemma, which states Faulhaber's formula (e.g., see [9]) for the sums of consecutive powers.

Lemma 21 (Sums of Consecutive Powers [9]). *Let N and m be any nonnegative integers. Then the familiar sum*

$$\theta_m(N) = \sum_{k=1}^{N} k^m$$

of the first N consecutive m-th powers of k is a rational polynomial in N, with $\theta_m(0) = 0$ (when $N = 0$, the empty sum is equal to zero). An explicit formula for $\theta_m(N)$ can be given in terms of the Bernoulli numbers B_k:

$$(3) \qquad \theta_m(N) = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} (-1)^k B_k \, N^{m+1-k}.$$

Corollary 22. *Let N be a positive integer, and $S(x)$ be any rational polynomial. Then the sum*

$$U(N) = \sum_{k=1}^{N} S(k)$$

is a rational polynomial in N, with $U(0) = 0$.

Proof. Let $S(x) = a_0 + a_1 x + \cdots + a_t x^t$, with $a_i \in \mathbf{Q}$. Then, in the notation of Lemma 21, the sum

$$U(N) = a_0 \sum_{k=1}^{N} 1 + a_1 \sum_{k=1}^{N} k + \cdots + a_t \sum_{k=1}^{N} k^t = a_0 \theta_0(N) + a_1 \theta_1(N) + \cdots + a_t \theta_t(N)$$

is a $\mathbf{Q}$-linear combination of rational polynomials in N with $N = 0$ as a root. $\qquad \square$

Let N be an unspecified positive integer and k be a variable that may take on the integer values $1, 2, \ldots, N$. Recall that we denote the set $\{1, 2, \ldots, k\}$ by $\mathbf{k}$. We define a sequence of multisets $A_0(N)$, $A_1(N)$, $A_2(N)$, ... by the following recursive rule: we set $A_0(N) = \{N\}$, and replace each occurrence of an integer k in the set $A_i(N)$ by all elements of the set $\mathbf{k}$ in $A_{i+1}(N)$. Thus

$$A_{i+1}(N) = \biguplus_{k=1}^{N} A_i(k),$$

where $\uplus$ denotes the disjoint union, or sum, of multisets. The first few multisets in this sequence are

$$A_0(N) = \{N\}, \quad A_1(N) = \biguplus_{k=1}^{N} A_0(k) = \mathbf{N} = \{1, 2, \ldots, N\}, \text{ and}$$

$$A_2(N) = \biguplus_{k=1}^{N} A_1(k) = \mathbf{1} \uplus \mathbf{2} \uplus \cdots \uplus \mathbf{N} = \{1\} \uplus \{1, 2\} \uplus \cdots \{1, 2, \ldots, N\}.$$

Corollary 23. *Let $1_{A_i(N)}(k)$ denote the multiplicity of the integer k in the multiset $A_i(N)$. For $i \geq 0$, let*

$$S_i(N) = \sum_{k=1}^{N} k \cdot 1_{A_i(N)}(k),$$

the sum of all elements of the multiset $A_i(N)$ counted with multiplicities. Then
(1) For each $i \geq 0$, we have

$$S_{i+1}(N) = \sum_{k=1}^{N} S_i(k).$$

(2) For each $i \geq 0$, the expression $S_i(N)$ is a rational polynomial in N, with $S_i(0) = 0$.

Proof. Part (1) follows immediately from the definition of A_i as a disjoint union of multisets and the additive property of the multiplicity function. For part (2), we note that $S_0(N) = N = \theta_0(N)$, and

$$S_1(N) = \sum_{k=1}^{N} k = \sum_{k=1}^{N} S_0(k) = \theta_1(N).$$

By Corollary 22 and part (1), it is clear that each subsequent $S_i(N)$ is a polynomial with the desired properties. $\qquad \square$

3.5. **Adding up.**

Proposition 24. *Let T be a canonical Gaussian partition of $V(n,q)$ given by*

$$T = d_1^1 d_2^{q^{d_1}} d_3^{q^{d_1+d_2}} \cdots d_k^{q^{d_1+\cdots+d_{k-1}}},$$

where $d_1 \geq \cdots \geq d_k$ and $d_1 + \ldots + d_k = n$. Then the number of all Gaussian partitions obtained from T is a rational polynomial in q. Moreover, $q = 1$ is a root of this polynomial.

