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An extremal problem for set families generated with the union and symmetric difference operations

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Let $\mathcal{G}$ be a family of sets and let $\cup^n\mathcal{G}$ be the family of sets obtained by taking all unions of k sets of $\mathcal{G}$ with $1 \le k \le n$. We define the *half-life* of $\mathcal{G}$ with respect to the union operation, denoted by $h_\cup(\mathcal{G})$, to be the smallest integer n such that some $x \in \cup_{A\in\mathcal{G}} A$ appears in at least half of the sets in $\cup^n\mathcal{G}$. If no such n exists, then we define it as ∞. We also define the *half-life* of $\mathcal{G}$ with respect to the symmetric difference operation in a similar fashion and denote it by $h_\Delta(\mathcal{G})$. In this paper, we establish several bounds for $h_\cup(\mathcal{G})$ and $h_\Delta(\mathcal{G})$. As a byproduct, we confirm Fránkl's union-closed conjecture for some special cases.

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1. Introduction

The symmetric difference of two sets A and B is $A\Delta B = (A\backslash B) \cup (B\backslash A)$. Let $\mathcal{G}$ be a family of sets and for any positive integer $n \le |\mathcal{G}|$, define

$$(1) \qquad \cup^n\mathcal{G} = \{A_{i_1} \cup \cdots \cup A_{i_k} : \ 1 \le k \le n \text{ and } A_{i_j} \in \mathcal{G} \text{ for } 1 \le j \le k\}$$

and

$$(2) \qquad \Delta^n\mathcal{G} = \{A_{i_1}\Delta \cdots \Delta A_{i_k} : \ 1 \le k \le n \text{ and } A_{i_j} \in \mathcal{G} \text{ for } 1 \le j \le k\}.$$

Definition 1. A family of sets $\mathcal{F}$ is *union-closed* (or Δ-closed) if it is closed under union (or symmetric difference), i.e, for any $A, B \in \mathcal{F}$, we have $A\cup B \in \mathcal{F}$ (or $A\Delta B \in \mathcal{F}$).

Let $\mathcal{F}$ be a union-closed family of sets. We call $\mathcal{F}$ *non-trivial* if it contains a non-empty set. A *generating set* of $\mathcal{F}$ is a subfamily of sets $\mathcal{G} \subseteq \mathcal{F}$ such that

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$\mathcal{F}$ is obtained by taking all the possible unions of sets in $\mathcal{G}$. Equivalently, we have

$$\mathcal{F} = \cup^m \mathcal{G}, \text{ where } m = |\mathcal{G}|.$$

For any family of sets $\mathcal{F}$, the *ground set* of $\mathcal{F}$ is defined by $\mathcal{F}_{\text{gd}} = \cup_{A \in \mathcal{F}} A$. The union-closed conjecture, due to Fránkl [4], can be stated as follows.

Conjecture 2. *For any non-trivial union-closed family of sets $\mathcal{F}$, there exists an element $x \in \mathcal{F}_{\text{gd}}$ which appears in at least half of the sets of $\mathcal{F}$.*

This simply stated conjecture turned out to be quite difficult to solve. Partial results that support Conjecture 2 can be found in [1, 3–5, 7–12] and the references therein.

If one takes a Δ-closed family of sets, then it is easy to show that the conclusion of Conjecture 2 holds. More precisely, the following proposition holds.

Proposition 3. *Let $\mathcal{F}$ be any non-trivial Δ-closed family of sets with ground set $\mathcal{F}_{\text{gd}}$. Then, any $x \in \mathcal{F}_{\text{gd}}$ appears in at least half of the sets in $\mathcal{F}$.*

Proof. Let $\mathcal{F} = \{A_1, \ldots, A_l\}$ be a non-trivial Δ-closed family of sets on the ground set $\mathcal{F}_g$. For any $x \in \mathcal{F}_g$, there exists at least one A_i containing x. Suppose that the number of sets in $\mathcal{F}$ containing x is less than $\frac{l}{2}$. Without loss of generality, assume that $x \in A_l$ and each $A_1, \ldots, A_{\lceil \frac{l+1}{2} \rceil}$ does not contain x. Then the sets $A_l \Delta A_i$, where $1 \leq i \leq \lceil \frac{l+1}{2} \rceil$, are pairwise distinct and each of them contains x. Since $\mathcal{F}$ is union-closed, all these sets are in $\mathcal{F}$, and this contradicts the assumption that the number of sets in $\mathcal{F}$ containing x is less than $\frac{l}{2}$. ☐

Let $\mathcal{G}$ be a non-empty family of $l \geq 2$ sets and let n be a nonnegative integer. We define the *half-life* of $\mathcal{G}$ with respect to $\cup$ to be the smallest integer n such that some $x \in \mathcal{G}_{\text{gd}}$ appears in at least half of the sets in $\cup^n \mathcal{G}$. If no such n exists, then we define it as ∞. We use $h_\cup(\mathcal{G})$ to denote the half-life of $\mathcal{G}$ with respect to $\cup$. The half-life of $\mathcal{G}$ with respect to Δ is defined similarly and denoted by $h_\Delta(\mathcal{G})$. The following problem will help us to understand the union-closed conjecture.

Problem 4. *Determine $h_\cup(\mathcal{G})$ and $h_\Delta(\mathcal{G})$.*

This rest of the paper is organized as follows. We give some bounds for $h_\Delta(\mathcal{G})$ and $h_\cup(\mathcal{G})$ in Sections 2 and 3 respectively. In Section 4, we confirm the union-closed conjecture for some special cases.

2. Some bounds on $h_\Delta(\mathcal{G})$

Let $\mathcal{G}$ be a family with $l \geq 2$ sets and $x \in \mathcal{G}_{\text{gd}}$. For any positive integer n, let $\Delta^n(\mathcal{G}, x)$ be the family of sets in $\Delta^n\mathcal{G}$ containing x. Let $\Delta^n(\mathcal{G}, \bar{x})$ be the family of sets in $\Delta^n\mathcal{G}$ not containing x. We define the *half-life* of $x \in \mathcal{G}_{\text{gd}}$ with respect to Δ, denoted by $h_\Delta(\mathcal{G}, x)$, to be the minimum between ∞ and the smallest integer n such that

$$\frac{|\Delta^n(\mathcal{G}, x)|}{|\Delta^n(\mathcal{G})|} \geq \frac{1}{2}.$$

Thus,

$$h_\Delta(\mathcal{G}) = \min_{x \in \mathcal{G}_{\text{gd}}} h_\Delta(\mathcal{G}, x).$$

If $l = 2$, then $h_\Delta(\mathcal{G}, x) = 1$ for any $x \in \mathcal{G}_{\text{gd}}$. So we assume $l \geq 3$.

