THE MINIMUM SIZE OF A FINITE SUBSPACE PARTITION

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Abstract. A *subspace partition* of $\mathbb{P} = \mathrm{PG}(n, q)$ is a collection of subspaces of $\mathbb{P}$ whose pairwise intersection is empty. Let $\sigma_q(n, t)$ denote the *minimum size* (i.e., minimum number of subspaces) in a subspace partition of $\mathbb{P}$ in which the largest subspace has dimension t. In this paper, we determine the value of $\sigma_q(n, t)$ for $n \leq 2t + 2$. Moreover, we use the value of $\sigma_q(2t + 2, t)$ to find the minimum size of a maximal partial t-spread in $\mathrm{PG}(3t + 2, q)$.

1. Introduction

A *subspace partition* (or partition) of $\mathbb{P} = \mathrm{PG}(n, q)$ is a collection of subspaces of $\mathbb{P}$ whose pairwise intersection is empty. Alternatively, we can think of $\mathbb{P}$ as the vector space of dimension $n+1$ over $\mathrm{GF}(q)$, denoted by $V = V(n+1, q)$. Then, a subspace partition of $\mathbb{P}$ is equivalent to a partition of V into a collection $\mathcal{S}$ of subspaces in such a way that each nonzero vector of V occurs in exactly one subspace in $\mathcal{S}$. The collection $\mathcal{S}$ is said to be a *vector space partition* (or simply a *partition*) of V. There is a rich literature about partitions of V (e.g., see [1, 3, 5, 14, 22] and the references therein).

Let $\sigma_q(n, t)$ denote the *minimum size* (i.e., minimum number of subspaces) in a subspace partition of $\mathbb{P}$ in which the largest subspace has dimension t. Since $\sigma_q(n, n) = 1$ and $\sigma_q(n, 0) = (q^{n+1} - 1)/(q - 1)$, we will focus on the case $0 < t < n$. Also note that if $t + 1$ divides $n + 1$, then $\sigma_q(n, t)$ is just the size of a *t-spread* of $\mathbb{P}$, i.e., a subspace partition of $\mathbb{P}$ in which all the subspaces have dimension t. For $0 < t < n$, A. Beutelspacher [2] established the following general lower bound:

$$\sigma_q(n, t) \geq q^{\lceil \frac{n+1}{2} \rceil} + 1.$$

In a recent manuscript, O. Heden and J. Lehmann [16] established new necessary conditions for the existence of certain subspace partitions. In particular they proved conditions for $\mathrm{PG}(2t-1, q)$ to admit partitions with subspaces of dimensions t and $d < t$ (see Theorem 11 in [16]). In the process, they also prove that for any partition Π of $\mathrm{PG}(n, q)$ such that t is the highest dimension that occurs in Π and $d < n - t$ is another dimension that occurs in Π,

$$|\Pi| \geq q^{t+1} + q^{d+1} + 1.$$

Their result is an improvement on a result of G. Spera [22] who proved that if Π is a partition of $\mathrm{PG}(n, q)$ such that s is the smallest dimension that occurs in Π, then $|\Pi| \geq q^{s+1} + 1$. In another related paper, A. Khare [20] established a sharp bound for the minimum number of subspaces needed to *cover* (not necessary partition) a given vector space V (finite or infinite) into subspaces with fixed co-dimension

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$k < \infty$. As observed above, if $V \cong \mathrm{PG}(n,q)$ and $k+1$ divides $n+1$, then a k-spread provides a minimum covering of V.

Let n and t be fixed integers such that $0 < t < n$. In this paper, we prove that (see Corollary 7)

$$\sigma_q(n,t) = q^{t+1} + 1 \text{ for } n < 2t+2,$$

and

$$\sigma_q(n,t) = q^{t+2} + q^{\lceil t/2 \rceil + 1} + 1 \text{ for } n = 2t+2.$$

We combine this result with a construction of P. Govaerts [13] to show (see Theorem 11) that the minimum size of a maximal partial t-spread in $\mathrm{PG}(3t+2,q)$ is given by $\sigma_q(2t+2,t)$.

