THE STRUCTURE OF THE MINIMUM SIZE SUPERTAIL OF A SUBSPACE PARTITION

E. NĂSTASE

MATHEMATICS DEPARTMENT
XAVIER UNIVERSITY
CINCINNATI, OHIO 45207, USA

P. SISSOKHO

MATHEMATICS DEPARTMENT
ILLINOIS STATE UNIVERSITY
NORMAL, ILLINOIS 61790, USA

(Dedicated to Professor Olof Heden on the occasion of his retirement)

ABSTRACT. Let $V = V(n, q)$ denote the vector space of dimension n over the finite field with q elements. A *subspace partition* $\mathcal{P}$ of V is a collection of nontrivial subspaces of V such that each nonzero vector of V is in exactly one subspace of $\mathcal{P}$. For any integer d, the *d-supertail* of $\mathcal{P}$ is the set of subspaces in $\mathcal{P}$ of dimension less than d, and it is denoted by ST. Let $\sigma_q(n, t)$ denote the minimum number of subspaces in any subspace partition of V in which the largest subspace has dimension t. It was shown by Heden et al. that $|ST| \geq \sigma_q(d, t)$, where t is the largest dimension of a subspace in ST. In this paper, we show that if $|ST| = \sigma_q(d, t)$, then the union of all the subspaces in ST constitutes a subspace under certain conditions.

1. INTRODUCTION

Let $V = V(n, q)$ denote a vector space of dimension n over the finite field with q elements. We use the term *d-subspace* to refer to a subspace of dimension d. For any subspace U of V, we let U^* denote the set of nonzero vectors in U. A *subspace partition* $\mathcal{P}$ of V, also known as a *vector space partition*, is a collection of nontrivial subspaces of V such that each vector of V^* is in exactly one subspace of $\mathcal{P}$ (e.g., see Heden [12] for a survey). The study of subspace partitions originated from the general problem of partitioning a finite (not necessarily abelian) group into subgroups that only intersect at the identity element (e.g.; see Zappa [21] for a survey). Subspace partitions can be used to construct translation planes, error-correcting codes, orthogonal arrays, and designs (e.g., see [1, 2, 3, 7, 10, 17, 18, 19]).

Suppose that there are m distinct dimensions, $d_1 < d_2 < \cdots < d_m$, that occur in a subspace partition $\mathcal{P}$, and let n_d denote the number of d-subspaces in $\mathcal{P}$. Then the expression $[d_1^{n_{d_1}}, \ldots, d_m^{n_{d_m}}]$ is called the *type* of $\mathcal{P}$. The general problem in this area is to

nastasee@xavier.edu, psissok@ilstu.edu.

find necessary and sufficient conditions for the existence of a subspace partition of V of a given type (e.g., see [6, 4, 8, 9, 19, 13] for the solution of some special cases). Two obvious necessary conditions for the existence of a subspace partition of type $[d_1^{n_{d_1}},\ldots,d_m^{n_{d_m}}]$ are the *packing condition*

$$\sum_{i=1}^{m} n_{d_i}(q^{d_i}-1)=q^n-1, \tag{1}$$

and the *dimension condition*

$$\begin{cases} n\geq d_i+d_j \text{ if } n_{d_i}{+}n_{d_j}\geq 2 \text{ and } i\neq j;\ \text{and}\\ n\geq 2d_i \text{ if } n_{d_i}\geq 2. \end{cases} \tag{2}$$

To the best of our knowledge, there are not many other known necessary conditions for the existence of a subspace partition $\mathcal{P}$ of V. Heden and Lehmann [13] derived some necessary conditions (see Lemma 10) by essentially counting in two ways tuples of the forms (H,U) and (H,W_1,W_2), where H is a hyperplane of V and U,W_1,W_2 are subspaces of $\mathcal{P}$ that are contained in H. Blinco et al. [5] and Heden [9, 11] derived some necessary conditions on the set T of subspaces of minimum dimension (called *tail* in [9]) of $\mathcal{P}$. The concept of tail was later generalized by Heden et al. [15], as we shall see below.

Let $\mathcal{P}$ be a subspace partition of $V=V(n,q)$ of type $[d_1^{n_{d_1}},\ldots,d_m^{n_{d_m}}]$. For any integer d such that $d_1<d\leq d_m$, the *d-supertail* of $\mathcal{P}$ is the set of subspaces in $\mathcal{P}$ of dimension less than d, and it is denoted by ST. The *size* of a subspace partition $\mathcal{P}$ is the number of subspaces in $\mathcal{P}$. For $1\leq t<n$, let $\sigma_q(n,t)$ denote the *minimum size* of any subspace partition of V in which the largest subspace has dimension t. The exact value of $\sigma_q(n,t)$ is given by the following theorem (see André [1] and Beutelspacher [3] for $n \pmod t\equiv 0$, and see [14, 20] for $n \pmod t\not\equiv 0$).

Theorem 1. *Let $n,k,t,$ and r be integers such that $0\leq r<t$, $k\geq 1$, and $n=kt+r$. Then*

$$\sigma_q(n,t)=\begin{cases} \dfrac{q^{kt}-1}{q^t-1} & \textit{for } r=0,\\ q^t+1 & \textit{for } r\geq 1 \textit{ and } 3\leq n<2t,\\ q^{t+r}\sum\limits_{i=0}^{k-2} q^{it}+q^{\lceil\frac{t+r}{2}\rceil}+1 & \textit{for } r\geq 1 \textit{ and } n\geq 2t. \end{cases}$$

The following theorem of Heden et al. [15] generalizes a theorem of Heden [11, Theorem 1].

Theorem 2. *Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of type $[d_1^{n_{d_1}},\ldots,d_m^{n_{d_m}}]$ and let $2\leq s\leq m$. If ST is d_s-supertail of $\mathcal{P}$, then*

$$|ST|\geq\sigma_q(d_s,d_{s-1})\ . \tag{3}$$

If equality holds in (3), then Theorem 2 has the following interesting corollary (see [15]).

Corollary 3. *If $|ST|=\sigma_q(d_s,d_{s-1})$ and $d_s\geq 2d_{s-1}$, then the union of the subspaces in ST forms a d_s-subspace.*

Note that the crucial part of the conclusion of Corollary 3 is that the set of all points covered by the subspaces in ST is a subspace. (In general, the d_s-supertail of a subspace partition of $V(n,q)$ need not be a subspace.) One outstanding question that remains is whether the conclusion of Corollary 3 holds for $d_{s-1} < d_s < 2d_{s-1}$. For the special case when ST is a simple tail, Heden [11, Theorem 3] proved the following theorem.