Proposition 5. *For any family $\mathcal{G}$ with $l \geq 3$ sets and for any $x \in \mathcal{G}_{\text{gd}}$, $h_\Delta(\mathcal{G}, x) \leq |\mathcal{G}|$ holds.*

Proof. Since $\Delta^l\mathcal{G}$ is Δ-closed for $l = |\mathcal{G}|$, Proposition 3 implies that

$$h_\Delta(\mathcal{G}, x) \leq |\mathcal{G}|. \qquad \square$$

In order to state our main result in this section, we need the following definition.

Definition 6. A family of non-empty sets $\mathcal{S} = \{A_1, \ldots, A_l\}$ is called *linearly independent* if for any integer j with $1 \leq j \leq l$ and for all indices $i_1, \ldots, i_s \in \{1, \ldots, l\} \setminus \{j\}$, we have $A_j \neq A_{i_1}\Delta \cdots \Delta A_{i_s}$.

Lemma 7 will be used in Theorems 8 and 9.

Lemma 7. *Let $\mathcal{S} = \{A_1, \ldots, A_l\}$ be a linearly independent family of sets. Let $1 \leq i_1 < \cdots < i_s \leq l$ and $1 \leq j_1 < \cdots < j_t \leq l$. Then, $A_{i_1}\Delta \cdots \Delta A_{i_s} = A_{j_1}\Delta \cdots \Delta A_{j_t}$ if and only if $\{i_1, \ldots, i_s\} = \{j_1, \ldots, j_t\}$.*

Proof. Let $I = \{i_1, \ldots, i_s\}$ and $J = \{j_1, \ldots, j_t\}$. Suppose that $I \neq J$ and

$$A_{i_1}\Delta \cdots \Delta A_{i_s} = A_{j_1}\Delta \cdots \Delta A_{j_t}. \tag{3}$$

We may assume that $I \cap J = \emptyset$. Indeed, for each $i \in I \cap J$ we can operate by $A_i\Delta$ on both sides of (3) and consider resulting sets $I' = I \setminus I \cap J$ and

$J' = J \setminus I \cap J$. With the assumption $I \cap J = \emptyset$, we can now operate by $A_{i_1}\Delta \cdots \Delta A_{i_{s-1}}\Delta$ on both sides of (3) to obtain

$$A_{i_s} = A_{i_1}\Delta \cdots \Delta A_{i_{s-1}}\Delta A_{j_1}\Delta \cdots \Delta A_{j_t}.$$

This contradicts the linear independence of $\mathcal{S}$. □

Theorem 8. *Let $\mathcal{S} = \{A_1, \ldots, A_l\}$ be a linearly independent family sets with $l \geq 3$. For $x \in \cup_{i=1}^{l} A_i$, let q_x be the number of sets in $\mathcal{S}$ containing x.*

(a) If q_x is even, then $h_\Delta(\mathcal{S}, x) \leq \lceil \frac{l-1}{2} \rceil$.

(b) If q_x is odd and $h_\Delta(\mathcal{S}, x) \neq l$, then $h_\Delta(\mathcal{S}, x) \leq \lfloor \frac{l-1}{2} \rfloor$.

Proof. Since $\Delta^l \mathcal{S}$ is Δ-closed, Proposition 3 yields

$$|\Delta^l(\mathcal{S}, x)| \geq |\Delta^l(\mathcal{S}, \bar{x})|. \tag{4}$$

We simply write q_x as q throughout the proof. Without loss of generality, we assume that $x \in A_i$ for $1 \leq i \leq q$ and $x \notin A_i$ for $i > q$. Note that a set in $\Delta^n(\mathcal{S}, x)$ must be of the form $A_{i_1}\Delta \cdots \Delta A_{i_s}$, where there are an odd number of indices $i_j \in \{i_1, \ldots, i_s\}$ such that $i_j \leq q$. Similarly, a set in $\Delta^n(\mathcal{S}, \bar{x})$ must be $\emptyset$ or of the form $A_{i_1}\Delta \cdots \Delta A_{i_s}$, where there are an even number of indices $i_j \in \{i_1, \ldots, i_s\}$ such that $i_j \leq q$.

(a) q is even. For any positive integer $n \leq l-2$, define a function f on $\Delta^n(\mathcal{S}, x)$ by

$$f(A_{i_1}\Delta \cdots \Delta A_{i_s}) = A_{j_1}\Delta \cdots \Delta A_{j_{l-s}}$$

where $\{j_1, \ldots, j_{l-s}\} = \{1, \ldots, l\} \setminus \{i_1, \ldots, i_s\}$. By Lemma 7, f is a one-to-one function. Since q is even, f is an onto function from $\Delta^n(\mathcal{S}, x)$ to $\Delta^l(\mathcal{S}, x) - \Delta^{l-n-1}(\mathcal{S}, x)$, which is the set

$$\{A_{i_1}\Delta \cdots \Delta A_{i_t} : x \in A_{i_1}\Delta \cdots \Delta A_{i_t},\ i_1 < \cdots < i_t,\ l-n \leq t \leq l\}.$$

Therefore,

$$|\Delta^n(\mathcal{S}, x)| = |\Delta^l(\mathcal{S}, x)| - |\Delta^{l-n-1}(\mathcal{S}, x)|. \tag{5}$$

Similarly, define a function g on $\Delta^n(\mathcal{S}, \bar{x})$ by

$$g(A_{i_1}\Delta \cdots \Delta A_{i_s}) = A_{j_1}\Delta \cdots \Delta A_{j_{l-s}}$$

where $\{j_1, \ldots, j_{l-s}\} = \{1, \ldots, l\} \setminus \{i_1, \ldots, i_s\}$, and

$$g(\emptyset) = A_1\Delta \cdots \Delta A_l \text{ if } n \geq 2.$$

Then g is a one-to-one and onto function from $\Delta^n(\mathcal{S},\bar{x})$ to $\Delta^l(\mathcal{S},\bar{x}) - \Delta^{l-n-1}(\mathcal{S},\bar{x})$. Therefore,

$$|\Delta^n(\mathcal{S},\bar{x})| = |\Delta^l(\mathcal{S},\bar{x})| - |\Delta^{l-n-1}(\mathcal{S},\bar{x})|. \tag{6}$$

By (4), (5) and (6),

$$|\Delta^n(\mathcal{S},x)| + |\Delta^{l-n-1}(\mathcal{S},x)| \geq |\Delta^n(\mathcal{S},\bar{x})| + |\Delta^{l-n-1}(\mathcal{S},\bar{x})|.$$

Thus, $|\Delta^n(\mathcal{S},x)| \geq |\Delta^n(\mathcal{S},\bar{x})|$ or $|\Delta^{l-n-1}(\mathcal{S},x)| \geq |\Delta^{l-n-1}(\mathcal{S},\bar{x}|$. If we take $n = \lfloor \frac{l-1}{2} \rfloor$, then we get

$$h_\Delta(\mathcal{S},x) \leq \left\lfloor \frac{l-1}{2} \right\rfloor \text{ or } h_\Delta(\mathcal{S},x) \leq \left\lceil \frac{l-1}{2} \right\rceil.$$

(b) q is odd. For any positive integer $n \leq l-1$, define a function f on $\Delta^n(\mathcal{S},x)$ by

$$f(A_{i_1}\Delta\cdots\Delta A_{i_s}) = A_{j_1}\Delta\cdots\Delta A_{j_{l-s}}$$

where $\{j_1,\ldots,j_{l-s}\} = \{1,\ldots,l\} \setminus \{i_1,\ldots,i_s\}$.

