

LEVEL MATRICES

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ABSTRACT. Let $n > 1$ and $k > 0$ be fixed integers. A matrix is said to be *level* if all its column sums are equal. A level matrix with m rows is called *reducible* if we can delete j rows, $0 < j < m$, so that the remaining matrix is level. We ask if there is a minimum integer $\ell = \ell(n, k)$ such that for all $m > \ell$, any $m \times n$ level matrix with entries in $\{0, \dots, k\}$ is reducible. It is known that $\ell(2, k) = 2k - 1$. In this paper, we establish the existence of $\ell(n, k)$ for $n \geq 3$ by giving upper and lower bounds for it. We then apply this result to bound the number of certain types of vector space multipartitions.

1. INTRODUCTION

Let $n > 1$ and $k > 0$ be integers. We define a k -*matrix* to be a matrix whose entries are in $\{0, 1, 2, \dots, k\}$. A matrix is said to be *level* if all its column sums are equal. A level matrix with m rows is called *reducible* if we can delete j rows, $0 < j < m$, so that the remaining matrix is level; otherwise it is *irreducible*. Note that if M is an irreducible matrix, then any matrix obtained from it by a permutation of rows or columns is also irreducible.

For $k = 1$ and any integer $n > 1$, the $n \times n$ identity matrix is irreducible. If $n > 4$, then we can construct an irreducible 1-matrix with n columns and $m > n$ distinct rows. Moreover, for any integers $k > 1$ and $n > 1$, we can construct an irreducible k -matrix with n columns and $m > n$ distinct rows.

In general, we do not require that irreducible k -matrices have distinct rows. We are interested in the following question.

Question. *Given integers $n > 1$ and $k > 0$, is there a minimum integer $\ell = \ell(n, k)$ such that for all $m > \ell$, any $m \times n$ level k -matrix is reducible? If $\ell(m, k)$ exists, then what can we say about its value?*

The exact value of $\ell(2, k)$ (see Theorem 1) follows from earlier work by Lambert [14]. Perhaps due to a wide range of notation and terminology in related areas, Lambert's result has been (independently) rediscovered by Diaconis et al. [6], and Sahs et al. [19]. In addition, M. Henk and R. Weismantel [13] gave improvements of Lambert's result.

Theorem 1 (Lambert [14]). *If $k > 1$, then $\ell(2, k) = 2k - 1$. Moreover, there are (up to row/column permutations) only two irreducible k -matrices with $2k - 1$ rows and 2 columns.*

However, to the best of our knowledge, the exact value of $\ell(n, k)$ is unknown for $n \geq 3$. In this paper, we prove the following theorem.

Theorem 2. *Let $n \geq 3$ and $k > 0$ be integers, and let $\epsilon > 0$ be any real number. There exist infinitely infinitely many values of n for which $\ell(n, 1) > e^{(1-\epsilon)\sqrt{n \ln n}}$. On the other hand,*

$$\ell(n, k) \leq \begin{cases} (2k)^3 & \text{if } n = 3, \\ k^{n-1} 2^{-n} (n+1)^{(n+1)/2} ((k+1)^n - k^n + 1) & \text{if } n > 3. \end{cases}$$

Let $M_{m,n}(\mathbb{Z})$ be the set of all $m \times n$ matrices with entries in \mathbb{Z} . In what follows, vectors are assumed to be column vectors (unless otherwise specified), and \mathbb{Z}^n denotes the set of all (column) vectors with n entries from \mathbb{Z} . Let $A \in M_{m,n}(\mathbb{Z})$ be a k -matrix and let $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{Z}^n$. We also let A_i denote the i th row of A .

For any $\vec{x} \in \mathbb{Z}^m$, we say \vec{x} is a *leveler* of A if \vec{x} has nonnegative entries and there exists a nonnegative $\alpha \in \mathbb{Z}$ such that $\vec{x}^T A = \alpha \mathbf{1}^T$, or equivalently, $A^T \vec{x} = \alpha \mathbf{1}$. In particular, given a leveler $\vec{x} = (x_1, x_2, \dots, x_m)^T \in \mathbb{Z}^m$ of A , we can form an $(x_1 + x_2 + \dots + x_m) \times n$ level k -matrix by taking x_1 copies of the first row of A , x_2 rows of the second row of A , etc. For the purposes of level k -matrices, this process will define this matrix up to a permutation of rows. In this way, each leveler represents a class of level k -matrices.