Theorem 4. *Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of type $[d_1^{n_{d_1}}, \ldots, d_m^{n_{d_m}}]$. If ST is the tail of $\mathcal{P}$ (i.e., ST is the set of d_1-subspaces) such that $|ST| = q^{d_1} + 1$ and $d_2 < 2d_1$, then ST is a d_1-spread (i.e., a subspace partition consisting of d_1-subspaces).*

In this paper we prove the following generalization of Theorem 4.

Theorem 5. *Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of type $[d_1^{n_{d_1}}, \ldots, d_m^{n_{d_m}}]$. Let $2 \leq s \leq m$, and suppose ST is a d_s-supertail of $\mathcal{P}$ such that $|ST| = \sigma_q(d_s, d_{s-1})$ and $d_s < 2d_{s-1}$. Furthermore, assume that one of the following conditions holds.*

(i) *$s - 1 \leq 2$, that is ST contains subspaces of at most 2 different dimensions.*
(ii) *$d_s = 2d_{s-1} - 1$.*
(iii) *All the subspaces in $\mathcal{P} \setminus ST$ have the same dimension.*

Then the union of the subspaces in ST forms a subspace W. Moreover,

(a) *$s - 1 = 1$, $n_1 = q^{d_1} + 1$, and $\dim W = 2d_1$, or*
(b) *$s - 1 = 2$, $n_1 = q^{d_2}$, $n_2 = 1$, and $\dim W = d_1 + d_2$.*

The following result is a consequence of Theorem 2, Corollary 3, and Theorem 5.

Corollary 6. *Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of type $[d_1^{n_{d_1}}, \ldots, d_m^{n_{d_m}}]$ and let $2 \leq s < m$. Let ST be the d_s-supertail of $\mathcal{P}$, let $\widehat{ST}$ be its d_{s+1}-supertail, and assume that $|ST| = \sigma_q(d_s, d_{s-1})$.*

(i) *If $2 \leq s \leq 3$, $d_s < 2d_{s-1}$, and $d_{s+1} < 2d_s$, then*

$$|\widehat{ST}| \geq \sigma_q(d_{s+1}, d_s) + \sigma_q(d_s, d_{s-1}).$$

(ii) *If $d_s \geq 2d_{s-1}$, or if $s = 3$ and $d_3 = d_2 + d_1$, then*

$$|\widehat{ST}| \geq \sigma_q(d_{s+1}, d_s) + \sigma_q(d_s, d_{s-1}) - 1.$$

Remark 7. *Note that the condition $s = 3$ in Corollary 6(ii) is equivalent to saying that ST contains subspaces of two different dimensions, namely d_1 and d_2.*

Theorem 5 can be viewed as a special case of the general question of determining nontrivial conditions under which a set of points of a projective space forms a subspace. We conjecture that Theorem 5 holds in all cases, and not just if (*i*), (*ii*), or (*iii*) holds.

The rest of the paper is organized as follows. In Section 2, we gather some known results that we shall use in Section 3 to first establish some auxiliary results, and then to prove our main results, i.e., Theorem 5 and Corollary 6. Finally, we include some supporting lemmas in Section 4 (Appendix).

2. Preliminaries

Let n be a positive integer and let q be a prime power. Set $\Theta_0 = 0$. For any integer i such that $1 \le i \le n$, let

$$\Theta_i = \frac{q^i - 1}{q - 1}$$

denote the number of points (i.e.,1-subspaces) in an i-subspace.

We will need the following elementary results (Proposition 8 and Proposition 9).

Proposition 8. *The number of hyperplanes containing a given d-subspace of $V(n,q)$ is Θ_{n-d}.*

Proposition 9. *If U is a subset of $V = V(n,q)$ containing Θ_d points and contained in precisely Θ_{n-d} hyperplanes, then U is a d-subspace of V.*

To state the next lemmas, we need the following definitions. For $n \ge 2$, let $\mathcal{P}$ be a subspace partition of $V = V(n,q)$ of type $[d_1^{n_{d_1}}, \ldots, d_m^{n_{d_m}}]$. For any hyperplane H of V, let $b_{H,x}$ be the number of x-subspaces in $\mathcal{P}$ that are contained in H and set $b_H = [b_{H,d_1}, \ldots, b_{H,d_m}]$. Define

$$\mathcal{B} = \{b_H : \ H \text{ is a hyperplane of } V\}.$$

For any $b \in \mathcal{B}$, let s_b denote the number of hyperplanes H of V such that $b_H = b$.

We will use Lemma 10 and Lemma 11 by Heden and Lehmann [13].

Lemma 10. *Let $\mathcal{P}$ be a subspace partition of $V(n,q)$, and let $\mathcal{B}$ and s_b be as defined earlier. If $\mathcal{P}$ contains two different subspaces, one of dimension d and another of dimension d', with $1 \le d, d' \le n-2$, then*
(i) $\sum_{b\in\mathcal{B}} s_b = \Theta_n$,
(ii) $\sum_{b\in\mathcal{B}} b_d \, s_b = n_d \Theta_{n-d}$,
(iii) $\sum_{b\in\mathcal{B}} \binom{b_d}{2} s_b = \binom{n_d}{2} \Theta_{n-2d}$,
(iv) $\sum_{b\in\mathcal{B}} b_d b_{d'} s_b = n_d n_{d'} \Theta_{n-d-d'}$.

Lemma 11. *Let $\mathcal{P}$ be a subspace partition of $V(n,q)$. If H is a hyperplane of V, then*

$$|\mathcal{P}| = 1 + \sum_{i=1}^{m} b_{H,d_i} q^{d_i}.$$

We will also use the following lemma due to Heden et al. [14].

Lemma 12. *Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of type $[d_1^{n_1}, \ldots, d_m^{n_m}]$ and let $2 \le s \le m$. If ST is a d_s-supertail of $\mathcal{P}$ and H is a hyperplane of V, then*

$$\sum_{i=1}^{s-1} (n_{d_i} - b_{H,d_i}) q^{d_i} = c_H \cdot q^{d_s},$$

where $c_H = q^{n-d_s} - \sum_{i=s}^{m} (n_{d_i} - b_{H,d_i}) q^{d_i - d_s}$ *is a nonnegative integer.*

Finally, we will need the following lemma due to Herzog and Schönheim [17], and independently Beutelspacher [3] and Bu [6].

Lemma 13. *Let n and d be integers such that $1 \leq d \leq n/2$. Then $V(n,q)$ admits a partition with one subspace of dimension $n-d$ and q^{n-d} subspaces of dimension d.*

3. Auxiliary Results and the proof of the main theorem

In this section, we use $\mathcal{H}$ to denote the set of all hyperplanes of V.