By Lemma 7, f is a one-to-one function. Since q is odd, f is an onto function from $\Delta^n(\mathcal{S},x)$ to $\Delta^l(\mathcal{S},\bar{x}) - \Delta^{l-n-1}(\mathcal{S},\bar{x})$, where $\Delta^0(\mathcal{S},\bar{x})$ is the family containing only the empty set $\emptyset$. Therefore,

$$|\Delta^n(\mathcal{S},x)| = |\Delta^l(\mathcal{S},\bar{x})| - |\Delta^{l-n-1}(\mathcal{S},\bar{x})|. \tag{7}$$

Similarly, define a function g on $\Delta^n(\mathcal{S},\bar{x})$ by

$$g(A_{i_1}\Delta\cdots\Delta A_{i_s}) = A_{j_1}\Delta\cdots\Delta A_{j_{l-s}}$$

where $\{j_1,\ldots,j_{l-s}\} = \{1,\ldots,l\} \setminus \{i_1,\ldots,i_s\}$, and

$$g(\emptyset) = A_1\Delta\cdots\Delta A_l \text{ if } n \geq 2.$$

Then g is a one-to-one and onto function from $\Delta^n(\mathcal{S},\bar{x})$ to $\Delta^l(\mathcal{S},x) - \Delta^{l-n-1}(\mathcal{S},x)$, where $\Delta^0(\mathcal{S},x)$ is the empty family. Therefore,

$$|\Delta^n(\mathcal{S},\bar{x})| = |\Delta^l(\mathcal{S},x)| - |\Delta^{l-n-1}(\mathcal{S},x)|. \tag{8}$$

By (4), (7) and (8),

$$|\Delta^n(\mathcal{S},x)| - |\Delta^n(\mathcal{S},\bar{x})| \leq |\Delta^{l-n-1}(\mathcal{S},x)| - |\Delta^{l-n-1}(\mathcal{S},\bar{x})|.$$

Swapping n with $l-n-1$ yields

$$|\Delta^n(\mathcal{S},x)| - |\Delta^n(\mathcal{S},\bar{x})| \geq |\Delta^{l-n-1}(\mathcal{S},x)| - |\Delta^{l-n-1}(\mathcal{S},\bar{x})|.$$

Thus,

$$|\Delta^n(\mathcal{S},x)| - |\Delta^n(\mathcal{S},\bar{x})| = |\Delta^{l-n-1}(\mathcal{S},x)| - |\Delta^{l-n-1}(\mathcal{S},\bar{x})|.$$

So if $h_\Delta(\mathcal{S},x) \neq l$, then there exists at least one $n \leq l-1$ such that

$$h_\Delta(\mathcal{S},x) \leq \min\{n, l-n-1\} \leq \left\lfloor \frac{l-1}{2} \right\rfloor. \qquad \square$$

Theorem 9. *Let $\mathcal{G} = \{A_1, \ldots, A_l\}$ be a linearly independent family of $l \geq 3$ sets. Let q_x be the number of sets in $\mathcal{G}$ containing x.*

(a) $h_\Delta(\mathcal{G},x) = 2$ if and only if $\frac{(l+1)-\sqrt{l-1}}{2} \leq q_x < \frac{l}{2}$.
(b) If $q_x = 1$, then $h_\Delta(\mathcal{G},x) = l$.
(c) If $q_x = 2$ and $l \geq 5$, then $h_\Delta(\mathcal{G},x) = \lceil \frac{l-1}{2} \rceil$.
(d) If $q_x = 3$ and $l \geq 7$, then $h_\Delta(\mathcal{G},x) = \lceil \frac{l-1-\sqrt{l-1}}{2} \rceil$.

Proof. We simply write q_x as q throughout the proof. Without loss of generality, we assume that $x \in A_i$ for $1 \leq i \leq q$ and $x \notin A_i$ for $i > q$.

Note that a set in $\Delta^n(\mathcal{G},x)$ must be of the form $A_{i_1}\Delta\cdots\Delta A_{i_s}$, where there are an odd number of indices $i_j \in \{i_1,\ldots,i_s\}$ such that $i_j \leq q$. Similarly, a set in $\Delta^n(\mathcal{G},\bar{x})$ must be $\emptyset$ or of the form $A_{i_1}\Delta\cdots\Delta A_{i_s}$, where there are an even number of indices $i_j \in \{i_1,\ldots,i_s\}$ such that $i_j \leq q$. Also recall that, by Lemma 7, all sets $A_{i_1}\Delta\cdots\Delta A_{i_s}$ with $i_1 < \cdots < i_s$ and $1 \leq s \leq l$ are pairwise distinct. So for $n \geq 2$, we obtain

$$(9) \qquad |\Delta^n(\mathcal{G},x)| = \sum_{k=0}^{\min\{\lfloor \frac{q-1}{2} \rfloor, \lfloor \frac{n-1}{2} \rfloor\}} \binom{q}{2k+1} \sum_{j=0}^{\min\{l-q, n-(2k+1)\}} \binom{l-q}{j},$$

and

$$(10) \qquad |\Delta^n(\mathcal{G},\bar{x})| = \sum_{k=0}^{\min\{\lfloor \frac{q}{2} \rfloor, \lfloor \frac{n}{2} \rfloor\}} \binom{q}{2k} \sum_{j=0}^{\min\{l-q, n-2k\}} \binom{l-q}{j}.$$

To show that $|\Delta^n(\mathcal{G},x)| \geq \frac{1}{2}|\Delta^n\mathcal{G}|$, it is equivalent to show that $|\Delta^n(\mathcal{G},x)| - |\Delta^n(\mathcal{G},\bar{x})| \geq 0$.

(a) Now assume $h_\Delta(\mathcal{G},x)=2$. Since

$$|\Delta^2\mathcal{G}| = 1+\binom{l}{1}+\binom{l}{2} \text{ and } |\Delta^2(\mathcal{G},x)| = q+q(l-q),$$

then $|\Delta^2(\mathcal{G},x)| \ge \frac{1}{2}|\Delta^2\mathcal{G}|$ if and only if $q+q(l-q) \ge \frac{1}{2}(1+l+\binom{l}{2})$. Solving the previous inequality for q yields $\frac{(l+1)-\sqrt{l-1}}{2} \le q \le \frac{(l+1)+\sqrt{l-1}}{2}$. Since $h_\Delta(\mathcal{G},x)=1$ when $q \ge \frac{l}{2}$, we conclude that $\frac{(l+1)-\sqrt{l-1}}{2} \le q < \frac{l}{2}$.