One way to classify level k -matrices then is to classify the levelers of the k -matrices A . To assist us in this analysis, we use the base field \mathbb{Q} and the following notation. For any $\vec{x} \in \mathbb{Q}^m$, we write $\vec{x} \geq \vec{0}$ if $x_i \geq 0$ for all $1 \leq i \leq m$. If $\vec{x}, \vec{y} \in \mathbb{Q}^m$, we write $\vec{x} \geq \vec{y}$ if $\vec{x} - \vec{y} \geq \vec{0}$, and write $\vec{x} > \vec{y}$ if $\vec{x} \geq \vec{y}$ and $\vec{x} \neq \vec{y}$. Finally, let $\vec{x} \in \mathbb{Z}^m$ be a leveler for A . We say \vec{x} is an *irreducible* leveler if, for any leveler $\vec{y} \in \mathbb{Z}^m$, we have $\vec{x} \geq \vec{y} \Rightarrow \vec{y} = \vec{x}$ or $\vec{y} = \vec{0}$. Note that \vec{x} is an irreducible leveler of A if and only if the corresponding matrix formed from A is an irreducible k -matrix.

Assume that the rows of $A \in M_{m,n}(\mathbb{Z})$ are distinct and $m \geq n$. Define

$$(1) \quad \mathcal{F}(A) = \{\vec{x} \in \mathbb{Q}^m \mid A^T \vec{x} = \mathbf{1} \text{ and } \vec{x} > \vec{0}\}.$$

Note that $\mathcal{F}(A)$ is a convex polytope in \mathbb{Q}^m since it is the intersection of the linear space $\{\vec{x} \in \mathbb{Q}^m \mid A^T \vec{x} = \mathbf{1}\}$ with the half-spaces $\mathcal{H}_i = \{\vec{x} \in \mathbb{Q}^m \mid x_i \geq 0\}$ for $1 \leq i \leq m$.

We say that $\vec{x} \in \mathcal{F}(A)$ is a *basic feasible solution* (BFS) if there exists a set of n indices $I = \{i_1, \dots, i_n\} \subseteq \{1, 2, \dots, m\}$ such that:

- (a) $x_i = 0$ for each $i \notin I$.
- (b) If C is the matrix with rows A_i for $i \in I$, then C is invertible. Thus, if $\vec{y} = C^{-1} \mathbf{1}$, then $x_{i_j} = y_j$ for $1 \leq j \leq n$.

Note that for any given set of n indices I , there is at most one BFS corresponding to it. We will use this property later.

Define

$$(2) \quad \mathcal{B}(A) = \{\vec{x} \in \mathcal{F}(A) \mid \vec{x} \text{ is a basic feasible solution in } \mathcal{F}(A)\}.$$

Let $\mathcal{C}(A) = \{q\vec{x} : q \geq 0, q \in \mathbb{Q}, \vec{x} \in \mathcal{F}(A)\}$ be the positive affine cone of $\mathcal{F}(A)$ in \mathbb{Q}^m . Then $\mathcal{C}(A)$ is a pointed rational cone generated by $\mathcal{B}(A)$, and $\mathcal{Z}(A) = \mathcal{C}(A) \cap \mathbb{Z}^m$ is exactly the set of levelers for A . By [17, Prop. 7.15], there exists a unique minimal

generating set of $\mathcal{Z}(A)$, which is called the *Hilbert basis* of $\mathcal{Z}(A)$. We have the following proposition.

Proposition 3. *If A is a matrix with nonnegative entries, then the Hilbert basis of $\mathcal{Z}(A)$ is the set of irreducible levelers of A .*

In Section 2, we prove our main theorem (Theorem 2) using tools from combinatorial optimization (in particular Carathéodory's Theorem). In Section 3, we apply Theorem 2 to prove some Ramsey-type statements about vector space multipartitions with respect to some irreducibility criteria that we shall define later.

2. PROOF OF THEOREM 2

The proof of our main theorem relies on Theorem 8 in Section 2.1, Theorem 9 in Section 2.2, and Theorem 10 in Section 2.3.

2.1. The first upper bound for $\ell(n, k)$.

For any matrix A , let $\ell(A)$ denote its number of rows, and let $|A|$ denote its determinant if A is a square matrix. For any rational vector \vec{x} , let \vec{x}_i denote its i th entry and let r_x be the smallest positive integer such that $r_x \vec{x}$ is integral, i.e., the entries of $r_x \vec{x}$ are all integers.