2. Main results

In our proofs, we use several results of Heden and Lehmann [16]. We start with some preliminary definitions introduced in [16].

Let $n \geq 2$ be an integer and let Π be a subspace partition of $\mathbb{P} = \mathrm{PG}(n,q)$ with m_i subspaces of dimension i, $0 \leq i \leq n-1$. Let H be any $(n-1)$-subspace of $\mathbb{P}$ and let $b_i \leq m_i$ be the number of subspaces of Π that are contained in H. We say that $(m_{n-1}, \dots, m_0)$ is the *type* of Π and $b = (b_{n-1}, \dots, b_0)$ is the *type of the hyperplane* H (with respect to Π). Let s_b denote the number of hyperplanes in $\mathbb{P}$ with type b and define the set

$$B = \{b : \; s_b > 0\}.$$

For $0 \leq i \leq n$, let

$$\theta_i = \frac{q^{i+1}-1}{q-1}$$

denote the number of points in an i-space of Π, and let

$$h_q(n,i) = \max\left\{0, \frac{q^{n-i}-1}{q-1}\right\}$$

be the number of $(n-1)$-spaces (or hyperplanes) in $\mathbb{P}$ that contain a fixed i-space of $\mathbb{P}$. Finally, for $n = 2t+2$, we define

$$\mu_q(n,t) = q^{t+2} + 1. \tag{1}$$

By using a construction of A. Beutelspacher [1] (which was rediscovered by T. Bu [5]), it is easy to see that there is a partial t-spread in $\mathrm{PG}(n,q)$ of size $\mu_q(n,t)$.

Lemma 1 (Heden and Lehmann [16]). *Let Π be a subspace partition of $\mathrm{PG}(n,q)$ and let $(b_{n-1}, \dots, b_0)$ be the type of the hyperplane H with respect to Π. Then the number of subspaces in Π is*

$$|\Pi| = 1 + \sum_{i=0}^{n-1} b_i q^{i+1}.$$

Lemma 2 (Heden and Lehmann [16]). *Let Π be a subspace partition of $\mathrm{PG}(n,q)$ of type $(m_{n-1}, \dots, m_0)$ and let $b = (b_{n-1}, \dots, b_0)$ be the* type of the hyperplane H *with respect to Π. Let s_b denote the number of hyperplanes in $\mathrm{PG}(n,q)$ with type b and suppose that $0 \leq d, \ell \leq n-2$. Then*

(i) $\sum_{b \in B} s_b = \frac{q^{n+1}-1}{q-1}$,

(ii) $\sum_{b \in B} b_d s_b = m_d h_q(n,d)$,

(iii) $\sum_{b \in B} \binom{b_d}{2} s_b = \binom{m_d}{2} h_q(n, 2d+1)$,

(iv) $\sum_{b \in B} b_d b_\ell s_b = m_\ell m_d h_q(n, d+\ell+1)$.

We will also use the next lemma of Beutelspacher [1] (also see Bu [5]).

Lemma 3 (Beutelspacher [1])**.** *Let n, d be integers such that $0 \leq d \leq (n-1)/2$. Then $\mathrm{PG}(n,q)$ admits a partition with one subspace of dimension $n-d-1$ and q^{n-d} subspaces of dimension d.*

We can now prove the following easy observation for the value of $\sigma_q(n,t)$ when $n < 2t+2$.