Lemma 14. *Let $\mathcal{P}$ be a subspace partition of $V = V(n,q)$ of type $[d_1^{n_{d_1}}, \ldots, d_m^{n_{d_m}}]$, where $1 \leq d_1 < \ldots < d_m$. Assume that $2 \leq s \leq m$, and let ST be a d_s-supertail of $\mathcal{P}$ such that $|ST| = \sigma_q(d_s, d_{s-1})$ and $d_s < 2d_{s-1}$. Then $d_s \leq d_{s-1} + d_1$.*

Proof. Suppose that $d_s > d_{s-1} + d_1$. Let $U, W \in ST$ be such that $\dim U = d_{s-1}$ and $\dim W = d_1$. Let B_W be a basis of W, B_U a basis of U, and consider a basis B of V obtained by extending $B_U \cup B_W$. Then $V' = \operatorname{span}(B \setminus B_W)$ is a subspace of V such that $\dim V' = n - d_1$, $U \subseteq V'$, and $V' \cap W = \emptyset$. Now let $\mathcal{P}'$ be the subspace partition induced by $\mathcal{P}$ in V', and let $ST' = \{A \cap V' \neq \{\mathbf{0}\} : \ A \in ST\}$, where $\mathbf{0}$ denotes the zero vector. Let X' be a subspace in $\mathcal{P}' \setminus ST'$ that has a minimum possible dimension. Since $X' = X \cap V'$ for some $X \in \mathcal{P} \setminus ST$ and $\dim X \geq d_s > d_{s-1} + d_1$, it follows that $\dim X' \geq \dim X - d_1 > d_{s-1}$. Moreover, the subspace $U' = U \cap V' = U$ is in ST' and $W' = W \cap V' = \{\mathbf{0}\}$. Thus, ST' is a supertail of $\mathcal{P}'$ with highest dimension $d_{s-1} = \dim U'$ and of size $|ST'| \leq |ST \setminus \{W\}| \leq |ST| - 1 = q^{d_{s-1}}$. This contradicts the fact that $|ST'| \geq \sigma_q(\dim X', d_{s-1}) = q^{d_{s-1}} + 1$. The lemma follows. □

Lemma 15. *Let $\mathcal{P}$ be a partition of $V(n,q)$ with supertail ST consisting of subspaces of dimensions at most t. For any $H \in \mathcal{H}$, let $\beta_H = \sum_{i \leq t} b_{H,i} q^i$ and let $\beta_0 = \min_{H \in \mathcal{H}} \beta_H$. Then $|ST| \geq \beta_0 + 1$.*

Proof. Suppose that $|ST| \leq \beta_0$. Then, applying the definition of β_H, implies that

$$\Theta_n |ST| \leq \Theta_n \beta_0 = \sum_{H \in \mathcal{H}} \beta_0 \leq \sum_{H \in \mathcal{H}} \beta_H = \sum_{H \in \mathcal{H}} \sum_{i=1}^{t} b_{H,i} q^i,$$

and thus with the use of Lemma 10, we obtain

$$\begin{aligned} \Theta_n |ST| \leq \sum_{i=1}^{t} q^i \Big(\sum_{H \in \mathcal{H}} b_{H,i} \Big) &\leq \sum_{i=1}^{t} q^i (n_i \Theta_{n-i}) \\ &\leq \sum_{i=1}^{t} n_i (\Theta_n - \Theta_i) \leq \Theta_n |ST| - \sum_{i=1}^{t} n_i \Theta_i, \end{aligned}$$

which is a contradiction since $\sum_{i=1}^{t} n_i \Theta_i$ is the number of points in ST, and this number is positive. □

Lemma 16. *Let $\mathcal{P}$ be a partition of $V(n,q)$ with a d-supertail ST such that t is the maximum dimension of any subspace in ST and $d < 2t$. For any $H \in \mathcal{H}$, let $\beta_H =$*

$\sum_{i=d_1}^{t} b_{H,i}q^i$ and $\beta_0 = \min_{H\in\mathcal{H}} \beta_H$. If ST has size q^t+1, then $\beta_0 = q^t$. Moreover, there exists an integer c_0 such that

$$\sum_{i=1}^{t} n_i\Theta_i = \frac{c_0 q^d - 1}{q-1}.$$

Proof. Let a be the minimum dimension of any subspace in ST First, suppose that for some $H \in \mathcal{H}$, we have $\beta_H < q^t$. Then by Lemma 12, there exists an integer c_H such that $\sum_{i=a}^{t}(n_i - b_{H,i})q^i = c_H q^d$. Hence, we have

$$\sum_{i=a}^{t} n_i q^{i-a} = c_H q^{d-a} + \beta_H q^{-a}.$$

Since $\beta_H < q^t$, we obtain $\beta_H q^{-a} < q^{t-a}$. Then, it follows from [15, Proof of Proposition 6 (Case 2)] that $|ST| > q^t + 1$. This is a contradiction and thus $\beta_H \geq q^t$ for all $H \in \mathcal{H}$. By Lemma 15, $\beta_0 \leq |ST| - 1 = q^t$, and thus $\beta_0 = q^t$.

Now by Lemma 12, there exists an integer c_0 such that $\sum_{i=a}^{t} n_i q^i - \beta_0 = c_0 q^d$. Since $\beta_0 = q^t$ and $\sum_{i=a}^{t} n_i = |ST| = q^t + 1$, it follows after some arithmetic that

$$\sum_{i=a}^{t} n_i\Theta_i = \frac{c_0 q^d - 1}{q-1}. \tag{4}$$

□

Let $\mathcal{P}$ be a partition of $V(n,q)$ with a d-supertail ST. If ST has type $[t^{q^t+1}]$, then we recall that it follows from Heden [11, Theorem 3] that the union of the subspaces in ST is a $2t$-subspace. The following lemma is an extension of that result.

