THE MAXIMUM SIZE OF A PARTIAL SPREAD II: UPPER BOUNDS

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ABSTRACT. Let n and t be positive integers with $t < n$, and let q be a prime power. A *partial* $(t-1)$-*spread* of $\mathrm{PG}(n-1,q)$ is a set of $(t-1)$-dimensional subspaces of $\mathrm{PG}(n-1,q)$ that are pairwise disjoint. Let $r \equiv n \pmod t$ with $0 \le r < t$, and let $\Theta_i = (q^i - 1)/(q-1)$. We essentially prove that if $2 \le r < t \le \Theta_r$, then the maximum size of a partial $(t-1)$-spread of $\mathrm{PG}(n-1,q)$ is bounded from above by $(\Theta_n - \Theta_{t+r})/\Theta_t + q^r - (q-1)(t-3) + 1$. We actually give tighter bounds when certain divisibility conditions are satisfied. These bounds improve on the previously known upper bound for the maximum size partial $(t-1)$-spreads of $\mathrm{PG}(n-1,q)$; for instance, when $\lceil \frac{\Theta_r}{2} \rceil + 4 \le t \le \Theta_r$ and $q > 2$. The exact value of the maximum size partial $(t-1)$-spread has been recently determined for $t > \Theta_r$ by the authors of this paper (see [21]).

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1. Introduction

Let n and t be positive integers with $t < n$, and let q be a prime power. Let $\mathrm{PG}(n-1,q)$ denote the $(n-1)$-dimensional projective space over the finite field $\mathbb{F}_q$. A *partial* $(t-1)$-*spread* S of $\mathrm{PG}(n-1,q)$ is a collection of $(t-1)$-dimensional subspaces of $\mathrm{PG}(n-1,q)$ that are pairwise disjoint. If S contains all the points of $\mathrm{PG}(n-1,q)$, then it is called a $(t-1)$-*spread*. It is well-known that a $(t-1)$-spread of $\mathrm{PG}(n-1,q)$ exists if and only if t divides n (e.g., see [3, p. 29]). Besides their traditional relevance to Galois geometry [6, 11, 13, 17], partial $(t-1)$-spreads are used to build byte-correcting codes (e.g., see [7, 16]), 1-perfect mixed error-correcting codes (e.g., see [15, 16]), orthogonal arrays (e.g., see [4]), and subspace codes (e.g., see [8, 10, 18]).

Convention: For the rest of the paper, we assume that q is a prime power, and n, t, and r are integers such that $n > t > r \ge 0$ and $r \equiv n \pmod t$. We also use $\mu_q(n,t)$ to denote the maximum size of any partial $(t-1)$-spread of $\mathrm{PG}(n-1,q)$.

The problem of determining $\mu_q(n,t)$ is a long standing open problem. Currently, the best general upper bound for $\mu_q(n,t)$ is given by the following theorem of Drake and Freeman [4].

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Theorem 1. *If $r > 0$, then $\mu_q(n,t) \le \frac{q^n - q^{t+r}}{q^t-1} + q^r - \lfloor \omega \rfloor - 1$, where $2\omega = \sqrt{4q^t(q^t - q^r) + 1} - (2q^t - 2q^r + 1)$.*

The following result is attributed to André [1] and Segre [22] for $r = 0$. For $r = 1$, it is due to Hong and Patel [16] when $q = 2$, and Beutelspacher [2] when $q > 2$.

Theorem 2. *If $0 \le r < t$, then $\mu_q(n,t) \ge \frac{q^n - q^{t+r}}{q^t-1} + 1$, and equality holds if $r \in \{0, 1\}$.*

In light of Theorem 2, it was later conjectured (e.g., see [5, 16]) that the value of $\mu_q(n,t)$ is given by the lower bound in Theorem 2. However, this conjecture was disproved by El-Zanati, Jordon, Seelinger, Sissokho, and Spence [9] who proved the following result.