Proposition 4. *Let n and t be fixed integers such that $0 < t < n \leq 2t+1$. Then*

$$\sigma_q(n,t) = q^{t+1} + 1.$$

Proof. Since $0 < t < n \leq 2t+1$, we have $n = t+a+1$ with $0 \leq a \leq t$. Let Π be an arbitrary subspace partition of $\mathbb{P} = \mathrm{PG}(n,q)$ whose largest subspace U has dimension t. Since $n > t$, we have $|\Pi| > 1$. So let $U' \in (\Pi \setminus \{U\})$ be a subspace of largest possible dimension. Then $\dim(U') \leq a$ since $n = t+a+1$. Since Π is arbitrarily chosen, counting the number of subspaces in Π yields

$$\sigma_q(n,t) \geq |\Pi| \geq 1 + \frac{\theta_n - \theta_t}{\theta_a} = 1 + q^{t+1}. \tag{2}$$

Now the proposition follows from (2) and the existence (see Lemma 3) of a partition Π_0 of $\mathrm{PG}(n,q)$ with one subspace of dimension t and q^{t+1} subspaces of dimension a. □

To prove our main result, Theorem 6, we first prove the following lemma which may be of independent interest.

Lemma 5. *Let n and $t \geq 1$ be fixed integers such that $n = 2t+2$. Let Π be a subspace partition of $\mathrm{PG}(n,q)$ with no subspace of dimension higher than t. Assume furthermore that Π contains two subspaces of dimensions t and d with $0 \leq d < t$. Then*

$$|\Pi| \geq q^{t+2} + q^{d+1} + 1.$$

Proof. Let Π be a subspace partition of $\mathrm{PG}(n,q)$ containing subspaces of dimension t and d with $0 \leq d < t$. Define

$$L = \frac{\theta_n - \theta_{t-1}\left(\mu_q(n,t) + q^{d+1}\right)}{\theta_t - \theta_{t-1}}. \tag{3}$$

We first show that the lemma holds if $m_t \leq L$. Note that Π is the disjoint union of $A = \{W \in \Pi : \dim(W) = t\}$ and $B = \{W \in \Pi : \dim(W) \leq t-1\}$. Since $|A| = m_t$, we have

$$\begin{aligned} |\Pi| = |A| + |B| &\geq m_t + \frac{\theta_n - m_t \cdot \theta_t}{\theta_{t-1}} \\ &= \frac{\theta_n - m_t(\theta_t - \theta_{t-1})}{\theta_{t-1}} \\ &\geq \frac{\theta_n - L(\theta_t - \theta_{t-1})}{\theta_{t-1}} \end{aligned}$$

$$= \frac{\theta_n - \left[\theta_n - \theta_{t-1}\left(\mu_q(n,t) + q^{d+1}\right)\right]}{\theta_{t-1}}$$
$$= \mu_q(n,t) + q^{d+1}. \tag{4}$$

This shows that the lemma holds for $m_t \le L$.

Now suppose that $m_t > L$. Since there exists a subspace of dimensions t and d in Π, we have $m_t > 0$ and $m_d > 0$. It follows from Lemma 2(iv) that

$$\sum_{b\in B} b_t b_d s_b = m_t m_d h_q(n, t+d+1) \neq 0. \tag{5}$$

Moreover,

$$\sum_{b\in B} b_t b_d s_b = \sum_{\substack{b\in B\\ 0\le b_t\le q-1}} b_t b_d s_b + \sum_{\substack{b\in B\\ b_t\ge q}} b_t b_d s_b. \tag{6}$$

If $\sum_{b\in B,\ b_t\ge q} b_t b_d s_b \neq 0$, then there exists $b \in B$ such that $b_t \ge q$, $b_d \ge 1$, and $s_b \ge 1$. In this case, Lemma 1 yields

$$|\Pi| = \sum_{i=0}^{n-1} b_i q^{i+1} + 1 \ge b_t q^{t+1} + b_d q^{d+1} + 1 \quad \ge \quad q^{t+2} + q^{d+1} + 1, \tag{7}$$

and the lemma follows. So we may assume that $\sum_{b\in B,\ b_t\ge q} b_t b_d s_b = 0$. We will show that this contradicts the assumption $m_t > L$. From (6) and Lemma 2(ii), we obtain

$$\begin{aligned}
(q-1)m_d h_q(n,d) &= \sum_{b\in B}(q-1)\cdot b_d s_b\\
&= \sum_{\substack{b\in B\\ 0\le b_t\le q-1}} (q-1)\cdot b_d s_b + \sum_{\substack{b\in B\\ b_t\ge q}} (q-1)\cdot b_d s_b\\
&\ge \sum_{\substack{b\in B\\ 0\le b_t\le q-1}} b_t\cdot b_d s_b + \sum_{\substack{b\in B\\ b_t\ge q}} b_t\cdot b_d s_b\\
&= \sum_{b\in B} b_t b_d s_b\\
&= m_t m_d h_q(n, t+d+1)
\end{aligned} \tag{8}$$