Lemma 17. *Let $\mathcal{P}$ be a partition of $V(n,q)$ with a d-supertail ST of type $[t^1, a^{q^t}]$, i.e., ST contains one subspace of dimension t and q^t subspaces of dimension a, where $t > a$. Then the union of the subspaces in ST forms a $t+a$-subspace.*

Proof. Recall that $\mathcal{H}$ denotes the set of all hyperplanes of V. Let $H \in \mathcal{H}$ be any hyperplane. It follows from Lemma 12 that there exists an integer $c_H \geq 0$ such that

$$(n_t - b_{H,t})q^t + (n_a - b_{H,a})q^a = c_H q^d.$$

Thus,

$$b_{H,a} = q^t + (1 - b_{H,t})q^{t-a} - c_H q^{d-a}, \tag{5}$$

where $0 \leq b_{H,a} \leq q^t$ and $b_{H,t} \in \{0,1\}$. Let $\mathcal{A}$ be the set of a-subspaces in ST, and let α_i denote the number of hyperplanes in V that contain exactly i members of $\mathcal{A}$. If $\alpha_i \neq 0$, then there exists a hyperplane $H \in \mathcal{H}$ that contains exactly $b_{H,a} = i$ members from $\mathcal{A}$. Thus, it follows from (5) that

$$\alpha_i \neq 0 \ \Rightarrow\ q^{t-a} \text{ divides } i. \tag{6}$$

Define the integers x, y, and z as follows:

$$x = \sum_{i=q^{t-a}}^{q^t} i\alpha_i, \quad y = \sum_{i=q^{t-a}}^{q^t} \binom{i}{2}\alpha_i, \quad z = \sum_{i=q^{t-a}}^{q^t} \alpha_i.$$

Then it follows from Lemma 10 that

$$x = \sum_{i=q^{t-a}}^{q^t} i\alpha_i = n_a\Theta_{n-a} \tag{7}$$

and

$$y = \sum_{i=q^{t-a}}^{q^t} \binom{i}{2}\alpha_i = \binom{n_a}{2}\Theta_{n-2a}. \tag{8}$$

Since $\sum_{i=0}^{q^t} \alpha_i = |\mathcal{H}| = \Theta_n$, it follows by (6) and the definition of z that

$$z = \sum_{i=q^{t-a}}^{q^t} \alpha_i = \Theta_n - \alpha_0. \tag{9}$$

Using (7)–(9) and the fact that $n_a = q^t$ and $\Theta_i = (q^i - 1)/(q-1)$, we obtain

$$\begin{aligned}
&\sum_{i=q^{t-a}}^{q^t} \alpha_i(i-q^{t-a})(i-q^t) \\
&= 2y + (1 - q^{t-a} - q^t)x + q^{2t-a}z \\
&= q^t(q^t-1)\Theta_{n-2a} + (1-q^{t-a}-q^t)q^t\Theta_{n-a} + q^{2t-a}(\Theta_n - \alpha_0) \\
&= \Theta_{n+t-a} - \Theta_{n+t-2a} - q^{2t-a}\alpha_0.
\end{aligned} \tag{10}$$

Note that

$$(i-q^{t-a})(i-q^t) \begin{cases} = 0 & \text{if } i = q^{t-a}, \\ < 0 & \text{if } q^{t-a} < i < q^t, \\ = 0 & \text{if } i = q^t. \end{cases} \tag{11}$$

Thus, it follows from (10) and (11) that

$$\sum_{i=q^{t-a}}^{q^t} \alpha_i(i-q^{t-a})(i-q^t) = \Theta_{n+t-a} - \Theta_{n+t-2a} - q^{2t-a}\alpha_0 \leq 0, \tag{12}$$

and it follows from (12) that

$$\alpha_0 \geq \Theta_{n-t} - \Theta_{n-t-a}. \tag{13}$$

Furthermore, since $b_{H,t} \in \{0,1\}$, it follows from (5) that for any hyperplane H, we have

$$b_{H,a} = 0 \ \Rightarrow \ b_{H,t} = 1. \tag{14}$$

Hence, if W_t is a t-subspace and W_a is an a-subspace in the supertail ST, then each of the α_0 hyperplanes that contain no a-subspace is a hyperplane that contains W_t but not W_a. As there are $\theta_{n-t} - \theta_{n-t-a}$ such hyperplanes, it follows that

$$\alpha_0 = \theta_{n-t} - \theta_{n-t-a}.$$

Since we considered any a-subspace of ST, the argument shows that the θ_{n-t-a} hyperplanes that contain some W_t must indeed contain all the a-spaces of ST. Thus all the subspaces of the supertail must be contained in the intersection T of these θ_{n-t-a} hyperplanes. Moreover, since θ_{n-t-a} hyperplanes intersect in a subspace of dimension at most $t+a$, it follows that T has dimension $t+a$ and is thus partitioned by the supertail. □

We are now ready to prove our main theorem.

Proof of Theorem 5. Let $\mathcal{P}$ be a partition of $V(n,q)$ with a d_s-supertail ST of size $|ST| = \sigma_q(d_s, d_{s-1}) = q^{d_{s-1}} + 1$. To simplify the notation, we set $d = d_s$ and $t = d_{s-1}$. Let k and r_d be integers such that $k \geq 1$, $n = kd + r_d$, and $1 \leq r_d < d$. Recall that if $d \geq 2t$, then Corollary 3 holds. So we may assume that $r_t = d - t$ satisfies $0 < r_t < t$.

Since $\mathcal{P}$ contains subspaces of dimensions d and t, it follows that $n \geq d+t$. We now show that $n \geq 2d$. By way of contradiction, assume that $n < 2d$. Then the dimension condition (see (2) in Section 1) implies that $\mathcal{P}$ contains at most one d-subspace. Thus, $\mathcal{P}$ contains exactly one d-subspace, Y, and its d-supertail is $ST = \mathcal{P}\setminus\{Y\}$. Since $n \geq d+t$, $|ST| = q^t + 1$ and the maximum dimension of any subspace in ST is (by definition) t, it follows that

$$q^t + 1 = |ST| \geq \frac{\left|\sum_{X\in ST} |X^*|\right|}{|V(t,q)^*|} = \frac{|V(n,q)^*| - |Y^*|}{|V(t,q)^*|} = \frac{(q^n-1)-(q^d-1)}{q^t-1} \geq q^d, \tag{15}$$

which is a contradiction since $d > t$ and $q \geq 2$. Thus, a d-supertail of a partition $\mathcal{P}$ of $V(n,q)$ cannot be of minimum size $\sigma_q(d,t) = q^t + 1$ if $n < 2d$. So we may assume that $n \geq 2d$, i.e., $k \geq 2$ (which we will use in the proof of part (iii) below). We now prove the theorem for each of the three conditions (i), (ii), and (iii) stated in the theorem.