Theorem 3. *If $n \ge 8$ and $n \bmod 3 = 2$, then $\mu_2(n,3) = \frac{2^n - 2^5}{7} + 2$.*

Recently, Kurz [19] proved the following theorem which upholds the lower bound for $\mu_q(n,t)$ when $q = 2$, $r = 2$, and $t > 3$.

Theorem 4. *If $n > t > 3$ and $n \bmod t = 2$, then $\mu_2(n,t) = \frac{2^n - 2^{t+2}}{2^t-1} + 1$.*

For any integer $i \ge 1$, let

$$\Theta_i = (q^i - 1)/(q - 1). \tag{1}$$

Still recently, the authors of this paper affirmed the conjecture (e.g., see [5, 16]) on the value of $\mu_q(n,t)$ for $t > \Theta_r$ and any prime power q, by proving the following general result (see [21]).

Theorem 5. *If $t > \Theta_r$, then $\mu_q(n,t) = \frac{q^n - q^{t+r}}{q^t-1} + 1$.*

In light of Theorem 5, it remains to determine the value of $\mu_q(n,t)$ for $2 \le r < t \le \Theta_r$. In this paper, we apply the *hyperplane averaging method* that we devised in [21] to prove the following results[1]. The rest of the paper is devoted to their proofs.

Theorem 6. *Let $c_1 \equiv (t-2) \pmod q$, $0 \le c_1 < q$, and $c_2 = \begin{cases} q & \text{if } q^2 \mid ((q-1)(t-2) + c_1) \\ 0 & \text{if } q^2 \nmid ((q-1)(t-2) + c_1). \end{cases}$ If $2 \le r < t \le \Theta_r$, then*

$$\mu_q(n,t) \le \frac{q^n - q^{t+r}}{q^t - 1} + q^r - (q-1)(t-2) - c_1 + c_2.$$

Consequently,

$$\mu_q(n,t) \le \frac{q^n - q^{t+r}}{q^t - 1} + q^r - (q-1)(t-3) + 1.$$

Remark 7. *The best possible bound in Theorem 6 is obtained when $t \equiv aq + 1 \pmod{q^2}$, $1 \le a \le q - 1$ (equivalently, when $t \equiv 1 \pmod q$ but $t \not\equiv 1 \pmod{q^2}$). In this case, we can check that $c_1 = q - 1$ and $c_2 = 0$, which implies that*

$$\mu_q(n,t) \le \frac{q^n - q^{t+r}}{q^t - 1} + q^r - (q-1)(t-1).$$

This was already noted in [21, Lemma 10 and Remark 11] *for $r \ge 2$ and $t = \Theta_r = (q^r - 1)/(q - 1)$.*

[1]Also see [20] for a recent preprint in this area.

Corollary 8. *Let $f_q(n,t)$ denote the upper bound for $\mu_q(n,t)$ in Theorem 1 and let $g_q(n,t)$ denote the upper bound for $\mu_q(n,t)$ in Theorem 6. Let c_1 and c_2 be as defined in Theorem 6. If $r \geq 2$ and $2r \leq t \leq \Theta_r$ then*

$$g_q(n,t) - f_q(n,p) = \left\lfloor \frac{q^r}{2} \right\rfloor - (q-1)(t-2) - c_1 + c_2.$$

Consequently, for $\lceil \frac{\Theta_r}{2} \rceil + 4 \leq t \leq \Theta_r$ with $q > 2$, and for $\lceil \frac{\Theta_r}{2} \rceil + 5 \leq t \leq \Theta_r$ with $q = 2$, we have

$$g_q(n,t) - f_q(n,p) < 0,$$

and thus the upper bound for $\mu_q(n,t)$ given in Theorem 6 is tighter than the Drake–Freeman bound in Theorem 1.

In Section 2, we present some auxiliary results from the area of subspace partitions, and in Section 3 we prove Theorem 6 and Corollary 8.

2. Subspace partitions

Let $V = V(n,q)$ denote the vector space of dimension n over $\mathbb{F}_q$. For any subspace U of V, let U^* denote the set of nonzero vectors in U. A *d-subspace* of $V(n,q)$ is a d-dimensional subspace of $V(n,q)$; this is equivalent to a *$(d-1)$-subspace* in $\mathrm{PG}(n-1,q)$.