Since $m_d > 0$, dividing both sides of (8) by m_d yields

$$m_t \le \frac{(q-1)\ h_q(n,d)}{h_q(n,t+d+1)} = \frac{(q-1)(q^{2t+2-d}-1)}{q^{t+1-d}-1}. \tag{9}$$

Since $0 \le d \le t-1$, the right side (9) is maximized when $d = t-1$. Hence

$$m_t \le \frac{(q-1)(q^{2t+2-(t-1)}-1)}{q^{t+1-(t-1)}-1} = \frac{(q-1)(q^{t+3}-1)}{q^2-1} = \frac{q^{t+3}-1}{q+1}.$$

Also, since $\mu_q(n,t) = q^{t+2}+1$ (see (1)) and L (defined in (3)) is minimized when $d = t-1$, the assumption $m_t > L$ yields

$$\begin{aligned}
m_t > L \quad &\ge \quad \frac{\theta_{2t+2} - \theta_{t-1}\cdot(\mu_q(n,t) + q^t)}{\theta_t - \theta_{t-1}}\\
&= \quad \frac{(q^{2t+3}-1) - (q^t-1)(q^{t+2}+q^t+1)}{(q^{t+1}-q^t)}
\end{aligned}$$

$$\geq \quad \frac{q^{t+3}-1}{q+1} \geq m_t,$$

which is a contradiction. Hence $\sum_{b\in B,\ b_t\geq q} b_t b_d s_b \neq 0$ and (7) holds. This concludes the proof of the lemma. □

Theorem 6. *Let n and $t \geq 1$ be fixed integers such that $n = 2t+2$. Then*

$$\sigma_q(n,t) = q^{t+2} + q^{\lceil t/2\rceil+1} + 1.$$

Proof. Let Π be a subspace partition of $\mathrm{PG}(n,q)$ in which the largest subspace has dimension t. Let $\beta = \lceil t/2\rceil$ and define the set

$$G = \{\dim(W) : \ W \in \Pi \text{ and } \beta \leq \dim(W) \leq t-1\}.$$

First, suppose that $G \neq \emptyset$. Then for any $d \in G$, Lemma 5 yields

$$|\Pi| \geq q^{t+2} + q^{d+1} + 1 \geq q^{t+2} + q^{\beta+1} + 1. \tag{10}$$

So, we may assume that $G = \emptyset$. Hence, all other subspaces in Π have dimensions at most $\beta - 1$. Recall from (1) that $\mu_q(n,t) = q^{t+2} + 1$. We consider the following two cases based on whether $m_t = \mu_q(n,t)$ or not.

Case 1: $m_t = \mu_q(n,t)$.

If $b_t \geq q+1$ for some $b \in B$, then

$$|\Pi| = \sum_{i=0}^{n-1} b_i q^{i+1} + 1 \geq b_t \cdot q^{t+1} + 1 \geq q^{t+2} + q^{t+1} + 1 \geq q^{t+2} + q^{\beta+1} + 1. \tag{11}$$

If $b_t \leq q$ for all $b \in B$, then

$$q \sum_{b\in B} s_b = \sum_{b\in B} q \cdot s_b \geq \sum_{b\in B} b_t s_b = m_t h_q(n,t). \tag{12}$$

Using Lemma 2(i) and (ii), we infer that (12) holds if and only if

$$\begin{aligned} & q\left(\frac{q^{n+1}-1}{q-1}\right) = q \sum_{b\in B} s_b \geq m_t h_q(n,t) = (q^{t+2}+1)\cdot \frac{q^{n-t}-1}{q-1} \\ \Leftrightarrow\ & q^{n+2} - q \geq q^{n+2} - q^{t+2} + q^{n-t} - 1 \\ \Leftrightarrow\ & q^{n-t} + q = q^{t+2} + q \leq q^{t+2} + 1, \end{aligned}$$

which is a contradiction since $q > 1$.