(b) If $q=1$, then for $2 \le n \le l-1$,

$$\begin{aligned}
|\Delta^n(\mathcal{G},x)| - |\Delta^n(\mathcal{G},\bar{x})| &= \sum_{j=0}^{n-1}\binom{l-1}{j} - \sum_{j=0}^{n}\binom{l-1}{j}\\
&= -\binom{l-1}{n}\\
&< 0.
\end{aligned}$$

By Proposition 5, $h_\Delta(\mathcal{G},x)=l$.

(c) If $q=2$ and $l \ge 5$, then for $2 \le n \le l-2$,

$$\begin{aligned}
|\Delta^n(\mathcal{G},x)| - |\Delta^n(\mathcal{G},\bar{x})| &= 2\sum_{j=0}^{n-1}\binom{l-2}{j} - \left(\sum_{j=0}^{n}\binom{l-2}{j} + \sum_{j=0}^{n-2}\binom{l-2}{j}\right)\\
&= \binom{l-2}{n-1} - \binom{l-2}{n}\\
&\ge 0,
\end{aligned}$$

where the last inequality holds if and only if $n \ge \lceil\frac{l-1}{2}\rceil$. So $h_\Delta(\mathcal{G},x)=\lceil\frac{l-1}{2}\rceil$.

(d) If $q=3$ and $l \ge 7$, then for $2 \le n \le l-3$,

$$\begin{aligned}
|\Delta^n(\mathcal{G},x)| - |\Delta^n(\mathcal{G},\bar{x})| &= 3\sum_{j=0}^{n-1}\binom{l-3}{j} + \sum_{j=0}^{n-3}\binom{l-3}{j}\\
&\quad - \left(\sum_{j=0}^{n}\binom{l-3}{j} + 3\sum_{j=0}^{n-2}\binom{l-3}{j}\right)\\
&= 2\binom{l-3}{n-1} - \binom{l-3}{n} - \binom{l-3}{n-2}\\
&\ge 0,
\end{aligned}$$

where the last inequality holds if and only if $\lceil \frac{l-1-\sqrt{l-1}}{2} \rceil \leq n \leq \lceil \frac{l-1+\sqrt{l-1}}{2} \rceil$. So $h_\Delta(\mathcal{G}, x) = \lceil \frac{l-1-\sqrt{l-1}}{2} \rceil$. □

Remark 10. It follows from the proof of Theorem 9 part (d) that $\frac{|\Delta^n(\mathcal{G},x)|}{|\Delta^n \mathcal{G}|}$ is not necessarily monotone with respect to n.

Theorem 11. *Let $\mathcal{G} = \{A_1, \ldots, A_m\}$ be a family of sets and let $l \geq 3$ be the maximum size of a linearly independent subset $\mathcal{S}$ of $\mathcal{G}$. For $x \in \cup_{i=1}^{l} A_i$, let q_x be the number of sets in $\mathcal{G}$ containing x.*

(a) If q_x is even, then $h_\Delta(\mathcal{G}, x) \leq l-1$.

(b) If q_x is odd and $\mathcal{S} \subset \mathcal{G}$, then $h_\Delta(\mathcal{G}, x) \leq l-1$.

Proof. Let $\mathcal{S} = \{A_1, \ldots, A_l\}$ be a maximum linearly independent subset of $\mathcal{G}$. Then for every non-empty set $A_j \in \mathcal{G}$, there exists $1 \leq i_1 < \cdots < i_s \leq l$ such that $A_j = A_{i_1} \Delta \cdots \Delta A_{i_s}$. Since

$$\Delta^l \mathcal{G} = \Delta^l \mathcal{S} = \{\emptyset\} \cup \{A_{i_1} \Delta \cdots \Delta A_{i_s} : 1 \leq i_1 < \cdots < i_s \leq l\},$$

both $\Delta^l \mathcal{G}$ and $\Delta^l \mathcal{S}$ are Δ-closed. Furthermore, $\Delta^{l-1}\mathcal{S} = \Delta^l \mathcal{S} \setminus \{A_1 \Delta A_3 \cdots \Delta A_l\}$ and $\Delta^{l-1}\mathcal{G}$ is either $\Delta^{l-1}\mathcal{S}$ or $\Delta^l \mathcal{S}$.

(a) Assume that q_x is even. If $\Delta^{l-1}\mathcal{G} = \Delta^l \mathcal{S}$, then $\Delta^{l-1}\mathcal{G}$ is Δ-closed, and it follows from Proposition 3 that $h_\Delta(\mathcal{G}, x) \leq l-1$. So we may assume that $\Delta^{l-1}\mathcal{G} = \Delta^{l-1}\mathcal{S}$.

Our goal now is to show that $|\Delta^{l-1}(\mathcal{G}, x)| \geq |\Delta^{l-1}\mathcal{G}|/2$. Since $\Delta^{l-1}(\mathcal{S}, x) = \Delta^{l-1}(\mathcal{G}, x)$, it suffices to show that

$$|\Delta^{l-1}(\mathcal{S}, x)| \geq |\Delta^{l-1}\mathcal{S}|/2.$$

By Proposition 3, $|\Delta^l(\mathcal{S}, x)| \geq |\Delta^l \mathcal{S}|/2$. Note that $|\Delta^l \mathcal{S}| - |\Delta^{l-1}\mathcal{S}| = 1$ since the only element in $\Delta^l \mathcal{S} \setminus \Delta^{l-1}\mathcal{S}$ is $A_1 \Delta \cdots \Delta A_l$.

Since q_x is even, then $A_1 \Delta \cdots \Delta A_l$ does not contain x, and consequently $|\Delta^{l-1}(\mathcal{S}, x)| = |\Delta^l(\mathcal{S}, x)|$. Hence, $|\Delta^{l-1}(\mathcal{S}, x)| \geq |\Delta^{l-1}\mathcal{S}|/2$.

(b) Assume that q_x is odd and $\mathcal{S} \subseteq \mathcal{G}$. We claim that $\Delta^{l-1}\mathcal{G} = \Delta^l \mathcal{S}$. Note that $\Delta^{l-1}\mathcal{S} \subseteq \Delta^{l-1}\mathcal{G} \subseteq \Delta^l \mathcal{S}$ and $\Delta^l \mathcal{S} \setminus \Delta^{l-1}\mathcal{S} = \{A_1 \Delta \cdots \Delta A_l\}$. Since $\mathcal{S} \subset \mathcal{G}$, then there exists $A_i \in \mathcal{G} \setminus \mathcal{S}$ such that A_i is the symmetric difference of at least two sets in $\mathcal{S}$; therefore, $A_1 \Delta \cdots \Delta A_l \in \Delta^{l-1}\mathcal{G}$ and consequently $\Delta^{l-1}\mathcal{G} = \Delta^l \mathcal{S}$. Then again it follows from Proposition 3 that $h_\Delta(\mathcal{G}, x) \leq l-1$ since $\Delta^{l-1}\mathcal{G}$ is Δ-closed. □