A *subspace partition* $\mathcal{P}$ of V, also known as a *vector space partition*, is a collection of nontrivial subspaces of V such that each vector of V^* is in exactly one subspace of $\mathcal{P}$ (e.g., see Heden [13] for a survey on subspace partitions). The *size* of a subspace partition $\mathcal{P}$, denoted by $|\mathcal{P}|$, is the number of subspaces in $\mathcal{P}$.

Suppose that there are s distinct integers, $d_s > \cdots > d_1$, that occur as dimensions of subspaces in a subspace partition $\mathcal{P}$, and let n_i denote the number of i-subspaces in $\mathcal{P}$. Then the expression $[d_s^{n_{d_s}}, \ldots, d_1^{n_{d_1}}]$ is called the *type* of $\mathcal{P}$.

Remark 9. *A partial $(t-1)$-spread of $\mathrm{PG}(n-1,q)$ of size n_t is a partial t-spread of $V(n,q)$ of size n_t. This is equivalent to a subspace partition of $V(n,q)$ of type $[t^{n_t}, 1^{n_1}]$, where $n_1 = \Theta_n - n_t\Theta_t$. We will use this subspace partition formulation in the proof of Lemma 14.*

Also, we will use the following theorem due to Heden [12] in the proof of Lemma 14.

Theorem 10. [12, Theorem 1] *Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of type $[d_s^{n_{d_s}}, \ldots, d_1^{n_{d_1}}]$, where $d_s > \ldots > d_1$. Then,*

(i) *if $q^{d_2-d_1}$ does not divide n_{d_1} and if $d_2 < 2d_1$, then $n_{d_1} \geq q^{d_1} + 1$.*
(ii) *if $q^{d_2-d_1}$ does not divide n_{d_1} and $d_2 \geq 2d_1$, then either $n_{d_1} = (q^{d_2}-1)/(q^{d_1}-1)$ or $n_{d_1} > 2q^{d_2-d_1}$.*
(iii) *if $q^{d_2-d_1}$ divides n_{d_1} and $d_2 < 2d_1$, then $n_{d_1} \geq q^{d_2} - q^{d_1} + q^{d_2-d_1}$.*
(iv) *if $q^{d_2-d_1}$ divides n_{d_1} and $d_2 \geq 2d_1$, then $n_{d_1} \geq q^{d_2}$.*

To state the next lemmas, we need the following definitions. Recall that for any integer $i \geq 1$,

$$\Theta_i = (q^i - 1)/(q-1).$$

Then, for $i \geq 1$, Θ_i is the number of 1-subspaces in an i-subspace of $V(n,q)$. Let $\mathcal{P}$ be a subspace partition of $V = V(n,q)$ of type $[d_s^{n_{d_s}}, \ldots, d_1^{n_{d_1}}]$. For any hyperplane H of V, let $b_{H,d}$ be the number of d-subspaces in $\mathcal{P}$ that are contained in H and set $b_H = [b_{H,d_s}, \ldots, b_{H,d_1}]$. Define the set $\mathcal{B}$ of *hyperplane types* as follows:

$$\mathcal{B} = \{b_H : \ H \text{ is a hyperplane of } V\}.$$

For any $b \in \mathcal{B}$, let s_b denote the number of hyperplanes of V of type b.

We will also use Lemma 11 and Lemma 12 by Heden and Lehmann [14] in the proof of Lemma 14.

Lemma 11. [14, Equation (1)] *Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of type $[d_s^{n_{d_s}}, \ldots, d_1^{n_{d_1}}]$. If H is a hyperplane of $V(n,q)$ and $b_{H,d}$ is as defined above, then*

$$|\mathcal{P}| = 1 + \sum_{i=1}^{s} b_{H,d_i} q^{d_i}.$$

Lemma 12. [14, Equation (2) and Corollary 5] *Let $\mathcal{P}$ be a subspace partition of $V(n,q)$, and let $\mathcal{B}$ and s_b be as defined above. Then*

$$\sum_{b \in \mathcal{B}} s_b = \Theta_n,$$

and for $1 \le d \le n-1$, we have

$$\sum_{b \in \mathcal{B}} b_d s_b = n_d \Theta_{n-d}.$$

3. Proofs of the main results

Recall that q is a prime power, and n, t, and r are integers such that $n > t > r \ge 0$, and $r \equiv n \pmod{t}$. To prove our main result, we first need to prove the following two technical lemmas.