(i) Suppose the supertail ST contains subspaces of at most two different dimensions $d_1 = a$ and $d_{s-1} = t$ such that $t > a$. Since n_i denotes the number of i-subspaces, we have

$$n_t + n_a = |ST| = q^t + 1, \tag{16}$$

where $n_t > 0$ and $n_a \geq 0$. Moreover, since $d = t + r_t$, Lemma 16 yields

$$n_t\Theta_t + n_a\Theta_a = \frac{c_0 q^d - 1}{q-1}, \tag{17}$$

where c_0 is a positive integer. Since $\Theta_i = (q^i-1)/(q-1)$, it follows from (16) and (17) that

$$n_t(q^{t-a}-1) = q^t(c_0 q^{r_t - a} - 1) + q^{t-a} - 1 \Rightarrow n_t = \frac{q^t(c_0 q^{r_t-a}-1)}{q^{t-a}-1} + 1.$$

Since $\gcd(q^t, q^{t-a}-1) = 1$, the above equation implies that $q^{t-a}-1$ divides $c_0 q^{r_t-a}-1$. Hence $n_t = q^t \cdot x + 1$, where $x = \dfrac{c_0 q^{r_t-a}-1}{q^{t-a}-1}$ is either 0 or 1 since $n_t \leq q^t + 1$. If $x = 0$, then $n_t = 1$ and $n_a = q^t$. In this case, it follows from Lemma 17 that the union of the subspaces in ST is a subspace of dimension $t + a = d_{s-1} + d_1$. If $x = 1$, then $n_t = q^t + 1$ and $n_a = 0$. In this case, ST contains only subspaces of dimension $t = d_{s-1}$ and Theorem 4 implies that ST is a d_{s-1}-spread.

(ii) If $t - r_t = 1$, then it follows from Lemma 14 that the smallest dimension in ST is $d_1 \geq r_t = t-1$. Thus ST contains subspaces of at most two different dimensions, namely t and $t-1$. Now the main theorem follows from Theorem 4 and part *(i)* above.

(iii) Recall that $d = d_s$, $t = d_{s-1}$, $r_t = d-t$, and $n \geq 2d$. Moreover, k and r_d are integers such that $k \geq 2$, $n = kd + r_d$, and $1 \leq r_d < d$. Let $\ell = q^{r_d} \sum_{i=0}^{k-2} q^{id}$, and let $\mathcal{P}$ be a partition of $V(n,q)$ with a d-supertail ST of minimum size q^t+1. Then $\mathcal{P} \setminus ST$ must be a partial d-spread. Note that

$$\ell = q^{r_d} \sum_{i=0}^{k-2} q^{id} \Rightarrow \ell q^d = \frac{q^d(q^{n-d} - q^{r_d})}{q^d - 1}. \tag{18}$$

Let $\mu_q(n,d)$ denote the maximum number of d-subspaces in any partial d-spread of $V(n,q)$. Then the upper bound given by Drake-Freeman [7, Corollary 8] implies that

$$\mu_q(n,d) < \ell q^d + \frac{q^{r_d} + q^{r_d - 1}}{2} + 1, \tag{19}$$

Since $n = kd + r_d$ and the maximum dimension in $\mathcal{P}$ is d, it follows from Theorem 1 that

$$|\mathcal{P}| \geq \ell q^d + q^{\lceil \frac{d+r_d}{2} \rceil} + 1. \tag{20}$$

By definition of $\mathcal{P}$ and ST, it follows that

$$|\mathcal{P}| = |\mathcal{P} \setminus ST| + |ST| = n_d + q^t + 1. \tag{21}$$

Hence, (19), (20), and (21) yield

$$q^{\lceil \frac{d+r_d}{2} \rceil} - q^t \leq n_d - \ell q^d < \frac{q^{r_d} + q^{r_d-1}}{2} + 1. \tag{22}$$

Since $0 \leq r_d < d$, $d = t + r_t$, and $1 \leq r_t < t$, it follows that

$$-q^d < q^{\lceil \frac{d+r_d}{2} \rceil} - q^t \text{ and } \frac{q^{r_d} + q^{r_d-1}}{2} < q^{r_d} < q^d. \tag{23}$$

Thus, (22) and (23) yield

$$-q^d < n_d - \ell q^d < q^d. \tag{24}$$

Next, it follows from Lemma 16 that there exists some integer c_0 such that the number of vectors in $\bigcup_{X \in ST} X$ is equal to

$$\sum_{i=a}^{t} n_i(q^i - 1) = c_0 q^d - 1, \tag{25}$$

where a is the smallest dimension of a subspace in ST. Since $\mathcal{P} \setminus ST$ is a partial d-spread of $V(n,q)$, it follows from (25) and from counting in two ways the number of nonzero vectors in $V(n,q)$ that

$$n_d(q^d - 1) + c_0 q^d - 1 = q^n - 1 \Rightarrow n_d(q^d-1) = q^d(q^{n-d} - c_0). \tag{26}$$

Hence, (26) implies that q^d divides n_d. Thus q^d divides $n_d - \ell q^d$. Now the second inequality in (24) implies that $n_d - \ell q^d = 0$, i.e.

$$|\mathcal{P} \setminus ST| = n_d = \ell q^d. \tag{27}$$

Since $n_d = \ell q^d$ and $d = t + r_t$, it follows from the first inequality in (22) that

$$q^{\lceil \frac{d+r_d}{2} \rceil} - q^t \leq n_d - \ell q^d = 0 \Rightarrow r_d \leq t - r_t$$

Since $d = t + r_t$, $|ST| = q^t + 1$, $n_d = \ell q^d$, and $r_d \leq t - r_t$, it follows from Lemma 19 (see the Appendix) that $W = \bigcup_{X \in ST} X$ is a subspace of dimension $d + r_d = t + r_t + r_d \leq 2t$. If $r_d = t - r_t$, then $\dim W = 2t$. Then ST is a subspace partition of W into t-subspaces only, since otherwise, counting the nonzero vectors in W in two different ways, i.e.,

$$q^{2t} - 1 = |W^*| = \sum_{X \in ST} |X^*| < |ST| \cdot |V(t,q)^*| = q^{2t} - 1,$$

yields a contradiction. We remark that if $r_d = t - r_t$, then $\frac{d+r_d}{2} = t$; thus the subspace partition $\mathcal{P}$ must be of type $[d^{\ell q^d}, t^{q^{t+1}}]$ and of minimum size (i.e., $|\mathcal{P}| = \sigma_q(n,d)$). Such partitions are known to exist and are discussed in [16] (in particular, see Theorem 2).