3. Some bounds on $h_\cup(\mathcal{G})$

Let $\mathcal{G}$ be a family with $l \geq 2$ sets and $x \in \mathcal{G}_{\rm gd}$. For any positive integer n, let $\cup^n(\mathcal{G},x)$ be the family of sets in $\cup^n\mathcal{G}$ containing x. Let $\cup^n(\mathcal{G},\bar{x})$ be the family of sets in $\cup^n\mathcal{G}$ not containing x. Recall that the *half-life* of $x \in \mathcal{G}_{\rm gd}$ with respect to $\cup$, denoted by $h_\cup(\mathcal{G},x)$, is the minimum between ∞ and the smallest integer n such that

$$\frac{|\cup^n(\mathcal{G},x)|}{|\cup^n(\mathcal{G})|} \geq \frac{1}{2}.$$

Thus,

$$h_\cup(\mathcal{G}) = \min_{x\in\mathcal{G}_{\rm gd}} h_\cup(\mathcal{G},x).$$

If the union-closed conjecture (Conjecture 2) is true, then $h_\cup(\mathcal{G}) \leq |\mathcal{G}|$. We know from Remark 10 that $|\Delta^n(\mathcal{G},x)|/|\Delta^n\mathcal{G}|$ is not necessarily monotone with respect to n. It is interesting to investigate whether or not $|\cup^n(\mathcal{G},x)|/|\cup^n\mathcal{G}|$ is monotone with respect to n. If $|\cup^n(\mathcal{G},x)|/|\cup^n\mathcal{G}|$ were monotone, then union-closed conjecture would hold if and only if $h_\cup(\mathcal{G}) \leq |\mathcal{G}|$.

We now provide two results about $h_\cup(\mathcal{G},x)$ when its value is significantly less than $|\mathcal{G}|$.

Proposition 12. *Let $\mathcal{G} = \cup_{i=1}^{l}\mathcal{G}_i$ be a family of sets, where $\mathcal{G}_i$ is a union closed family of sets and let x be an element in the ground set of $\mathcal{G}$. Assume that $|\mathcal{G}_i| = g > 1$ for $1 \leq i \leq l$ and $A_{i_1}\cup\cdots\cup A_{i_s} \neq A_{j_1}\cup\cdots\cup A_{j_k}$ for $\{i_1,\ldots,i_s\} \neq \{j_1,\ldots,j_k\}$, where $A_{i_p} \in \mathcal{G}_{i_p}$ for $1 \leq p \leq l$. Finally, assume that there is an i_0, with $1 \leq i_0 \leq l$, such that x belongs to all sets in $\mathcal{G}_{i_0}$ and x does not belong to any other set. Then $h_\cup(\mathcal{G},x) \leq l-1$ whenever $l \geq 1+\ln 2/\ln(1+1/g)$. Moreover, the union-closed conjecture is true for $\cup^l\mathcal{G}$, and thus $h_\cup(\mathcal{G},x) \leq l = |\mathcal{G}|/g$ holds in general.*

Proof. Since all $A_{i_1}\cup\cdots\cup A_{i_s}$ are pairwise distinct where $i_1 < \cdots < i_s$ and each A_{i_j} is from $\mathcal{G}_j$, $1 \leq j \leq s$ and $1 \leq s \leq l$, then

$$|\cup^n\mathcal{G}| = \sum_{j=1}^{n} g^j\binom{l}{j}.$$

Since a set in $\cup^n\mathcal{G}$ containing x must be of the form

$$A_{i_0} \text{ or } A_{i_0}\cup A_{i_1}\cup\cdots\cup A_{i_s},$$

then

$$|\cup^n(\mathcal{G},x)| = \sum_{j=0}^{n-1} g\cdot g^j\binom{l-1}{j}.$$

So,

$$\begin{aligned}
2|\cup^n(\mathcal{G},x)| &= g+\sum_{j=1}^{n-1}\left(g^j\binom{l-1}{j-1}+g^j\binom{l-1}{j}\right)\\
&\quad+\sum_{j=1}^{n-1}\left(g^{j+1}-g^j\right)\binom{l-1}{j}+g^{n-1+1}\binom{l-1}{n-1}\\
&\geq g+\sum_{j=1}^{n-1}g^j\binom{l}{j}+\sum_{j=1}^{n-1}g^j\binom{l-1}{j}+g^{n-1+1}\binom{l-1}{n-1}\\
&=|\cup^n\mathcal{G}|+g+\sum_{j=1}^{n-1}g^j\binom{l-1}{j}+g^{n-1+1}\binom{l-1}{n-1}-g^n\binom{l}{n},
\end{aligned}$$

here we used the assumption $g>1$.

Case 1. If $n=l-1$, then

$$\begin{aligned}
2|\cup^{l-1}(\mathcal{G},x)| &\geq |\cup^{l-1}\mathcal{G}|+g+\sum_{j=1}^{l-2}g^j\binom{l-1}{j}+g^{l-1}(l-1)-g^{l-1}l\\
&\geq |\cup^{l-1}\mathcal{G}|+\sum_{j=0}^{l-1}g^j\binom{l-1}{j}-g^{l-1}+g^{l-1}(l-1)-g^{l-1}l\\
&=|\cup^{l-1}\mathcal{G}|+(1+g)^{l-1}-g^{l-1}+g^{l-1}(l-1)-g^{l-1}l\\
&=|\cup^{l-1}\mathcal{G}|+(1+g)^{l-1}-2g^{l-1}.
\end{aligned}$$

Therefore, $2|\cup^{l-1}(\mathcal{G},x)|\geq|\cup^{l-1}\mathcal{G}|$ when $l\geq 1+\ln 2/\ln(1+1/g)$.

Case 2. If $n=l$, then

$$2|\cup^{l}(\mathcal{G},x)|\geq|\cup^{l}\mathcal{G}|+g+g^l\binom{l-1}{l-1}-g^l\binom{l}{l}>|\cup^l\mathcal{G}|.$$

Therefore, we have shown that $h_\cup(\mathcal{G})\leq l-1$ when $l>1+\ln 2/\ln(1+1/g)$. Moreover, the union-closed conjecture is true for $\cup^l\mathcal{G}$ and thus, $h_\cup(\mathcal{G})\leq l$ in general. □

Remark 13. If $g = 1$ in Proposition 12, then $\cup^i\mathcal{G} = \Delta^i\mathcal{G}$ for any i, and the case is covered in Theorem 9 (b).