Lemma 13. *Let x be an integer such that $0 < x < q^r$. For any positive integer i, let $\delta_i = q^i \cdot \lceil x q^{-i} \Theta_i \rceil - x\Theta_i$. Then the following properties hold:*

(i) $\lceil x q^{-t} \Theta_t \rceil = \lceil \frac{x}{q-1} \rceil$.

(ii) *for $1 \le i \le t$, we have $0 \le \delta_i < q^i$, $q \mid (x + \delta_{i+1})$, and $\delta_i = q^{-1}(x + \delta_{i+1}) \bmod q^i$.*

(iii) *$\delta_i = 0$ if and only if $q^i \mid x$.*

Proof. Let α and β be integers such that $x = \alpha(q-1) + \beta$, $\alpha \ge 0$, and $0 \le \beta < q-1$. Since $0 < x < q^r$ and $r < t$ hold by hypothesis, it follows that

$$0 \le \alpha < x < q^r < q^t \text{ and } \alpha(q-1) \le x < q^r < q^t. \tag{2}$$

If $\beta = 0$, then by (2), we obtain

$$\lceil x q^{-t} \Theta_t \rceil = \left\lceil \frac{\alpha(q^t - 1)}{q^t} \right\rceil = \left\lceil \alpha - \frac{\alpha}{q^t} \right\rceil = \alpha = \left\lceil \frac{x}{q-1} \right\rceil. \tag{3}$$

Now suppose $1 \le \beta < q-1$. First, since $\beta \ge 1$, it follows from (2) that

$$\begin{aligned}\lceil x q^{-t} \Theta_t \rceil = \left\lceil \frac{[\alpha(q-1) + \beta](q^t - 1)}{q^t(q-1)} \right\rceil &\ge \left\lceil \frac{[\alpha(q-1) + 1](q^t - 1)}{q^t(q-1)} \right\rceil \\ &= \left\lceil \alpha + \frac{(q^t - 1) - \alpha(q-1)}{q^t(q-1)} \right\rceil \\ &= \alpha + 1. \end{aligned} \tag{4}$$

Second, since $\beta < q-1$, it follows from (2) and the properties of the ceiling function that

$$\lceil xq^{-t}\Theta_t \rceil = \left\lceil \frac{[\alpha(q-1)+\beta](q^t-1)}{q^t(q-1)} \right\rceil \leq \left\lceil \frac{(\alpha+1)(q^t-1)}{q^t} \right\rceil = \left\lceil \alpha+1-\frac{\alpha+1}{q^t} \right\rceil = \alpha+1. \tag{5}$$

Then (4) and (5) imply that for $1 \leq \beta < q-1$,

$$\lceil xq^{-t}\Theta_t \rceil = \alpha+1 = \left\lceil \frac{x}{q-1} \right\rceil,$$

which completes the proof of (i).

We now prove (ii). Since $0 \leq \lceil a \rceil - a < 1$ holds for any real number a, we have

$$0 \leq \lceil q^{-i}x\Theta_i \rceil - q^{-i}x\Theta_i < 1 \Longrightarrow \delta_i = q^i\lceil xq^{-i}\Theta_i \rceil - x\Theta_i < q^i \text{ and } \delta_i \geq 0.$$