Finally, if $r_d < t - r_t$, then $\dim W = t + r_t + r_d < 2t$ and it follows from the dimension condition (see (2) in Section 1) that ST is a subspace partition of W with exactly one t-subspace and with other subspaces of dimension at most $r_t + r_d$. Since $\dim W = t + r_t + r_d < 2t$ and $|ST| = q^t + 1$, counting in two ways the nonzero vectors in W yields

$$\begin{aligned} q^{t+r_t+r_d} - 1 = |W^*| &= \sum_{X \in ST} |X^*| \\ &\leq |V(t,q)^*| + (|ST| - 1) \cdot |V(r_t + r_d, q)^*| = q^{t+r_t+r_d} - 1. \end{aligned} \tag{28}$$

If some subspace in ST has dimension less than $r_t + r_d$, then inequality in (28) becomes strict and we obtain a contradiction. Thus, ST contains one t-subspace and q^t subspaces of dimension $r_t + r_d$. In this case, we also remark that if $r_d = t - r_t - 1$, then the subspace partition $\mathcal{P}$ is of type $[d^{\ell q^d}, t^1, (t-1)^{q^t}]$ and of minimum size (e.g., see Theorem 4 in [16] for the existence of such partitions). However, if $r_d < t - r_t - 1$, then the resulting subspace partitions are not necessarily of minimum size. For instance, if $n = 34$, $k = 3$, $d = 11$, $r_d = 1$, $t = 7$, and $r_t = 4$, we can apply (several times) Lemma 13 to construct a subspace partition of $V(34,q)$ of type $[11^{q^{23}+q^{12}}, 7^1, 5^{q^7}]$ and size $q^{23} + q^{12} + q^7 + 1$, which is larger than $\sigma_q(34,11) = q^{23} + q^{12} + q^6 + 1$. □

We are now ready to prove Corollary 6.

Proof. Let $\mathcal{P}$ be a subspace partition of $V = V(n,q)$ of type $[d_1^{n_1}, \ldots, d_m^{n_m}]$. Let ST be the d_s-supertail of $\mathcal{P}$, let $\widehat{ST}$ be its d_{s+1}-supertail, and assume that ST has size $\sigma_q(d_s, d_{s-1})$. We shall prove the following statements:

(i) If $2 \leq s \leq 3$, $d_s < 2d_{s-1}$, and $d_{s+1} < 2d_s$, then

$$|\widehat{ST}| \geq \sigma_q(d_{s+1}, d_s) + \sigma_q(d_s, d_{s-1}).$$

(ii) If $d_s \geq 2d_{s-1}$, or if $s = 3$ and $d_3 = d_2 + d_1$, then

$$|\widehat{ST}| \geq \sigma_q(d_{s+1}, d_s) + \sigma_q(d_s, d_{s-1}) - 1.$$

By using the hypotheses of Corollary 3 and Theorem 5, we can infer that $W = \bigcup_{X \in ST} X$ is a subspace of V and $\dim W \geq d_s$. Thus, it follows from the definitions of $\mathcal{P}$ and ST that

$$\mathcal{P}' = \{X \in \mathcal{P} : X \notin ST\} \cup \{W\} \tag{29}$$

is a subspace partition of V. Since $\mathcal{P}$ contains a d_s-subspace and $\dim X < d_s$ for any $X \in ST$, it follows from the definition in (29) that $\mathcal{P}'$ also contains a d_s-subspace Y. Let $\widehat{ST'}$ be the d_{s+1}-supertail of $\mathcal{P}'$ and let h be the the largest dimension of a subspace in $\widehat{ST'}$. Since $Y \in \widehat{ST'}$, it follows that $d_s = \dim Y \leq h < d_{s+1}$.

If $W \notin \widehat{ST'}$, then $h = \dim Y = d_s$, and $\widehat{ST}$ is the disjoint union of $\widehat{ST'}$ and ST. Thus,

$$|\widehat{ST}| = |\widehat{ST'}| + |ST| \geq \sigma_q(d_{s+1}, d_s) + \sigma_q(d_s, d_{s-1}). \tag{30}$$

Observe that $W \notin \widehat{ST'}$ if and only if $\dim W \geq d_{s+1}$. In this case, $d_{s+1} \leq \dim W < 2d_s$.

If $W \in \widehat{ST'}$, then $h = \dim W < d_{s+1}$, and $\widehat{ST}$ is the disjoint union of $\widehat{ST'} \setminus \{W\}$ and ST. Thus,

$$|\widehat{ST}| \geq (|\widehat{ST'}| - 1) + |ST| \geq \sigma_q(d_{s+1}, \dim W) + \sigma_q(d_s, d_{s-1}) - 1. \tag{31}$$

If $d_s < 2d_{s-1}$ and $d_{s+1} < 2d_s$, then $\dim W < d_{s+1} < 2\dim W$, and it follows from Theorem 1 that

$$\sigma_q(d_{s+1}, \dim W) = q^{\dim W} + 1 > q^{d_s} + 1 = \sigma_q(d_{s+1}, d_s). \tag{32}$$

Thus, from (31) and the strict inequality of (32), we obtain that

$$|\widehat{ST}| \geq \sigma_q(d_{s+1}, d_s) + \sigma_q(d_s, d_{s-1}). \tag{33}$$

Now part (i) of the corollary follows from (30), (32), and (33).

Finally, $\dim W = d_s$ if $d_s \geq 2d_{s-1}$ (by Corollary 3), or if $d_s = d_1 + d_{s-1}$ (by Theorem 5). Thus part (ii) of the corollary follows from (31). $\square$

4. Appendix

The following two lemmas (Lemma 18 and Lemma 19) are direct adaptations of [16, Lemma 7 and Proposition 1]. For the sake of completeness, we repeat their proofs here. In the following, let $\mathcal{D}$ denote the family of d-subspaces in a partition $\mathcal{P}$ of $V = V(n, q)$ with minimum size d-supertail ST. Let α_i denote the number of hyperplanes in V that contain exactly i members of $\mathcal{D}$.

Lemma 18. *Let n, k, d, and r_d be integers such that $k \geq 2$, $n = kd + r_d$, $d = t + r_t$, $1 \leq r_d < d$, and $1 \leq r_t < t$. Let $\ell = q^{r_d} \sum_{i=0}^{k-2} q^{id}$, and let $\mathcal{P}$ be a partition of $V = V(n, q)$ whose largest subspace dimension is d and such that $n_d = \ell q^d$. Assume, furthermore that $\mathcal{P}$ has a d-supertail ST of minimum size $|ST| = q^t + 1$ and with largest subspace dimension t. Then, the following conclusions hold.*

(a) *If $\alpha_i \neq 0$, then $\delta = \ell - q^{r_d} \leq i \leq \ell$.*

(b) *The extremal case* $\alpha_\delta \neq 0$ *occurs if there exists a hyperplane* H *of* V *such that all members of* ST *are subspaces of* H.

(c) *The extremal case* $\alpha_\ell \neq 0$ *occurs if there exists a hyperplane* H *of* V *such that the number of non-zero vectors in* $\bigcup_{X\in ST}(X\cap H)$ *equals* $q^{d+r_d-1}-1$.