Proposition 14. *Let $\mathcal{G} = \cup_{i=1}^l \mathcal{G}_i$ be a family of sets where each $\mathcal{G}_i$ is union closed and $A \cap B = \emptyset$ for $A \in \mathcal{G}_i$ and $B \in \mathcal{G}_j$ $(i \neq j)$. Assume there is an i_0 and an $x \in \mathcal{G}_{i_0}$ such that x is in at least $(|\mathcal{G}_{i_0}| + 1)/2$ of the sets of $\mathcal{G}_{i_0}$. Then the union-closed conjecture is true for $\cup^l\mathcal{G}$ and thus, $h_\cup(\mathcal{G}, x) \leq l$.*

Proof. Let $|\mathcal{G}_i| = k_i$. Without loss of generality, let x be in at least $(k_1+1)/2$ sets of $\mathcal{G}_1$. We will show that $|\cup^l(\mathcal{G}, x)| \geq \frac{|\cup^l\mathcal{G}|}{2}$.

Form a complete l partite graph H whose parts are the $\mathcal{G}_i$ families $1 \leq i \leq l$. Then $|\cup^l\mathcal{G}|$ is the total number of cliques in H and $|\cup^l(\mathcal{G}, x)|$ is the total number of cliques containing a vertex $A \in \cup_{i=1}^l \mathcal{G}_i$ such that $x \in A$. So

$$\begin{aligned}|\cup^l\mathcal{G}| = {} & \sum_{i=1}^l k_i + \sum_{1\leq i_1<i_2\leq l} k_{i_1}k_{i_2} + \cdots \\ & + \sum_{1\leq i_1<\cdots<i_{l-1}\leq l} k_{i_1}k_{i_2}\cdots k_{i_{l-1}} + k_1k_2\cdots k_l,\end{aligned}$$

and

$$\begin{aligned}|\cup^l(\mathcal{G}, x)| = {} & \frac{1+k_1}{2} + \frac{1+k_1}{2}\sum_{i=2}^l k_i + \frac{1+k_1}{2}\sum_{2\leq i_1<i_2\leq l} k_{i_1}k_{i_2} + \cdots \\ & + \frac{1+k_1}{2}\sum_{2\leq i_1<\cdots<i_{l-2}\leq l} k_{i_1}k_{i_2}\cdots k_{i_{l-2}} + \frac{1+k_1}{2}k_2\cdots k_l \\ = {} & \frac{1}{2} + \frac{1}{2}\Big(k_1 + \sum_{i=2}^l k_i\Big) + \frac{1}{2}\Big(k_1\sum_{i=2}^l k_i + \sum_{2\leq i_1<i_2\leq l} k_{i_1}k_{i_2}\Big) + \cdots \\ & + \frac{1}{2}\Big(k_1\sum_{2\leq i_1<\cdots<i_{l-2}\leq l} k_{i_1}k_{i_2}\cdots k_{i_{l-2}} + k_2\cdots k_{l-1}\Big) + \frac{k_1}{2}k_2\cdots k_{l-1}.\end{aligned} \tag{11}$$

From (11) and the expression of $|\cup^l\mathcal{G}|$ above, we obtain

$$\begin{aligned}|\cup^l(\mathcal{G}, x)| = {} & \frac{1}{2} + \frac{1}{2}\sum_{i=1}^l k_i + \frac{1}{2}\sum_{1\leq i_1<i_2\leq l} k_{i_1}k_{i_2} + \cdots \\ & + \frac{1}{2}\sum_{1\leq i_1<\cdots<i_{l-1}\leq l} k_{i_1}k_{i_2}\cdots k_{i_{l-1}} + \frac{1}{2}k_1k_2\cdots k_l \\ > {} & \frac{|\cup^l\mathcal{G}|}{2}.\end{aligned}$$

□

4. Some special cases of union-closed conjecture

Let $\mathcal{G} = \{A_1, \ldots, A_l\}$ be a family of l sets. We follow the same notations as in the previous section. Note that $\cup^l \mathcal{G}$ is the union-closed family of sets generated by the sets in $\mathcal{G}$.

Definition 15. We say that $\{\{A_{i_1}, \ldots, A_{i_s}\}, \{A_{j_1}, \ldots, A_{j_t}\}\}$ is an overcount in $\cup^n \mathcal{G}$ (resp. $\cup^n(\mathcal{G}, x)$) if the following conditions hold

(i) $A_{i_1} \cup \cdots \cup A_{i_s}$ and $A_{j_1} \cup \cdots \cup A_{j_t}$ are in $\cup^n \mathcal{G}$ (resp. $\cup^n(\mathcal{G}, x)$),
(ii) $\{A_{i_1}, \ldots, A_{i_s}\} \neq \{A_{j_1}, \ldots, A_{j_t}\}$,
(iii) $A_{i_1} \cup \cdots \cup A_{i_s} = A_{j_1} \cup \cdots \cup A_{j_t}$.

We define an auxiliary graph $H^n = (V, E^n)$ (resp. $H^n_x = (V, E^n_x)$) corresponding to the overcounts in $\cup^n \mathcal{G}$ (resp. $\cup^n(\mathcal{G}, x)$) as follows. Let

$$V = \{\{A_{i_1}, \ldots, A_{i_s}\} : \ 1 \leq i_1 < \cdots < i_s \leq n \leq l\}$$

and join $L \in V$ and $R \in V$ by an edge in H^n (resp. H^n_x) if $\{L, R\}$ is an overcount in $\cup^n \mathcal{G}$ (resp. $\cup^n(\mathcal{G}, x)$).

Definition 16. A set $\mathcal{O}$ of overcounts in $\cup^n \mathcal{G}$ (or $\cup^n(\mathcal{G}, x)$) is independent if the corresponding edges in graph H^n (resp. H^n_x) do not induce a cycle.

Lemma 17. *Let $\mathcal{G} = \{A_1, \ldots, A_l\}$ be a family of l sets. Let $x \in \mathcal{G}_{\mathrm{gd}}$ and c_x be the maximum number of independent overcounts in $\cup^n(\mathcal{G}, x)$. Let c be the maximum number of independent overcounts in $\cup^n \mathcal{G}$. Suppose that x is in q sets $A_i \in \mathcal{G}$. Then, $2|\cup^n (\mathcal{G}, x)| - |\cup^n \mathcal{G}| \geq 0$ if and only if*

$$c - 2c_x \geq 2\sum_{i=1}^{n} \binom{l-q}{i} - \sum_{i=1}^{n} \binom{l}{i}.$$

Proof. Without loss of generality, we assume that $x \in A_i$ for $1 \leq i \leq q$ and $x \notin A_i$ for $i > q$. Let us estimate $|\cup^n \mathcal{G}|$ and $|\cup^n (\mathcal{G}, x)|$.