By the definition of δ_i, we have that

$$x+\delta_{i+1} = x + q^{i+1}\cdot\lceil xq^{-i-1}\Theta_{i+1}\rceil - x\Theta_{i+1} = q(q^i\cdot\lceil xq^{-i-1}\Theta_{i+1}\rceil - x\Theta_i),$$

and thus,

$$\begin{aligned} q^{-1}(x+\delta_{i+1}) &\equiv q^i\cdot\lceil xq^{-i-1}\Theta_{i+1}\rceil - x\Theta_i \\ &\equiv -x\Theta_i \\ &\equiv q^i\cdot\lceil xq^{-i}\Theta_i\rceil - x\Theta_i \\ &\equiv \delta_i \pmod{q^i}. \end{aligned} \tag{6}$$

Finally, we prove (iii). Since $\gcd(q^i, \Theta_i) = 1$ for any positive integer i, we have

$$\delta_i = q^i\cdot\lceil xq^{-i}\Theta_i\rceil - x\Theta_i = 0 \Longleftrightarrow \lceil xq^{-i}\Theta_i\rceil = xq^{-i}\Theta_i \Longleftrightarrow q^i|x.$$

□

We now prove our main lemma.

Lemma 14. *Let x be a positive integer such that $q \mid x$ and $q^2 \nmid x$. Let $\ell = (q^{n-t}-q^r)/(q^t-1)$. If $r \geq 2$ and $t \geq \Theta_r - \lceil x/(q-1)\rceil + 2$, then $\mu_q(n,t) \leq \ell q^t + x$.*

Proof. If $x \geq q^r$, then Theorem 1 implies the nonexistence of a partial t-spread of size $\ell q^t + x$. Thus, we can assume that $x < q^r$.

Recall that $\Theta_i = (q^i-1)/(q-1)$ for any integer $i \geq 1$. For an integer i, with $2 \leq i \leq t$, let

$$\delta_i = q^i\cdot\lceil xq^{-i}\Theta_i\rceil - x\Theta_i. \tag{7}$$

Applying Lemma 13(i), we let

$$h := \lceil q^{-t}x\Theta_t\rceil = \left\lceil \frac{x}{q-1} \right\rceil. \tag{8}$$

The proof is by contradiction. So assume that $\mu_q(n,t) > \ell q^t + x$. Then $\mathrm{PG}(n-1,q)$ has a $(t-1)$-partial spread of size $\ell q^t + 1 + x$. Thus, it follows from Remark 9 that there exists a subspace partition $\mathcal{P}_0$ of $V(n,q)$ of type $[t^{n_t}, 1^{n_1}]$, with

$$n_t = \ell q^t + 1 + x, \text{ and}$$

$$n_1 = q^t\Theta_r - x\Theta_t = q^t(\Theta_r - \lceil q^{-t}x\Theta_t\rceil) + (q^t\lceil q^{-t}x\Theta_t\rceil - x\Theta_t) = q^t(\Theta_r - h) + \delta_t, \tag{9}$$

where h is given by (8) and δ_t is given by (7).

We will prove by induction that for each integer j with $0 \le j \le t-2$, there exists a subspace partition $\mathcal{P}_j$ of $H_j \cong V(n-j,q)$ of type

$$[t^{m_{j,t}}, (t-1)^{m_{j,t-1}}, \ldots, (t-j)^{m_{j,t-j}}, 1^{m_{j,1}}], \tag{10}$$

where $m_{j,t}, \ldots, m_{j,t-j}$ are nonnegative integers such that

$$\sum_{i=t-j}^{t} m_{j,i} = n_t = \ell q^t + 1 + x, \tag{11}$$

and where $m_{j,1}$ and c_j are integers such that

$$m_{j,1} = c_j q^{t-j} + \delta_{t-j}, \text{ and } 0 \le c_j \le \max\{\Theta_r - h - j, 0\}. \tag{12}$$

The base case, $j = 0$, holds since $\mathcal{P}_0$ is a subspace partition of $H_0 = V(n,q)$ with type $[t^{n_t}, 1^{n_1}]$, and letting $m_{0,t} = n_t$ and $m_{0,1} = n_1$, $\mathcal{P}_0$ is of type given in (10), and it satisfies the properties given in (11) and (12).