Case 2: $m_t \leq \mu_q(n,t) - 1$. In this case, each subspace in Π, other than the m_t subspaces, has dimension at most $\beta - 1$ (so at most $\theta_{\beta-1}$ points). Therefore, we can estimate the number of subspaces in Π as follows

$$\begin{aligned} |\Pi| \quad & \geq \quad m_t + \frac{\theta_n - m_t\cdot\theta_t}{\theta_{\beta-1}} \\ & = \quad \frac{\theta_n - m_t(\theta_t - \theta_{\beta-1})}{\theta_{\beta-1}} \\ & \geq \quad \frac{\theta_n - (\mu_q(n,t)-1)\cdot(\theta_t - \theta_{\beta-1})}{\theta_{\beta-1}} \\ & = \quad \frac{(q^{2t+3}-1) - q^{t+2}(q^{t+1}-q^{\beta})}{q^{\beta}-1} \\ & \geq \quad q^{t+2} + q^{\beta+1} + q. \end{aligned} \tag{13}$$

Now it follows from (10), (11), and (13) that

$$|\Pi| \geq q^{t+2} + q^{\beta+1} + 1$$

holds in all cases. Since Π is an arbitrarily chosen subspace partition, we obtain

$$\sigma_q(n,t) \geq q^{t+2} + q^{\beta+1} + 1. \tag{14}$$

Moreover, it follows from Lemma 3 that there exists a partition Π_0 of $\mathrm{PG}(2t{+}2, q)$ into one subspace W of dimension $t+1$ and q^{t+2} subspaces of dimension t. If t is even, then $t+2 = 2(\beta+1)$ and we can partition W into a β-spread containing $q^{\beta+1}+1$ subspaces. If t is odd then $t+2 = 2\beta+1$ and we use Lemma 3 again to partition W into one subspace of dimension β and $q^{\beta+1}$ subspaces of dimension $\beta-1$. This shows that

$$q^{t+2} + q^{\beta+1} + 1 = |\Pi_0| \geq \sigma_q(n,t). \tag{15}$$

Finally (14) and (15) yield

$$\sigma_q(n,t) = q^{t+2} + q^{\beta+1} + 1.$$

□

Proposition 4 and Theorem 6 lead directly to the following corollary.

Corollary 7. *Let n and t be fixed integers such that $0 < t < n$. Then*

$$\sigma_q(n,t) = q^{t+1} + 1 \quad \textit{for } n < 2t+2,$$

and

$$\sigma_q(n,t) = q^{t+2} + q^{\lceil t/2 \rceil + 1} + 1 \quad \textit{for } n = 2t+2.$$

Proof. This follows directly from Proposition 4 and Theorem 6. □

We conclude this section by proposing the following conjecture.

Conjecture 8. *Let n, k, and t be positive integers such that $n = k(t+1)$. If $k \geq 2$ then*

$$\sigma_q(n,t) = \frac{q^{(t+1)+1}(q^{(k-1)(t+1)} - 1)}{q^{t+1} - 1} + q^{\lceil t/2 \rceil + 1} + 1.$$

Note that Conjecture 8 holds for $k = 2$ (see Theorem 6) and $\sigma_q(n,t) = q^{t+1} + 1$ for $k = 1$ (see Proposition 4).