For any vector $\vec{x} \in \mathcal{F}(A)$ (thus, $A^T \vec{x} = \mathbf{1}$ and $\vec{x} > \vec{0}$) and any integer $r > 0$ such that $r\vec{x}$ is an integral vector, let $L(A, r, \vec{x})$ be the matrix obtained by stacking rx_i copies of A_i for $1 \leq i \leq m$. Note that we define $L(A, r, \vec{x})$ up to a permutation of rows.

Lemma 4. *If $A \in M_{n,n}(\mathbb{Z})$ is an invertible k -matrix and $\vec{x} \in \mathcal{F}(A)$, then $L(A, r_x, \vec{x})$ is irreducible.*

Proof. If the lemma does not hold, then there exists $\vec{y} \in \mathbb{Z}^n$, with $\vec{0} < \vec{y} < r_x \vec{x}$, such that $A^T \vec{y}$ is level, say with column sums $t > 0$. Then $A^T \vec{y} = t\mathbf{1}$, so $A^T t^{-1} \vec{y} = \mathbf{1} = A^T \vec{x}$. Thus $t^{-1} \vec{y} = \vec{x}$. But then $t\vec{x} = \vec{y}$ is integral, so by the definition of r_x we have $t \geq r_x$. This contradicts the assumption that $\vec{y} < r_x \vec{x}$. \square

A *convex combination* of the vectors $\vec{x}^{(1)}, \dots, \vec{x}^{(t)}$ is an expression of the form

$$\lambda_1 \vec{x}^{(1)} + \dots + \lambda_t \vec{x}^{(t)} \text{ with } \lambda_i \in \mathbb{R}^+ \text{ for } 1 \leq i \leq t, \text{ and } \sum_{i=1}^t \lambda_i = 1.$$

If $\lambda_i \in \mathbb{Q}^+$ for all $1 \leq i \leq t$, then the convex combination is called *rational*.

Lemma 5. *Let $n > 1$ and $k \geq 1$ be integers. Suppose that H is an irreducible k -matrix with n columns, at least 3 rows, and a 0-entry in each row. Then there exists a k -matrix $A = A(H)$ with n columns such that*

- (i) *A has rank n and $m = \frac{1}{2}((k+1)^n - k^n - 1)$ distinct rows,*
- (ii) *$H = L(A, r_h, \vec{h})$, where $\vec{h} \in \mathcal{F}(A)$ is a rational convex combination of the BFS in $\mathcal{B}(A)$.*

Proof. Let $R = (a_1, \dots, a_n)$ be a row of H and define the complement of R to be the vector $R^c = (t - a_1, \dots, t - a_n)$, where $t = \max_{1 \leq i \leq n} a_i$. Since H is an irreducible k -matrix with at least 3 rows and a 0-entry in each row, the following conditions hold.

- (C₁) $a_j = 0$ for some j , $1 \leq j \leq n$;
- (C₂) R is not the zero row vector;
- (C₃) R^c is not a row of H .

Note that (C₁) implies that no row of H is equal to its complement, and (C₃) does not hold if H has exactly two rows (namely R and R^c).

Let $A = A(H)$ be any k -matrix obtained by stacking together one copy of each distinct row of H and all the row vectors of length n with entries in $[0, k] = \{0, 1, \dots, k\}$ such that the conditions (C₁)–(C₃) hold for every row vector R of A . Note that there are $(k+1)^n$ row vectors of length n with entries in $[0, k]$, k^n such row vectors that do not satisfy (C₁), and one row vector that does not satisfy (C₂). Moreover, for any two distinct row vectors R_i and R_j in A , the row vectors R_i, R_i^c, R_j, R_j^c are all distinct and satisfy (C₁) and (C₂). Thus, the number of rows of A is

$$m = \frac{1}{2} ((k+1)^n - k^n - 1).$$

Furthermore, the matrix A has rank n since the linear span, \mathcal{S} , of its rows contains all the rows of the $n \times n$ identity matrix I_n . To see this, first note that $\mathbf{1}^T = (1, \dots, 1) \in \mathcal{S}$ since H is a level matrix and its distinct rows are in A . Next, it follows from the definition of A that if the row e_i of I_n is not a row of A , then its complement $e_i^c = (1, \dots, 1) - e_i$ is. Since $\mathbf{1}^T$ and e_i^c are in \mathcal{S} , then the linear combination $\mathbf{1}^T - e_i^c = e_i$ is also in \mathcal{S} . Thus, all the rows of I_n are in \mathcal{S} .