For the inductive step, suppose that for some j, with $0 \le j < t-2$, we have constructed a subspace partition $\mathcal{P}_j$ of $H_j \cong V(n-j,q)$ of the type given in (10), and with the properties given in (11) and (12). We then use Lemma 12 to determine the average, $b_{avg,1}$, of the values $b_{H,1}$ over all hyperplanes H of H_j. We have

$$\begin{aligned} b_{avg,1} := \frac{m_{j,1}\Theta_{n-1-j}}{\Theta_{n-j}} &= \left(c_j q^{t-j} + \delta_{t-j}\right)\left(\frac{q^{n-1-j}-1}{q^{n-j}-1}\right) \\ &< (c_j q^{t-j} + \delta_{t-j})q^{-1} \\ &= c_j q^{t-j-1} + q^{-1}\delta_{t-j}. \end{aligned} \tag{13}$$

It follows from (13) that there exists a hyperplane H_{j+1} of H_j with

$$b_{H_{j+1},1} \le b_{avg,1} < c_j q^{t-j-1} + q^{-1}\delta_{t-j}. \tag{14}$$

Next, we apply Lemma 11 to the subspace partition $\mathcal{P}_j$ and the hyperplane H_{j+1} of H_j to obtain:

$$\begin{aligned} 1 + b_{H_{j+1},1}\, q + \sum_{i=t-j}^{t} b_{H_{j+1},i}\, q^i &= |\mathcal{P}_j| \\ &= n_t + m_{j,1} \\ &= \ell q^t + 1 + x + c_j q^{t-j} + \delta_{t-j}, \end{aligned} \tag{15}$$

where $0 \le c_j \le \max\{\Theta_r - h - j, 0\}$. Simplifying (15) yields

$$b_{H_{j+1},1} + \sum_{i=t-j}^{t} b_{H_{j+1},i}\, q^{i-1} = \ell q^{t-1} + c_j q^{t-j-1} + q^{-1}(x + \delta_{t-j}). \tag{16}$$

Then, it follows from Lemma 13(ii) and (16) that

$$b_{H_{j+1},1} \equiv q^{-1}(x+\delta_{t-j}) \equiv \delta_{t-j-1} \pmod{q^{t-j-1}}. \tag{17}$$

Since $0 \le q^{-1}\delta_{t-j} < q^{t-j-1}$ by Lemma 13(ii), it follows from (14) and (17) that there exists a nonnegative integer c_{j+1} such that

$$\begin{aligned} &b_{H_{j+1},1} = c_{j+1}q^{t-j-1} + \delta_{t-j-1} \text{ and} \\ &0 \le c_{j+1} \le \max\{c_j - 1, 0\} \le \max\{\Theta_r - h - j - 1, 0\}. \end{aligned} \tag{18}$$

Let $\mathcal{P}_{j+1}$ be the subspace partition of H_{j+1} defined by:

$$\mathcal{P}_{j+1} = \{W \cap H_{j+1} : \ W \in \mathcal{P}_j\},$$

and by the definition made in (18), let $m_{j+1,1} = b_{H_{j+1},1}$. Since $t - j > 2$ and $\dim(W \cap H_{j+1}) \in \{\dim W, \dim W - 1\}$ for each $W \in \mathcal{P}_j$, it follows that $\mathcal{P}_{j+1}$ is a subspace partition of H_{j+1} of type

$$[t^{m_{j+1,t}}, (t-1)^{m_{j+1,t-1}}, \ldots, (t-j-1)^{m_{j+1,t-j-1}}, 1^{m_{j+1,1}}], \tag{19}$$

where $m_{j+1,t}, m_{j+1,t-1}, \ldots, m_{j+1,t-j-1}$ are nonnegative integers such that

$$\sum_{i=t-j-1}^{t} m_{j+1,i} = \sum_{i=t-j}^{t} m_{j,i} = n_t. \tag{20}$$

The inductive step follows since $\mathcal{P}_{j+1}$ is a subspace partition of $H_{j+1} \cong V(n-j-1,q)$ of the type given in (19), which satisfies the conditions in (18) and (20).