3. An application to maximal partial t-spreads

Let $\mathbb{P} = \mathrm{PG}(n,q)$ denote the projective space of dimension n over the Galois field $\mathrm{GF}(q)$. A *partial t-spread* of $\mathbb{P}$ is a collection $\mathcal{S} = \{W_1, \ldots, W_k\}$ of t-dimensional subspaces of $\mathbb{P}$ such that $W_i \cap W_j = \emptyset$ for $i \neq j$. The number $|\mathcal{S}|$ is called the *size* of $\mathcal{S}$. If $\mathbb{P} = \bigcup_{W \in \mathcal{S}} W$, then $\mathcal{S}$ is called a *spread*. It is well-known that a spread exists if and only if $t+1$ divides $n+1$.

A *maximal* partial t-spread is one which cannot be extended to a larger one. The problem of classifying the maximal partial t-spreads of $\mathbb{P}$ has been extensively studied (see [9, 11, 13, 15, 18, 19]). It has applications in the construction of error-correcting codes [6, 8], orthogonal arrays [7, 10], and factorial designs [21].

Let n and t be fixed integers and let k and r be the unique integers defined by $n - t = k(t+1) + r - 1$ and $0 \leq r \leq t$. We let $\tau_q(n,t)$ denote the minimum number of subspaces in any maximal partial t-spread of $\mathbb{P}$. The maximal partial t-spread

$\mathcal{S}$ of $\mathbb{P}$ such that $|\mathcal{S}| = \tau_q(n,t)$, is called a *minimum size* maximal partial t-spread. Beutelspacher [1] showed that for $r = 0$ and any positive integers k and t,

$$\tau_q(n,t) = \frac{q^{k(t+1)} - 1}{q^{t+1} - 1}.$$

For $r > 0$, P. Govaerts [13] proved several results related to the number $\tau_q(n,t)$. In particular, he provided an upper bound for $\tau_q(n,t)$ by constructing a class of small (not necessarily minimum) size of maximal partial t-spreads of $\mathbb{P}$. We will use his bound in the case $r = 1$. For $n = k(t+1)$, define

$$\mu_q(n,t) = \frac{q^{(t+1)+1}(q^{(k-1)(t+1)} - 1)}{q^{t+1} - 1} + 1.$$

Lemma 9 (Govaerts [13])**.** *Let n, k, and $t \geq 0$ be fixed integers and write $n = k(t+1) + t$. If $k \geq 2$ then there exist (see page 610 in [13] for a construction) maximal partial t-spreads of $\mathrm{PG}(n,q)$ of size $\mu_q(n-t,t) + q^{\lceil t/2 \rceil + 1}$. Consequently,*

$$\tau_q(n,t) \leq \mu_q(n-t,t) + q^{\lceil t/2 \rceil + 1}.$$

We can apply our main result, Theorem 6, to determine the value of $\tau_q(3t+2,t)$. Our strategy is due to Govaerts but we replace his set-partition based analysis with the more appropriate subspace-partition analysis. We first introduce the relevant definitions. A set of points B of $\mathbb{P}$ is called a *blocking set* with respect to the t-spaces of $\mathbb{P}$ if $W \cap B \neq \emptyset$ for any t-spaces W in $\mathbb{P}$. Note that any $(n-t)$-space of $\mathbb{P}$ is a blocking set with respect to the t-spaces of $\mathbb{P}$. Such blocking sets are called *trivial*. The following lemma follows from the results of Govaerts (see case 2, page 612 in [13]).

Lemma 10 (Govaerts [13])**.** *Let n, k, and t be positive integers such that $n = k(t+1)+t$. If $k \geq 2$ and $\mathcal{S}$ is a minimum size maximal partial t-spread of $\mathrm{PG}(n,q)$, then $\bigcup_{W \in \mathcal{S}} W$ contains a trivial blocking set.*

We can use Lemma 10 with $k = 2$ to prove the following theorem.