By definition of A , it follows that H can be obtained (up to a row permutation) by stacking some $v_i \geq 0$ copies of A_i (the i th row of A) for $1 \leq i \leq m$. Hence, the vector $\vec{v} = (v_1, \dots, v_m)^T$ satisfies $A^T \vec{v} = t \mathbf{1}$ for some positive integer t . If we let $\vec{h} = \frac{1}{t} \vec{v}$, then $H = L(A, t, \vec{h}) = L(A, r_h, \vec{h})$, where $r_h = t$ follows from the irreducibility of H . Moreover, $A^T \vec{h} = \mathbf{1}$ and $h_i \geq 0$ for $1 \leq i \leq m$. Thus, $\vec{h} \in \mathcal{F}(A)$. It follows from the theory of polytopes (e.g., see [18, Theorem 2.3]) that \vec{h} is a rational convex combination of the BFS in $\mathcal{B}(A)$. \square

We define the *complement of a matrix* B , denoted by B^c , to be the matrix obtained by replacing each entry b of B by $t - b$, where t is the maximum entry in B .

In the next lemma, we give an upper bound on the number of rows in the matrix $L(A, r_y, \vec{y})$ for the case when A is an invertible k -matrix and $\vec{y} \in \mathcal{F}(A)$.

Lemma 6. *Let $n > 1$ and $k \geq 1$ be integers. If $A \in M_{n,n}(\mathbb{Z})$ is an invertible k -matrix, $\vec{y} \in \mathcal{F}(A)$, and $L_A = L(A, r_y, \vec{y})$, then*

$$\ell(L_A) \leq (k/2)^{n-1} (n+1)^{(n+1)/2}.$$

Proof. Since $\vec{y} \in \mathcal{F}(A)$, then $A^T \vec{y} = \mathbf{1}$. Recall that r_y is by definition the smallest positive integer such that $r_y \vec{y}$ is an integral vector. Since $|A^T| \cdot \vec{y}$ is an integral vector by Cramer's rule, it follows that $r_y \leq |A^T|$.

Since A is a k -matrix, we may assume that its largest entry is k , otherwise A is a k' -matrix with largest entry $k' < k$. By definition, $L_A = L(A, r_y, \vec{y})$ is also a k -matrix and L_A^c denotes its complement. Since L_A is irreducible by Lemma 4, then we can directly verify that L_A^c is also irreducible. Moreover, if we let A^c be the complement of A , then there exists $y' \in \mathcal{F}(A^c)$ such that $L_A^c = L_{A^c} = L(A^c, r_{y'}, \vec{y}')$, $r_{y'} \vec{y}' = r_y \vec{y}$, and $\vec{z}' = |(A^c)^T| \cdot ((A^c)^T)^{-1} \mathbf{1} = |(A^c)^T| \cdot \vec{y}'$ is an integral vector (thus, $r_{y'} \leq |(A^c)^T|$). If $\vec{w} = (a_1, \dots, a_n)$ is the i th row vector of A^T , then $\vec{w}^c = (k - a_1, \dots, k - a_n)$ is the i th row vector of $(A^c)^T$. Since $A^T \vec{y} = \mathbf{1}$ and $(A^c)^T \vec{y}' = \mathbf{1}$, it follows that $\vec{w} \cdot \vec{y} = 1$ and $\vec{w}^c \cdot \vec{y}' = 1$. By using these observations and $r_{y'} \vec{y}' = r_y \vec{y}$, we obtain

$$r_y + r_{y'} = \vec{w} \cdot r_y \vec{y} + \vec{w}^c \cdot r_{y'} \vec{y}' = \sum_{j=1}^n a_j r_y y_j + \sum_{j=1}^n (k - a_j) r_y y_j = r_y k \sum_{j=1}^n y_j,$$

so that

$$\sum_{j=1}^n y_j = \frac{r_y + r_{y'}}{r_y k}.$$

Thus, the number of rows of L_A (or L_{A^c}) is by definition

$$\begin{aligned} \ell(L_A) &= r_y \sum_{j=1}^n \vec{y}_j = \frac{r_y + r_{y'}}{k} \leq \frac{|A^T| + |(A^c)^T|}{k} \\ (3) \qquad &\leq \frac{2(k/2)^n (n+1)^{(n+1)/2}}{k} \\ &= (k/2)^{n-1} (n+1)^{(n+1)/2}, \end{aligned}$$

where the inequality (3) holds since $(k/2)^n (n+1)^{(n+1)/2}$ is an upper bound for the determinant of any invertible $n \times n$ matrix in which all entries are real and the absolute value of any entry is at most k (see [3, 9]). \square

To prove the main theorem in this section, we also use a theorem of Carathéodory, which we shall state after a few definitions, following the account of Ziegler [20].