Thus far, we have shown that the desired subspace partition $\mathcal{P}_j$ of H_j exists for any integer j such that $0 \le j \le t-2$. Since $q^2 \nmid x$ by hypothesis, Lemma 13(iii) implies that $\delta_{t-j} \ne 0$ for $j \in [0, t-2]$. Thus, $m_{j,1} = c_j q^{t-j} + \delta_{t-j} \ne 0$ for $j \in [0, t-2]$. If $j \in [\Theta_r - h, t-2]$, then it follows from (12) that $c_j = 0$, and thus, $m_{j,1} = \delta_j \ne 0$. In particular, since $t \ge \Theta_r - h + 2$, we have $c_{t-2} = 0$ and $m_{t-2,1} = \delta_2 \ne 0$. For the final part of the proof, we set $j = t-2$, and then show that the existence of the subspace partition $\mathcal{P}_{t-2}$ of H_{t-2} leads to a contradiction.

It follows from the above observations and Lemma 13(ii) that

$$m_{t-2,1} = \delta_2 = q^2\lceil xq^{-2}\Theta_2\rceil - x\Theta_2 \text{ and } 0 < \delta_2 < q^2. \tag{21}$$

Since $m_{t-1,2} > 0$, the smallest dimension of a subspace in $\mathcal{P}_{t-2}$ is 1. So let $s \ge 2$ be the second smallest dimension of a subspace in $\mathcal{P}_{t-2}$. (Note that the existence of s follows from (11).) To derive the final contradiction, we consider the following cases.

Case 1: $s \ge 3$.

Then by applying Theorem 10(ii)&(iv) to the subspace partition $\mathcal{P}_{t-2}$ with $d_2 = s$ and $d_1 = 1$, we obtain $m_{t-2,1} \ge \min\{(q^s-1)/(q-1), 2q^{s-1}, q^s\} > q^2$, which contradicts the fact that $m_{t-2,1} < q^2$ given by (21).

Case 2: $s = 2$.

Since $q \mid x$ by hypothesis, it follows from (21) that $q \mid m_{t-2,1}$. Thus, by applying Theorem 10(iv) to $\mathcal{P}_{t-2}$ with $d_2 = s = 2$ and $d_1 = 1$, we obtain $m_{t-2,1} \ge q^2$, which contradicts the fact that $m_{t-2,1} < q^2$ given by (21). □

We are now ready to prove Theorem 6 and Corollary 8.

Proof of Theorem 6. Recall that

$$c_1 \equiv t-2 \pmod q,\ 0 \le c_1 < q, \text{ and } c_2 = \begin{cases} q & \text{if } q^2 \mid ((q-1)(t-2)+c_1), \\ 0 & \text{if } q^2 \nmid ((q-1)(t-2)+c_1). \end{cases} \tag{22}$$

Define

$$x := q^r - (q-1)(t-2) - c_1 + c_2. \tag{23}$$

Since $r \ge 2$, it follows from (22) and (23) that:

(a) If $q^2 \mid ((q-1)(t-2)+c_1)$, then $c_2 = q$, and also, $q^2 \mid (q^r - (q-1)(t-2) - c_1)$. Thus, $x \equiv q \not\equiv 0 \pmod{q^2}$.

(b) If $q^2 \nmid ((q-1)(t-2)+c_1)$, then $c_2 = 0$, and also, $q^2 \nmid (q^r - (q-1)(t-2) - c_1)$. Thus, $x = q^r - (q-1)(t-2) - c_1 \not\equiv 0 \pmod{q^2}$.

Thus, $q^2 \nmid x$ holds in all cases.

Also, since $c_1 \equiv t-2 \pmod q$ by (22), we have $t-2 = \alpha q + c_1$ for some nonnegative integer α. Thus, it follows from (23) that

$$x = q^r - \alpha q(q-1) - c_1 q + c_2. \tag{24}$$

Since $c_2 \in \{0, q\}$ by (22), it follows from (24) that $q \mid x$.