Proof. For any subspace U of V and any hyperplane H of V, let $B_H(U)$ denote the set of all points (i.e., 1-subspaces) of U that are not in H. Then elementary linear algebra arguments yield

$$|B_H(U)| = \begin{cases} 0 & \text{if } U \subseteq H, \\ q^{\dim U - 1} & \text{otherwise.} \end{cases} \tag{34}$$

If $\alpha_i \neq 0$ then there is at least one hyperplane H containing i members of $\mathcal{D}$. From (34) we then get that

$$(n_d - i)q^{d-1} \leq |B_H(V)| = q^{n-1}$$

and thus, since $n = kd + r_d$ and $\ell = q^{r_d}\sum_{i=0}^{k-2} q^{id}$, we obtain

$$i \geq n_d - q^{(k-1)d+r_d} = \ell q^d - q^{(k-1)d+r_d} = \ell - q^{r_d}. \tag{35}$$

We now show by contradiction that $i \leq \ell$ for all $\alpha_i \neq 0$. Assume that $i \geq \ell+1$ for some H. Since $\mathcal{D}$ denotes the set of members of $\mathcal{P}$ that have dimension d, we have $|\mathcal{P}\setminus\mathcal{D}| = q^t + 1$. Then it follows from Lemma 11 and the fact that $d > t$ that

$$|\mathcal{P}| \geq i\cdot q^d + 1 \geq (\ell+1)q^d + 1 > \ell q^d + q^t + 1 = |\mathcal{D}| + |\mathcal{P}\setminus\mathcal{D}| = |\mathcal{P}|, \tag{36}$$

which is a contradiction. Now conclusion (a) of the theorem follows from (35) and (36).

Next, we can infer from the analysis leading to (35) that the case $\alpha_\delta \neq 0$ with $\delta = \ell - q^{r_d}$ occurs if all members of ST are contained in some hyperplane H. This proves conclusion (b). Finally, if $\alpha_\ell \neq 0$ occurs, then by definition of α_i, there exists a hyperplane H of V that contains exactly ℓ subspaces of $\mathcal{D}$. Let $\mathcal{D}' \subseteq \mathcal{D}$ be the set containing those ℓ subspaces. Since $\mathcal{P}$ is a subspace partition of V, counting the nonzero vectors of H in two ways yields

$$|H^*| = \left|\bigcup_{X\in\mathcal{D}'}(X\cap H)^*\right| + \left|\bigcup_{X\in\mathcal{D}\setminus\mathcal{D}'}(X\cap H)^*\right| + \left|\bigcup_{X\in ST}(X\cap H)^*\right|, \tag{37}$$

where $(X\cap H)^*$ is the set of nonzero vectors in $X\cap H$. Since $\mathcal{D}$ contains ℓq^d subspaces of dimension d, $\dim H = kd + r_d - 1$, and $\dim(X\cap H) = d-1$ for all $X \in (\mathcal{D}\setminus\mathcal{D}')$, it follows from (37) that

$$\left|\bigcup_{X\in ST}(X\cap H)^*\right| = (q^{kd+r_d-1}-1) - \ell(q^d-1) - (\ell q^d - \ell)(q^{d-1}-1) = q^{d+r_d-1}-1,$$

where we used the fact that $\ell = q^{r_d}\sum_{i=0}^{k-2} q^{id} = q^{r_d}\frac{q^{(k-1)d}-1}{q^d-1}$. This proves conclusion (c). □

Lemma 19. *Let* n, k, d, *and* r_d *be integers such that* $k \geq 2$, $n = kd + r_d$, $d = t + r_t$, $1 \leq r_d < d$, *and* $1 \leq r_t < t$. *Let* $\ell = q^{r_d}\sum_{i=0}^{k-2} q^{id}$, *and let* $\mathcal{P}$ *be a partition of* $V(n,q)$ *whose largest subspace dimension is* d *and such that* $n_d = \ell q^d$. *Assume, furthermore* *that $\mathcal{P}$ has a d-supertail ST of minimum size $|ST| = q^t + 1$ and with largest subspace dimension t. Then the union of the subspaces in ST is itself a $d + r_d$-subspace.*

Proof. We again let $\mathcal{D}$ denote the family of d-subspaces in $\mathcal{P}$, and we let α_i denote the number of hyperplanes in $V = V(n, q)$ that contain exactly i members of $\mathcal{D}$. It is trivial that $\alpha_i \geq 0$ for all i; a fact that we will use later for all i. From Lemma 18, we know that

$$\alpha_i \neq 0 \qquad \Longrightarrow \qquad \delta = \ell - q^{r_d} \leq i \leq \ell.$$

We first define the integers x, y and z by

$$x = \sum_{i=\delta}^{\ell} i\alpha_i, \qquad y = \sum_{i=\delta}^{\ell} \binom{i}{2}\alpha_i, \text{ and } z = \sum_{i=\delta}^{\ell} \alpha_i.$$

Each member of $\mathcal{D}$ is a d-subspace and is thus contained in exactly $(q^{(k-1)d+r_d} - 1)/(q-1)$ hyperplanes. By double counting incidences (H, U), for $H \in \mathcal{H}$ with $U \in \mathcal{D}$ and $U \subseteq H$, we obtain

$$x = \sum_{i=\delta}^{\ell} i\alpha_i = n_d \cdot \Theta_{(k-1)d+r_d}. \tag{38}$$

Any two members of $\mathcal{D}$ are contained in $(q^{(k-2)d+r_d} - 1)/(q - 1)$ hyperplanes. Thus, by double counting incidences, we get

$$y = \sum_{i=\delta}^{\ell} \binom{i}{2}\alpha_i = \binom{n_d}{2}\Theta_{(k-2)d+r_d}. \tag{39}$$

Furthermore, by counting the number of hyperplanes in V, we obtain

$$z = \sum_{i=\delta}^{\ell} \alpha_i = \Theta_{kd+r_d}. \tag{40}$$

Observe that (38), (39), and (40) imply that the constants x, y and z are independent of the particular choice of subspace of $\mathcal{P}$ with $n_d = \ell q^d$ and minimum size d-supertail. Moreover,

$$\sum_{i=\delta}^{\ell} \alpha_i(i - \delta)(i - \ell) = 2y + x - (\delta + \ell)x + \delta\ell z. \tag{41}$$

Also note the following facts that we shall use later:

$$(i - \delta)(i - \ell) \begin{cases} = 0 & \text{if } i = \delta, \\ < 0 & \text{if } \delta < i < \ell, \\ = 0 & \text{if } i = \ell. \end{cases} \tag{42}$$

Since $n = kd + r_d$, $d = t + r_t$, and $1 \leq r_d \leq t - r_t$, we can use Lemma 13 to construct a partition $\mathcal{P}_0$ of $V(n, q)$ with ℓq^d subspaces of dimension d, one t-subspace, and q^t subspaces of dimension $r_d + r_t$. (Note that if $r_d = t - r_t$, $\mathcal{P}_0$ has type $[d^{\ell q^d}, t^{q^t+1}]$.)