Theorem 11. *For any positive integer t, we have*

$$\tau_q(3t+2,t) \geq \sigma_q(2t+2,t).$$

Proof. Let $\mathcal{S}$ be a minimum size maximal partial t-spread in $\mathrm{PG}(3t+2,q)$. Then by Lemma 10, $A = \bigcup_{W \in \mathcal{S}} W$ contains a trivial blocking set. In other words, there exists a $(2t+2)$-space $B \subseteq A$. Let

$$\Pi_{\mathcal{S}} = \{W \cap B : \ W \in \mathcal{S}\}.$$

Since B is a blocking set with respect to t-spaces, we have $W \cap B \neq \emptyset$ for any $W \in \mathcal{S}$. Thus, $\Pi_{\mathcal{S}}$ is a subspace partition of $B \cong \mathrm{PG}(2t+2,q)$ containing subspaces of dimensions at most t. If $\Pi_{\mathcal{S}}$ contains a t-subspace, then it follows from Theorem 6 and the minimality of $\mathcal{S}$ that

$$\tau_q(3t+2,t) = |S| = |\Pi_{\mathcal{S}}| \geq \sigma_q(2t+2,t).$$

If $\Pi_{\mathcal{S}}$ contains no t-subspace, then each subspace in $\Pi_{\mathcal{S}}$ has dimension at most $t-1$ (and contains at most θ_{t-1} points). So we can estimate the number of subspaces in $\Pi_{\mathcal{S}}$ to obtain

$$\tau_q(3t+2,t) = |S| = |\Pi_{\mathcal{S}}| \quad \geq \quad \left\lceil \frac{\theta_{2t+2}}{\theta_{t-1}} \right\rceil$$

$$\begin{aligned} &= \left\lceil \frac{q^{(2t+2)+1}-1}{q^t-1} \right\rceil \\ &> q^{t+2}+q^{\lceil t/2\rceil+1}+1 = \sigma_q(2t+2,t). \end{aligned}$$

This concludes the proof of the theorem. □

We can now prove the following corollary which determines the number $\tau_q(3t+2,t)$ for all $t \geq 1$. The cases $1 \leq t \leq 2$ were already known from the work of Govaerts [13].

Corollary 12. *Let $t \geq 1$ be a fixed integer. Then*

$$\tau_q(3t+2,t) = \sigma_q(2t+2,t) = q^{t+2}+q^{\lceil t/2\rceil+1}+1.$$

Proof. This is a direct consequence of Theorem 6, Lemma 9, and Theorem 11. □

We believe that if Conjecture 8 is true, it can be combined with Lemma 9 to prove that

$$\tau_q(n,t) = \sigma_q(n-t,t) = \frac{q^{(t+1)+1}(q^{(k-1)(t+1)}-1)}{q^{t+1}-1}+q^{\lceil t/2\rceil+1}+1,$$

for any integers $k \geq 2$ and $t \geq 1$ such that $n = k(t+1)+t$.

We remark that the cases for $k = 1$ and $1 \leq r \leq t$, i.e., $2t+1 \leq n \leq 3t$, have proved to be difficult. In particular, for $n = 3$ and $t = 1$, Glynn [12] established the following lower bound

$$\tau_q(3,1) \geq 2q,$$

while Gács and Szönyi [11] later proved the following upper bound

$$\tau_q(3,1) \leq \begin{cases} (2\ln q+1)q+1, & \text{if } q \text{ odd} \\ (6.1\ln q+1)q+1, & \text{if } q > q_0 \text{ even}, \end{cases}$$

Although the gap between these bounds is somewhat considerable, they are (as far as we know) the best bounds for $\tau_q(3,1)$.

Furthermore, there are (e.g., see Hirschfeld [17]) maximal partial 1-spreads of $\mathrm{PG}(3,q)$ of size q^2-q+2 for any $q > 3$, and of size 7 for $q = 3$. For a while, it was generally believed that these maximal partial 1-spreads have largest possible size among all maximal partial 1-spreads which are not 1-spreads. However, for $q = 7$, Heden [15] constructed a maximal partial 1-spread of size 45. All the maximal partial 1-spreads of $\mathrm{PG}(3,q)$ of size 45 have subsequently been classified by Blokhuis, Brouwer, and Wilbrink [4].

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