Let $S = \{\vec{x}^{(1)}, \dots, \vec{x}^{(t)}\}$ be a set of vectors from \mathbb{R}^n . The *affine hull* of S is

$$\text{Aff}(S) = \{\lambda_1 \vec{x}^{(1)} + \dots + \lambda_t \vec{x}^{(t)} : \lambda_i \in \mathbb{R} \text{ and } \sum_{i=1}^t \lambda_i = 1\}.$$

The *convex hull* of S , which we denote by $\text{Conv}(S)$, is the set of all its convex combinations. A set I of vectors in \mathbb{R}^n is *affinely independent* if every proper subset of I has a smaller affine hull. The dimension of an affine hull G is $g - 1$, where g is the cardinality

of largest affinely independent subset $I \subseteq G$. Finally, the dimension of a convex hull $\text{Conv}(S)$ is the dimension of the corresponding affine hull $\text{Aff}(S)$.

Theorem 7 (Carathéodory's Theorem [20]). *Let $S = \{\vec{x}^{(1)}, \dots, \vec{x}^{(t)}\}$ be a set of vectors from \mathbb{R}^n such that $\text{Conv}(S)$ has dimension d . If $\vec{h} \in \text{Conv}(S)$, then \vec{h} is the convex combination of at most $d + 1$ properly chosen vectors from S .*

We can now prove the following theorem.

Theorem 8. *Let $n > 1$ and $k > 0$ be integers. If H is an irreducible k -matrix with n columns, then*

$$\ell(H) \leq k^{n-1} 2^{-n} (n+1)^{(n+1)/2} ((k+1)^n - k^n + 1).$$

Proof. Let $R = (a_1, \dots, a_n)$ be a row of H such that $a_0 = \min_{1 \leq j \leq n} a_j$ is positive. Then it is easy to verify that the matrix H' obtained from H by replacing R with $(a_1 - a_0, \dots, a_n - a_0)$ is also an irreducible k -matrix with the same number of rows as H . Since we are interested in bounding $\ell(n, k)$, the maximum number of rows of an irreducible k -matrix with n columns, we may assume (w.l.o.g.) that each row of H contains a 0-entry. Since the theorem holds (by inspection) if H has fewer than 3 rows, we may also assume that H has at least 3 rows.

Thus, it follows from Lemma 5 that there exists a matrix $A = A(H)$ with m rows such that $H = L(A, r_h, \vec{h})$ for some $\vec{h} \in \mathcal{F}(A)$. Moreover, there exist nonnegative rational numbers $\lambda_1, \dots, \lambda_t$, such that $\sum_{j=1}^t \lambda_j = 1$ and $\vec{h} = \sum_{j=1}^t \lambda_j \vec{x}^{(j)}$, where $\vec{x}^{(j)} \in \mathcal{B}(A)$. For $1 \leq j \leq t$, recall that $r_j = r_{x^{(j)}}$ is the smallest positive integer such that $r_j \vec{x}^{(j)}$ is an integral vector. Thus,

$$(4) \quad r_h \vec{h} = \sum_{j=1}^t \left(\lambda_j \frac{r_h}{r_j} \right) r_j \vec{x}^{(j)}.$$

If $\lambda_j \frac{r_h}{r_j} > 1$ for some j , then it follows from (4) that $r_h \vec{h} > r_j \vec{x}^{(j)}$ since $\vec{h} \neq \vec{x}^{(j)}$. This would imply that matrix $H' = L(A, r_h, \vec{h}) - L(A, r_j, \vec{x}^{(j)})$ (where the subtraction is done componentwise) is a proper level k -submatrix of $H = L(A, r_h, \vec{h})$, which contradicts the irreducibility of H . Hence, we must have $\lambda_j \frac{r_h}{r_j} \leq 1$ for all j . Then this fact and (4) yield

$$(5) \quad \begin{aligned} \ell(H) = r_h \sum_{i=1}^m h_i &= r_h \sum_{j=1}^t \left(\lambda_j \sum_{i=1}^m \vec{x}_i^{(j)} \right) \\ &= \sum_{j=1}^t \left(\lambda_j \frac{r_h}{r_j} \sum_{i=1}^m r_j \vec{x}_i^{(j)} \right) \\ &\leq \sum_{j=1}^t \sum_{i=1}^m r_j \vec{x}_i^{(j)}. \end{aligned}$$