Remark 25. *The Gaussian partitions obtained from T according to the rules in Section 3.3 are necessarily non-canonical by Proposition 19.*

Proof. In the special cases of a canonical Gaussian partition T for which the only splits that adhere to the dimension rule create other canonicals, or where all $d_i = 1$, the polynomial in question is the zero polynomial. Henceforth, we assume that it is possible to obtain non-canonical Gaussian partitions from T. Since sequences of splits are applied to one dimension d_i at a time, it suffices to show that the number of Gaussian partitions obtained from one d_i is a polynomial $P_i(q)$ of the desired type. Again, we single out the cases where it is not possible to fit any splits to d_i, and declare that in these cases $P_i(q) = 1$. However, there will be at least one factor $P_i(q)$ that is a rational polynomial with 1 as a root by our assumption. The total number for T will be the product $P_T(q)$ of all $P_i(q)$.

Thus, assume that the Gaussian partitions obtained from T only contain changes to d_i, and that q^u is the largest power of q in the exponent of d_i. Let $N = q^u - 1$ and recall the notation of Corollary 23. Now, several different splits (a,b) may be allowed for d_i (i.e., we have $a + b = d_i$ and $d_i > a \geq b \geq d_{i+1}$), but any given Gaussian partition may contain only one of these, possibly applied several times (see Section 3.3). Repeated applications of a split (a,b) results in a total of $N = S_0(N)$ possible Gaussian partitions on a spine, where the set of exponents of a is $\{1, 2, \ldots, N\} = A_1(N)$. If there are $t_0 = t_0(i)$ possible splits (a,b) of d_i, then there must be $t_0 S_0(N)$ Gaussian partitions that have exactly one more kind of split than T has in their construction, because spines are disjoint by Proposition 20.

For any one of the first splits (a,b) of d_i, let (c,d) be one of the next generation of splits (that is, $c + d = a$, and $a > c \geq d \geq b$). Then each a^k on the spine will generate a new set of exponents for c on a spinelet, namely, $\mathbf{k} = \{1, 2, \ldots, k\}$. The multiset $A_2(N)$ will contain all exponents of c thus generated, and $S_1(N)$ new Gaussian partitions that contain only (a,b) and (c,d) in their construction sequence (starting from T) will be created. If there are $t_1 = t_1(i)$ possible sets of back-to-back splits (a,b) and (c,d) applied to d_i, then the number of Gaussian partitions that are obtained from T by splitting d_i with only two kinds of new splits is $t_1 S_1(N)$, as once again Proposition 20 tells us that there cannot be any repetitions of Gaussian partitions when different sequences of splits are employed. We continue in this manner as far as possible.

If $t_r = t_r(i)$ denotes the number of all distinct back-to-back split sequences of length $r+1$ applied to d_i, and if the maximum possible length of such sequences is $m = m(i)$, then the total number of Gaussian partitions that can be obtained from T by splitting only d_i is given by the rational polynomial

$$P_i(q) = t_0 S_0(N) + \cdots + t_{m-1} S_{m-1}(N),$$

which must have $q = 1$ as a root by Corollary 23. The product

$$P_T(q) = \prod_{i=1}^{k} P_i(q)$$

is hence the total number of Gaussian partitions obtained from T, a rational polynomial with $P_T(1) = 0$. Note that the numbers u, N, m, and $t_1, \ldots, t_{m-1}$ depend on i, but we are suppressing references to i for simplicity of notation. $\square$

As an immediate corollary of Proposition 24, we obtain the proof of our main result.

Proof of Theorem 1. The total number of canonical Gaussian partitions is $\mathrm{p}(n)$, and each canonical partition T produces $P_T(q)$ non-canonical partitions. Then the total number of Gaussian partitions of $V(n, q)$ is given by

$$R(q) = \mathrm{p}(n) + \sum_T P_T(q),$$

which is a rational polynomial in q with $R(1) = \mathrm{p}(n) + \sum_T 0 = \mathrm{p}(n)$. $\square$

4. Concluding Comments

We can relax the conditions that splits may only be applied to canonical subspace partitions, and have to preserve the subspaces that contain basis elements, as long as they do not violate the dimension rule and the left-right rule. Splits can certainly be applied in a haphazard fashion to $V(n, q)$ and produce a number of additional partitions. We call the larger set of partitions obtained by applying sequences of unrestricted splits to $V(n, q)$ the *extended subspace partitions*. Thus, the corresponding types are called the *extended Gaussian partitions* of $V(n, q)$.

The lattice of all subspace partitions of $V(n, q)$ regardless of construction, including those subspace partitions that are yet to be discovered, were surprisingly shown to have a count that is congruent to $\mathrm{p}(n)$ modulo $q - 1$ in our paper [2]. However, we do not know whether the number of extended Gaussian partitions for $V(n, q)$ is a rational polynomial in q that equals $\mathrm{p}(n)$ when we set $q = 1$.

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