In order to show that the right side of (41) is equal to zero, we consider the partition $\mathcal{P}_0$. From the construction of the partition $\mathcal{P}_0$, it follows that the points in the d-supertail constitute a $d+r_d$-subspace W. Any hyperplane $H \in \mathcal{H}$ either contains W or intersects W in $(q^{\dim W-1} - 1)$ non-zero vectors. These are the two extremal cases discussed in parts (b) and (c) of Lemma 18. So for the partition $\mathcal{P}_0$, we have $\alpha_i = 0$ for $\delta < i < \ell$. Then, it follows from (42) that the left side of (41) is equal to zero. Thus, we obtain from (41) that for any partition $\mathcal{P}$,

$$\sum_{i=\delta+1}^{\ell-1} \alpha_i (i-\delta)(i-\ell) = 0.$$

As $\alpha_i \geq 0$, we may thus conclude from the equation above and (42) that

$$\delta < i < \ell \qquad \Longrightarrow \qquad \alpha_i = 0.$$

Hence, we can now use (38) and (40) (or refer to the partition $\mathcal{P}_0$, which must have the same solution α_δ and α_ℓ to these two equations) to calculate α_δ (and α_ℓ). We then get that

$$\alpha_\delta = \Theta_{(k-1)d}. \tag{43}$$

Let $\gamma = q^{(k-1)d+r_d}$. Since

$$\ell = q^{r_d} \sum_{i=0}^{k-2} q^{id} = q^{r_d} \frac{q^{(k-1)d} - 1}{q^d - 1},$$

it follows that

$$|\mathcal{D}| - \gamma = \ell q^d - q^{(k-1)d+r_d} = \ell - q^{r_d} = \delta. \tag{44}$$

Let $\mathcal{H}_0$ denote the set of all hyperplanes of V that intersect γ members of $\mathcal{D}$. Since a hyperplane of V either contains a given subspace or intersects it, it follows from (44) that $\mathcal{H}_0$ can be also defined as the set of all hyperplanes of V that contain δ members of $\mathcal{D}$. Thus it follows from the definition of α_i that $\alpha_\delta = |\mathcal{H}_0|$. Let

$$W = \bigcap_{H \in \mathcal{H}_0} H \quad \text{and} \quad R = \bigcup_{X \in ST} X.$$

Since $\mathcal{P}$ is a subspace partition of V and $\mathcal{D}$ contains ℓq^d subspaces of dimension d, the number of non zero vectors in R is

$$|R^*| = |V^*| - \sum_{U \in \mathcal{D}} U^* = q^n - 1 - \ell q^d (q^d - 1) = q^{d+r_d} - 1. \tag{45}$$

Moreover, it follows from Lemma 18 (b) that

$$R \subseteq \bigcap_{H \in \mathcal{H}_0} H = W, \tag{46}$$

and it follows from (43) that

$$\dim W \leq n - (k-1)d = d + r_d. \tag{47}$$

Thus, it follows from (45)–(47) that $R = W$ is a $d + r_d$-subspace. $\square$

Acknowledgement: We thank the referees for their valuable comments which helped improve the presentation and simplify some proofs (in particular, Lemma 17).

References

[1] J. André, Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe, *Math Z.* 60 (1954), 156–186.

[2] R. Baer, Partitionen endlicher Gruppen, *Math. Z.* 75 (1960/61), 333–372.

[3] A. Beutelspacher, Partial spreads in finite projective spaces and partial designs, *Math. Zeit.* 145 (1975), 211–229.

[4] A. Beutelspacher, Partitions of finite vector spaces: an application of the Frobenius number in geometry, *Arch. Math.* 31 (1978), 202–208.

[5] A. Blinco, S. El-Zanati, G. Seelinger, P. Sissokho, L. Spence, C. Vanden Eynden, On vector space partitions and uniformly resolvable designs, *Des. Codes Cryptogr.* 48 (2008), 66–77.

[6] T. Bu, Partitions of a vector space, *Disc. Math.* 31 (1980), 79–83.

[7] D. Drake and J. Freeman, Partial t-spreads and group constructible (s, r, μ)-nets, *J Geom* 13 (1979), 211–216.

[8] S. El-Zanati, G. Seelinger, P. Sissokho, L. Spence, and C. Vanden Eynden, Partitions of finite vector spaces into subspaces, *J. Combin. Des.* 16 (2008), 329–341.

[9] O. Heden, Partitions of finite abelian groups, *Europ. J. Combin.* 7 (1986), 11–25.

[10] O. Heden, A survey on prefect codes, *Adv. Math. Commun.* 2 (2008), 223–247.

[11] O. Heden, On the length of the tail of a vector space partition, *Discrete Math.* 309 (2009), 6169–6180.

[12] O. Heden, A survey of the different types of vector space partitions, *Disc. Math. Algo. Appl.*, 4 (2012), 1–14.

[13] O. Heden and J. Lehmann, Some necessary conditions for vector space partitions, *Discrete Math.* 312(2012), 351–361.

[14] O. Heden, J. Lehmann, E. Năstase, and P. Sissokho, Extremal sizes of subspace partitions, *Des. Codes and Cryptogr.* 64(3) (2012), 265–274.

[15] O. Heden, J. Lehmann, E. Năstase, and P. Sissokho, The supertail of a subspace partition, *Des. Codes Cryptogr.* 69 (2013), 305–316

[16] O. Heden, J. Lehmann, E. Năstase, and P. Sissokho, On the type(s) of minimum size subspace partitions, *Disc. Math.* 332 (2014), 1–9.

[17] M. Herzog and J. Schönheim, Group partition, factorization and the vector covering problem, *Canad. Math. Bull.* 15(2) (1972), 207–214.

[18] S. Hong and A. Patel, A general class of maximal codes for computer applications *IEEE Trans. Comput.* C-21 (1972), 1322–1331.

[19] B. Lindström, Group partitions and mixed perfect codes, *Canad. Math. Bull.* 18 (1975), 57–60.

[20] E. Năstase and P. Sissokho, The minimum size of a finite subspace partition, *Lin. Alg. and its Appl.*, 435 (2011), 1213–1221.

[21] G. Zappa, Partitions and other coverings of finite groups, *Illinois J. Math.*, 47 (2003), 571–580.