Since $\vec{x}^{(j)} \in \mathcal{B}(A)$, there exists a matrix $L_j = L(A, r_j, \vec{x}^{(j)})$ such that $\ell(L_j) = \sum_{i=1}^m r_j \vec{x}_i^{(j)}$ for $1 \leq j \leq t$. Moreover, it follows from Theorem 7 that $t \leq d + 1$, where d is the dimension of the polytope generated by the vectors in $\mathcal{B}(A)$. Thus, it follows from (5) and the preceding observations that

$$(6) \quad \ell(H) \leq (d + 1) \max_j \ell(L_j).$$

Since $\vec{x}^{(j)} \in \mathcal{B}(A)$ is a BFS, there exists a subset $I \subseteq \{1, \dots, m\}$ of n indices such that $\vec{x}_i^{(j)} = 0$ for $i \notin I$. Let $A^{(j)}$ be the $n \times n$ matrix containing the rows A_i for each $i \in I$. Then $A^{(j)}$ is invertible and $\vec{y}^{(j)} = ((A^{(j)})^T)^{-1} \mathbf{1}$ satisfies $x_i^{(j)} = x_i$ for all $i \in I$. Hence, $L_j = L(A, r_{x^{(j)}}, \vec{x}^{(j)})$ and $L'_j = L(A^{(j)}, r_{y^{(j)}}, \vec{y}^{(j)})$ are the same matrices up to a permutation of rows. Thus, it follows from Lemma 6 that

$$(7) \quad \ell(L_j) = \ell(L'_j) \leq (k/2)^{n-1} (n+1)^{(n+1)/2}.$$

Since d is at most the number of distinct rows in A , it follows from Lemma 5 that

$$(8) \quad d + 1 \leq \frac{(k+1)^n - k^n - 1}{2} + 1 = \frac{(k+1)^n - k^n + 1}{2},$$

It now follows from (6), (8), and (7) that

$$\ell(H) \leq k^{n-1} 2^{-n} (n+1)^{(n+1)/2} ((k+1)^n - k^n + 1),$$

which concludes the proof. \square

2.2. The second upper bound for $\ell(n, k)$.

In this section, we establish another upper bound for $\ell(n, k)$ that is better than the upper bound provided by Theorem 9 for $n \in \{2, 3\}$.

Theorem 9. *Let $n > 1$ and $k > 0$ be integers. If H is an irreducible k -matrix with n columns, then $\ell(H) < (2k)^{2^{n-1}-1}$.*

Proof. For convenience define $r_n = 2^{n-1} - 1$. Then $r_2 = 1$ and $r_{n+1} = 2r_n + 1$.

The proof will be by induction on $n \geq 2$, where the $n = 2$ case follows from Theorem 1. Assume the statement of the theorem holds for n , and let H be an irreducible k -matrix of size $m \times (n+1)$. Let \widehat{H} be the matrix obtained from H by deleting the last column of H .

Note that if a level matrix is reducible, then its rows can be rearranged to form a stack of two level matrices, and this division can be continued until a stack of irreducible matrices is attained. Rearrange the rows of H to form a new matrix consisting of a stack of k -matrices $M^{(i)}$ of size $m_i \times (n+1)$, $1 \leq i \leq t$, such that each matrix $\widehat{M}^{(i)}$ formed by deleting the last column of $M^{(i)}$ is an irreducible matrix. For $1 \leq i \leq t$, let a_i be the common column sum of $\widehat{M}^{(i)}$, and let b_i be the sum of the entries in the last column of $M^{(i)}$. Note that for $1 \leq i \leq t$, we have $m_i \leq (2k)^{r_n}$ by the induction hypothesis. Since H is a k -matrix, both a_i and b_i cannot exceed $K = (2k)^{r_n} k$.