Moreover, since $0 \le c_1 \le q-1$ and $c_2 \in \{0, q\}$, we obtain

$$\begin{aligned} x = q^r - (q-1)(t-2) - c_1 + c_2 &\ge q^r - (q-1)(t-2) - (q-1) \\ \implies \frac{x}{q-1} &\ge \frac{q^r-1}{q-1} + \frac{1}{q-1} - t + 1 \\ \implies \left\lceil \frac{x}{q-1} \right\rceil &\ge \frac{q^r-1}{q-1} - t + 2 \\ \implies t &\ge \Theta_r - \left\lceil \frac{x}{q-1} \right\rceil + 2. \end{aligned} \tag{25}$$

Since the hypothesis holds from the above observations, Lemma 14 yields

$$\mu_q(n,t) \le \ell q^t + x = \frac{q^n - q^{t+r}}{q^t - 1} + q^r - (q-1)(t-2) - c_1 + c_2.$$

Moreover, since $-q+1 \le -c_1 + c_2 \le q$, it follows that

$$\begin{aligned} \mu_q(n,t) &\le \frac{q^n - q^{t+r}}{q^t - 1} + q^r - (q-1)(t-2) - c_1 + c_2 \\ &\le \frac{q^n - q^{t+r}}{q^t - 1} + q^r - (q-1)(t-2) + q \\ &= \frac{q^n - q^{t+r}}{q^t - 1} + q^r - (q-1)(t-3) + 1, \end{aligned}$$

which concludes the proof of Theorem 6. □

Proof of Corollary 8. Let $f_q(n,t)$ and $g_q(n,t)$ be as defined in the statement of the corollary. Then

$$g_q(n,t) = \frac{q^n - q^{t+r}}{q^t - 1} + q^r - (q-1)(t-2) - c_1 + c_2, \tag{26}$$

where c_1 and c_2 are as in (22), and

$$f_q(n,t) = \frac{q^n - q^{t+r}}{q^t - 1} + q^r - \lfloor \omega \rfloor - 1, \tag{27}$$

where $2\omega = \sqrt{4q^t(q^t - q^r) + 1} - (2q^t - 2q^r + 1)$.

If $r \geq 1$ and $t \geq 2r$, then it is straightforward to show that (e.g.,see [19, Lemma 2])

$$\lfloor \omega \rfloor = \left\lfloor \frac{q^r - 2}{2} \right\rfloor = \left\lfloor \frac{q^r}{2} \right\rfloor - 1. \tag{28}$$

Now it follows from (26)–(28) that if $t \geq 2r$, then

$$g_q(n,t) - f_q(n,p) = \left\lfloor \frac{q^r}{2} \right\rfloor - (q-1)(t-2) - c_1 + c_2. \tag{29}$$

We now prove the second part of the corollary for $q > 2$. If $\lceil \frac{\Theta_r}{2} \rceil + 4 \leq t \leq \Theta_r$, then by applying (29) with $0 \leq c_1 < q$ and $c_2 \in \{0, q\}$, we obtain

$$\begin{aligned} g_q(n,t) - f_q(n,p) &\leq \left\lfloor \frac{q^r}{2} \right\rfloor - (q-1)(t-2) + q \\ &\leq \left\lfloor \frac{q^r}{2} \right\rfloor - (q-1)\left(\left\lceil \frac{\Theta_r}{2} \right\rceil + 2\right) + q \\ &= \left\lfloor \frac{q^r}{2} \right\rfloor - (q-1)\left\lceil \frac{q^r - 1}{2(q-1)} \right\rceil - q + 2 \\ &\leq \frac{q^r}{2} - (q-1)\left(\frac{q^r - 1}{2(q-1)}\right) - q + 2 \\ &= 5/2 - q < 0 \qquad \text{(since } q > 2\text{)}. \end{aligned}$$

If $q = 2$, then by doing the same analysis as above with $t \geq \lceil \frac{\Theta_r}{2} \rceil + 5$ instead of $t \geq \lceil \frac{\Theta_r}{2} \rceil + 4$, we obtain $g_q(n,t) - f_q(n,p) < 0$. This completes the proof of the corollary.

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