Note that every set in $\cup^n \mathcal{G}$ is of the form $A_{i_1} \cup \cdots \cup A_{i_s}$, where $1 \leq s \leq n$. If $A_{i_1} \cup \cdots \cup A_{i_s}$ are pairwise distinct, then $|\cup^n \mathcal{G}| = \sum_{i=1}^{n} \binom{l}{i}$. In general,

$$|\cup^n \mathcal{G}| = \sum_{i=1}^{n} \binom{l}{i} - c. \tag{12}$$

Similarly, every set in $\cup^n(\mathcal{G}, x)$ is of the form $A_{i_1} \cup \cdots \cup A_{i_s}$, where $1 \leq s \leq n$ and $1 \leq i_1 \leq q$. If all these $A_{i_1} \cup \cdots \cup A_{i_s}$ are distinct, then $|\cup^n (\mathcal{G}, x)| =$

$\sum_{i=1}^{n}\binom{l}{i} - \sum_{i=1}^{n}\binom{l-q}{i}$. In general

$$|\cup^n (\mathcal{G}, x)| = \sum_{i=1}^{n}\binom{l}{i} - \sum_{i=1}^{n}\binom{l-q}{i} - c_x. \tag{13}$$

By (12) and (13), we see that $2|\cup^n (\mathcal{G}, x)| - |\cup^n \mathcal{G}| \geq 0$ is equivalent to $c - 2c_x \geq 2\sum_{i=1}^{n}\binom{l-q}{i} - \sum_{i=1}^{n}\binom{l}{i}$. □

By taking $l = n$ in the above lemma, we obtain the following results confirming some special cases of the union-closed conjecture.

Theorem 18. *Let $\mathcal{G} = \{A_1, \ldots, A_l\}$ be a family of l sets. Let $x \in \cup_{i=1}^{l} A_i$ and c_x be the maximum number of independent overcounts in $\cup^l(\mathcal{G}, x)$. Let c be the maximum number of independent overcounts in $\cup^l\mathcal{G}$ and $\gamma = c - c_x$. Suppose that x is in q sets $A_i \in \mathcal{G}$.*

(1) The union-closed conjecture is true for $\cup^l\mathcal{G}$ if and only if there exists an $x \in \cup_{i=1}^{l} A_i$ such that $c \geq 2c_x - 2^l + 2^{l-q+1} - 1$.
(2) In particular, the union-closed conjecture is true for $\cup^l\mathcal{G}$ if one of the following conditions holds:
  - *(2.a) $c_x \leq 2^l - 2^{l-q+1} + 1$.*
  - *(2.b) $\gamma \geq 2^{l-q} - 1 - q$.*
  - *(2.c) $2^{l-q} \leq q + 1$.*
  - *(2.d) $|\cup^l \mathcal{G}| \geq 2^{l-q+1} - 2$.*

Proof. Taking $n = l$ in (12) and (13), we have

$$|\cup^l \mathcal{G}| = 2^l - 1 - c \text{ and } |\cup^l (\mathcal{G}, x)| = 2^l - 2^{l-q} - c_x. \tag{14}$$

(1) By (14), we see that $2|\cup^l (\mathcal{G}, x)| - |\cup^l \mathcal{G}| \geq 0$ is equivalent to $c \geq 2c_x - 2^l + 2^{l-q+1} - 1$.

(2.a) Note that $c \geq c_x$. If $c_x \leq 2^l - 2^{l-q+1} + 1$, then (14) yield

$$2|\cup^l (\mathcal{G}, x)| - |\cup^l \mathcal{G}| \geq 2^l - 2^{l-q+1} + 1 - c_x \geq 0.$$

(2.b) Note that $|\cup^l(\mathcal{G}, x)| \geq q$, so $c_x \leq 2^l - 2^{l-q} - q$. If $\gamma = c - c_x \geq 2^{l-q} - 1 - q$, then (14) yield

$$2|\cup^l (\mathcal{G}, x)| - |\cup^l \mathcal{G}| \geq 2^l - 2^{l-q+1} + 1 - c_x + \gamma \geq 0,$$

because $c_x \leq 2^l - 2^{l-q} - q$ and $\gamma \geq 2^{l-q} - 1 - q$.

(2.c) Since $c_x \leq 2^l - 2^{l-q} - q$ and $2^{l-q} \leq q+1$, then equations (14) yield

$$2|\cup^l(\mathcal{G},x)| - |\cup^l\mathcal{G}| \geq 2^l - 2^{l-q+1} + 1 - c_x \geq 0.$$

(2.d) If $|\cup^l\mathcal{G}| \geq 2^{l-q+1} - 2$, then equation (14) yields $c \leq 2^l - 2^{l-q+1} + 1$. Since $c_x \leq c$, then $c_x \leq 2^l - 2^{l-q+1} + 1$ and (2.d) follows from (2.a). □

The next corollary follows directly from Theorem 18 (2.d) and the fact that the union-closed conjecture holds for the families with a generating family of pairwise disjoint sets.

Corollary 19. *The union-closed conjecture holds for a union-closed family $\mathcal{F}$ of sets if $\mathcal{F}$ has a generating family of sets $\mathcal{G}$ with $|\mathcal{G}| \leq \log_2(|\mathcal{F}|+2)+1$.*

Given a family of sets $\mathcal{G}$ with l sets, we say that $\cup^l\mathcal{G}$ satisfies the *averaged Fránkl's property* if

$$\sum_{x\in\mathcal{G}_{gd}} (2|\cup^l(\mathcal{G},x)| - |\cup^l\mathcal{G}|) \geq 0.$$

Satisfying the averaged Fránkl's property clearly implies satisfying the union-closed conjecture. As observed in [2], there are many families $\mathcal{G}$ with l sets such that $\cup^l\mathcal{G}$ satisfying the union-closed conjecture, but the averaged Fránkl's property fails.

For any family of sets $\mathcal{G} = \{A_1, \ldots, A_l\}$, recall that the ground set of $\mathcal{G}$ is $\mathcal{G}_{\text{gd}} = \cup_{i=1}^l A_i$. For any $x \in \mathcal{G}_{\text{gd}}$, we let

$$q_x(\mathcal{G}) = \{A :\ x \in A \in \mathcal{G}\} \text{ and } q_{\min}(\mathcal{G}) = \min_{x\in\mathcal{G}_{\text{gd}}} q_x(\mathcal{G}).$$

Furthermore, we sometimes write q_x (resp. $q_{\min}$) instead of $q_x(\mathcal{G})$ (resp. $q_{\min}(\mathcal{G})$) if the family $\mathcal{G}$ is clear from the context.