Let M be the K -matrix of size $t \times 2$ with i th row vector (a_i, b_i) , $1 \leq i \leq t$. Since H is level, so is M . Since $m_i \leq (2k)^{r_n}$, we also have

$$m = \sum_{i=1}^t m_i \leq \sum_{i=1}^t (2k)^{r_n} = t(2k)^{r_n}.$$

Since H is irreducible, then M is also irreducible, and $t \leq 2K - 1$ by Theorem 1. Thus

$$m \leq t(2k)^{r_n} < 2K(2k)^{r_n} = 2k((2k)^{r_n})^2 = (2k)^{2r_n+1} = (2k)^{r_{n+1}}.$$

□

2.3. The lower bound.

In this section we construct examples of irreducible matrices, each of whose number of rows is greater than some exponential function of its number of columns.

Theorem 10. *Let $\epsilon > 0$ be any real number. There exist infinitely many irreducible 1-matrices with n columns and more than $e^{(1-\epsilon)\sqrt{n \ln n}}$ rows.*

Proof. We write $f(x) \sim g(x)$ if $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$.

Let p_i denote the i th prime, and for $x \geq 2$, let $t = \pi(x)$ be the number of primes not exceeding x . Let J_r be the $r \times r$ matrix with 0's on its main diagonal and 1's everywhere else. Set $r_i = p_i + 1$, $n = n(x) = \sum_{i=1}^t r_i$, and consider the $n \times n$ matrix

$$A = A(x) = \begin{bmatrix} J_{r_1} & 0 & 0 & \cdots & 0 \\ 0 & J_{r_2} & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 0 & J_{r_t} \end{bmatrix},$$

where the 0's represent zero matrices of the appropriate sizes. By repeating each row of J_{r_i} in A exactly P/p_i times, where $P = P(x) = \prod_{i=1}^t p_i$, we get a level matrix $A^* = A^*(x)$. Since p_1, p_2, \dots, p_t are relatively prime in pairs, the matrix A^* is irreducible, and the number of columns of A^* is still n .

Let $\theta = \theta(x) = \ln(\prod_{i=1}^t p_i)$. The prime number theorem states that $\pi(x) \sim x/\ln x$, and this is equivalent to $\theta \sim x$ (see [1], Th. 4.4). It also follows from [15] that $\sum_{i=1}^t p_i \sim x^2/\ln(x^2)$, and from [1], Th. 12, that $\sum_{i=1}^t \frac{1}{p_i} \sim \ln \ln x$.

Now the number of rows of A^* is

$$m = m(x) = \sum_{i=1}^t \frac{P}{p_i} (p_i + 1).$$

Thus

$$m = P \sum_{i=1}^t (1 + 1/p_i) \text{ and } \ln m = \theta(x) + \ln \left(\sum_{i=1}^t (1 + 1/p_i) \right).$$

These relations yield $\ln m \sim x$ and $n \sim x^2/\ln(x^2)$.

Notice that $f(y) = \sqrt{y \ln y}$ is increasing for $y \geq 1$. Let y be such that $x = \sqrt{y \ln y}$, and note that $x \rightarrow \infty$ if and only if $y \rightarrow \infty$. Then $\ln m \sim x$ and $n \sim x^2 / \ln(x^2)$ imply that

$$\ln m \sim \sqrt{y \ln y} \text{ and } n \sim \frac{x^2}{\ln x^2} = \frac{y \ln y}{\ln(y \ln y)} = \frac{y \ln y}{\ln y + \ln \ln y} \sim y.$$

Hence $\ln m \sim \sqrt{n \ln n}$. Thus, for any $\epsilon > 0$, there exists an integer $n = n(x, \epsilon)$ such that $\ln m / \sqrt{n \ln n} > 1 - \epsilon$, or

$$m > e^{(1-\epsilon)\sqrt{n \ln n}}.$$

□

2.4. Proof of Theorem 2.

The first part Theorem 2 follows directly from Theorem 10. The upper bound for $\ell(n, k)$ follows from Theorem 8 when $n = 3$, and from Theorem 9 when $n > 3$.

3. APPLICATION TO MULTIPARTITIONS OF FINITE VECTOR SPACES

Let $V = V(n, q)$, where $V(n, q)$ denotes the n -dimensional vector space over the finite field with q elements. We will consider multisets of nonzero subspaces of V such that each nonzero element of V is in the same number of subspaces, counting multiplicities. More explicitly, a *multipartition* P of V is a pair (F, α) , where F is a finite set, α is a function from F to the set of nonzero subspaces of V , and there exists a positive integer λ such that whenever v is a nonzero elements of V we have

$$|\{f \in F : v \in \alpha(f)\}| = \lambda.$$

In this case, we call P a λ -*partition*.

A number of papers have been written about 1-partitions, usually just called “partitions”, (e.g., see [2, 4, 7, 10, 16] and [11] for a survey), and at least one about multipartitions ([8]). A general question in this area is to classify the multipartitions of V .

If V has a λ -partition P and a μ -partition Q , then a $(\lambda + \mu)$ -partition of V may be formed by combining P and Q in the obvious way. We denote this by $P + Q$. Conversely, it may be possible to break a multipartition into smaller multipartitions. Thus, it is of interest to investigate multipartitions that cannot be broken up any further. We call a multipartition P of V *irreducible* if there do not exist multipartitions Q_1 and Q_2 of V such that $P = Q_1 + Q_2$. Clearly any multipartition of V can be written as a sum of irreducible multipartitions of V .

Let S be a set. Call (F, α) a *level family* of S if α is a function from F into $2^S \setminus \{\emptyset\}$ for which there exists a positive integer λ such that if $x \in S$, then $|\{f \in F : \alpha(f) = x\}| = \lambda$. We call λ the *height* of the family. Call the level family (F, α) with height λ *reducible* if there exists a subset F' of F and an integer λ' , $0 < \lambda' < \lambda$, such that $(F', \alpha|_{F'})$ is a level family of S with height λ' ; otherwise call (F, α) *irreducible*.

Corollary 11. *If S has $n \geq 2$ elements and (F, α) is an irreducible family of S , then*

$$|F| \leq (n+1)^{(n+1)/2}.$$

Thus, a finite set S has only finitely-many irreducible families.

Proof. Let s_1, \dots, s_n be the distinct elements of S , and let f_1, \dots, f_m be the distinct elements of F . Define the 1-matrix B with entries $b_{1,1}, \dots, b_{m,n}$ by letting $b_{ij} = 1$ if $s_j \in \alpha(f_i)$, and $b_{ij} = 0$ otherwise. By the definition of an irreducible family all the column sums of B are equal, and B is an irreducible matrix. Then by setting $k = 1$ in Theorem 2, we obtain $|F| = m \leq (n+1)^{(n+1)/2}$. \square

Corollary 12. *If (F, α) be an irreducible λ -partition of $V(n, q)$ for some integer λ , then*

$$|F| \leq q^{(n-1)q^{n-1}/2}.$$

Thus, $V(n, q)$ has finitely-many irreducible λ -partitions.

Proof. Let W_1, \dots, W_t be the distinct 1-dimensional subspaces of $V(n, q)$, where $t = (q^n - 1)/(q - 1)$, and let f_1, \dots, f_m be the distinct elements of F . Define the 1-matrix B with entries $b_{1,1}, \dots, b_{m,t}$ by letting $b_{ij} = 1$ if $W_j \subseteq \alpha(f_i)$ and $b_{ij} = 0$ otherwise. Then all the column sums of B are λ , and B is an irreducible matrix. Since $t + 1 = q^{n-1}$, setting $k = 1$ in Theorem 2 yields

$$|F| = m \leq (t+1)^{(t+1)/2} = q^{(n-1)q^{n-1}/2}.$$

\square

4. CONCLUSION

Our main question (see page 1) is still open in general. For example, if $A \in M_{m,n}(\mathbb{Z})$ is a k -matrix such that $A^T \vec{x} = \mathbf{1}$ for some $\vec{x} > \vec{0}$, then we know by Lemma 6 that the number of rows, $\ell(L_A)$, of the irreducible matrix $L_A = L(A, r_x, \vec{x})$ satisfies $\ell(L_A) \leq (k/2)^{n-1} (n+1)^{(n+1)/2}$. However, the small cases that we have checked suggest that $\ell(L_A)$ is much smaller. It would be interesting to find the exact value of $\ell(L_A)$ or improve its upper bound. Such an improvement would also give a better upper bound for the general value of $\ell(n, k)$ in Theorem 2.

In Proposition 3, we characterized the set $\mathcal{Z}(A)$ of levelers of a given matrix by a cone whose Hilbert basis is the set of irreducible levelers $\mathcal{I}(A) \subseteq \mathcal{Z}(A)$. It would be interesting to investigate if this characterization can shed more light on the study of Hilbert bases (e.g., see [5, 12]) in certain cases.

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