Let $\mathcal{O}$ be a maximum independent set of overcounts in $\cup^l\mathcal{G}$. Then for any overcount $W = \{L, R\} \in \mathcal{O}$, we let $S_W = \bigcup_{A\in L} A = \bigcup_{A\in R} A$. Then S_W is a union of some sets $A_i \in \mathcal{G}$. Define the average size of a set S_W over all $W \in \mathcal{O}$ by

$$\overline{s}(\mathcal{G}) = \frac{1}{|\mathcal{O}|}\sum_{W\in\mathcal{O}} |S_W|. \tag{15}$$

Let c_x be the maximum number of independent overcounts in $\cup^l(\mathcal{G},x)$ and define the average value of c_x over all $x \in \mathcal{G}_{\text{gd}}$ by

$$\overline{c}(\mathcal{G}) = \frac{1}{|\mathcal{G}_{\text{gd}}|}\sum_{x\in\mathcal{G}_{\text{gd}}} c_x. \tag{16}$$

Theorem 20. *Let $\mathcal{G}$ be a family of l sets with $g = |\mathcal{G}_{\mathrm{gd}}|$. Let $\overline{s} = \overline{s}(\mathcal{G})$ and $\overline{c} = \overline{c}(\mathcal{G})$ be as defined in (15) and (16) respectively. Then $g/\overline{s} \geq 1$ always holds.*
(i) The averaged Fránkl's property is true for $\cup^l\mathcal{G}$ if and only if

$$(2 - g/\overline{s})\overline{c} \leq 1 + 2^l - (2^l/g) \sum_{x \in \mathcal{G}\mathrm{gd}} \frac{1}{2^{q_x - 1}}.$$

In particular, the averaged Fránkl's property is true for $\cup^l\mathcal{G}$ if $g/\overline{s} \geq 2$.
(ii) The union-closed conjecture is true for $\cup^l\mathcal{G}$ if $1 \leq (g/\overline{s}) < 2$, and there exists $x \in \mathcal{G}_{\mathrm{gd}}$ satisfying

$$c_x \leq \min\left\{\frac{2^l - 2^{l-q_x+1} + 1}{2 - g/\overline{s}}, \overline{c}\right\}.$$

In particular, the union-closed conjecture is true for $\cup^l\mathcal{G}$ if $\overline{c} \leq \frac{2^l - 2^{l-q_x+1}+1}{2-g/\overline{s}}$.
(iii) The union-closed conjecture is true for $\cup^l\mathcal{G}$ for any positive number $\epsilon < 1$ with

$$1 + \epsilon \leq g/\overline{s} < 2 \text{ and } q_{\min} \geq 1 - \log_2(\epsilon).$$

Moreover, by combining (i) and (iii), it follows that the union-closed conjecture is true for $\cup^l\mathcal{G}$ whenever $g/\overline{s} > 1$ and $q_{\min} \geq 1 - \log_2(g/\overline{s} - 1)$.

Proof. Let $\mathcal{O}$ be a maximum independent set of overcounts in $\cup^l\mathcal{G}$. For any $x \in \mathcal{G}_{\mathrm{gd}}$, let $\mathcal{O}_x \subseteq \mathcal{O}$ denote the (possibly empty) set of all those overcounts $W = \{L, R\} \in \mathcal{O}$ for which $x \in S_W = \bigcup_{A \in L} A = \bigcup_{A \in R} A$. We count in two ways the number of pairs (x, W) such that $x \in \mathcal{G}_{\mathrm{gd}}$ and $W \in \mathcal{O}_x$. Then we have

$$\sum_{W \in \mathcal{O}} |S_W| = \sum_{x \in \mathcal{G}_{\mathrm{gd}}} |\mathcal{O}_x|. \tag{17}$$

Let $c = |\mathcal{O}|$, and let C_x be a maximum independent set of overcounts in $\cup^l(\mathcal{G}, x)$ with $c_x = |C_x|$. If $|C_x| > |\mathcal{O}_x|$ then $\mathcal{O}' = (\mathcal{O} \setminus \mathcal{O}_x) \cup C_x$ is also an independent set of overcounts in $\cup^l\mathcal{G}$ with $|\mathcal{O}'| > |\mathcal{O}|$, which contradicts $\mathcal{O}$ being of maximum size. So we may assume that $c_x = |C_x| = |\mathcal{O}_x|$ for any $x \in \mathcal{G}_{\mathrm{gd}}$. Then (17) yields

$$\sum_{W \in \mathcal{O}} |S_W| = \sum_{x \in \mathcal{G}_{\mathrm{gd}}} c_x. \tag{18}$$

Now, it follows from (18) and the definitions of $\overline{s}$ and $\overline{c}$ (see (15) and (16)) that

$$c = (g/\overline{s}) \cdot \overline{c}. \tag{19}$$

In general $g \geq \overline{s}$ since $S_W \subseteq \mathcal{G}_{\text{gd}}$ for all $W \in \mathcal{O}$.

(i) Applying (14), we have $\sum_{x\in\mathcal{G}_{gd}}(2|\cup^l(\mathcal{G},x)| - |\cup^l \mathcal{G}|) \geq 0$ if and only if

$$\sum_{x\in\mathcal{G}_{gd}}(2|\cup^l(\mathcal{G},x)| - |\cup^l \mathcal{G}|) = \sum_{x\in\mathcal{G}_{gd}}(2(2^l - 2^{l-q_x} - c_x) - (2^l - 1 - c)) \geq 0$$

if and only if

$$c \geq \frac{2\sum_{x\in\mathcal{G}_{\text{gd}}} c_x}{g} + \frac{2^l}{g}\sum_{x\in\mathcal{G}_{\text{gd}}}\frac{1}{2^{q_x-1}} - 2^l - 1,$$

if and only if

$$c \geq 2\overline{c} + \frac{2^l}{g}\sum_{x\in\mathcal{G}_{\text{gd}}}\frac{1}{2^{q_x-1}} - 2^l - 1.$$

By (19), the above inequality holds if and only if

$$(2 - g/\overline{s})\overline{c} \leq 1 + 2^l - \frac{2^l}{g}\sum_{x\in\mathcal{G}\text{gd}}\frac{1}{2^{q_x-1}}.$$

If there exists $x \in \mathcal{G}_{\text{gd}}$ satisfying

$$c_x \leq \min\left\{\frac{2^l - 2^{l-q_x+1} + 1}{2 - g/\overline{s}}, \overline{c}\right\},$$

then by (19), we obtain

$$c = (g/\overline{s}) \cdot \overline{c} \geq (g/\overline{s})c_x \geq 2c_x - 2^l + 2^{l-q_x+1} - 1,$$

because $c_x \leq (2^l - 2^{l-q_x+1} + 1)/(2 - g/\overline{s})$ holds by hypothesis. Now (ii) follows from Theorem 18 (2.b).

To prove (iii), first note $2 - g/\overline{s} \leq 1 - \epsilon$ since (by hypothesis) $g/\overline{s} \geq 1 + \epsilon$. Consequently, the sufficient condition in (ii), namely $\overline{c} \leq (2^l - 2^{l-q_x+1} + 1)/(2 - g/\overline{s})$, holds if

$$\overline{c} \leq (1-\epsilon)^{-1} \cdot (2^l - 2^{l-q_x+1} + 1). \tag{20}$$

Now (20) holds because

$$\bar{c} \leq 2^l \leq (1-\epsilon)^{-1} \cdot (2^l - 2^{l-q_x+1} + 1),$$

where the last inequality holds since $\epsilon < 1$ and

$$q_x \geq q_{\min} \geq 1 - \log_2(\epsilon) \Rightarrow 2^l \leq (1-\epsilon)^{-1} \cdot (2^l - 2^{l-q_x+1} + 1).$$

The proof of part (iii) is now complete.

The last statement of the theorem is a straightforward combination of (i) and (iii). □

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