ON THE EXISTENCE OF A RAINBOW 1-FACTOR IN 1-FACTORIZATIONS OF $K_{rn}^{(r)}$

S. EL-ZANATI, M. PLANTHOLT, P. SISSOKHO, AND L. SPENCE

DEPARTMENT OF MATHEMATICS
ILLINOIS STATE UNIVERSITY
NORMAL, IL 61790-4520, USA.

Abstract. Let $\boldsymbol{\mathcal{F}}$ be a 1-factorization of the complete uniform hypergraph $\mathcal{G} = K_{rn}^{(r)}$ with $r \geq 2$ and $n \geq 3$. We show that there exists a 1-factor of $\mathcal{G}$ whose edges belong to n different 1-factors in $\boldsymbol{\mathcal{F}}$. Such a 1-factor is called a "rainbow" 1-factor or an "orthogonal" 1-factor.

Keywords: Rainbow 1-factor; Orthogonal 1-factor; Rainbow Matching.

1. Introduction

A *hypergraph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V}$ of vertices and a set $\mathcal{E}$ of subsets of $\mathcal{V}$ called edges. A *1-factor* of a hypergraph $(\mathcal{V}, \mathcal{E})$ is a collection of pairwise disjoint edges which partition $\mathcal{V}$. A *1-factorization* of $(\mathcal{V}, \mathcal{E})$ is a partition of $\mathcal{E}$ into 1-factors. For positive integers $r \geq 2$ and n, the *complete r-uniform hypergraph* on n vertices is the hypergraph $K_n^{(r)}$, with a vertex set $\mathcal{V}$ of order n and an edge set $\mathcal{E}$ consisting of all r-subsets of $\mathcal{V}$. Note that $K_n^{(2)}$ is K_n, the simple complete graph of order n. In order for $K_n^{(r)}$ to contain a 1-factor, it is clearly necessary that r divides n. In 1973, Baranyai [2] showed that $K_{rn}^{(r)}$ has a 1-factorization.

The following question is attributed to Rosa in 1977 (see [3]): Given a 1-factorization $\boldsymbol{\mathcal{F}}$ of $K_{rn}^{(r)}$, $n \geq 3$, prove there exists a 1-factor $\mathcal{F}$ in $K_{rn}^{(r)}$ whose edges belong to n different 1-factors of $\boldsymbol{\mathcal{F}}$. Such a 1-factor $\mathcal{F}$ would commonly be said to be *suborthogonal* or *orthogonal* to $\boldsymbol{\mathcal{F}}$ (see [1] for a survey of orthogonal factorizations of graphs). Woolbright made the first progress on this problem. In 1978, he showed [4] that there exists a 1-factor in $K_{rn}^{(r)}$ whose edges belong to at least $n-1$ different 1-factors of $\boldsymbol{\mathcal{F}}$. And in 1998, Woolbright with Fu [3] proved Rosa's question in the case $r = 2$ (i.e., for 1-factorizations of K_{2n}). In this manuscript, we provide a proof of Rosa's general conjecture.

Before proceeding, we note that Woolbright and Fu [3] offer a coloring version of Rosa's question. If the edges of $K_{rn}^{(r)}$ are colored so that two edges receive the same color if and only if they belong to the same 1-factor in $\boldsymbol{\mathcal{F}}$, then the edges of the orthogonal 1-factor $\mathcal{F}$ receive distinct colors. They call $\mathcal{F}$ a *rainbow*. We shall refer to such an $\mathcal{F}$ as a *rainbow 1-factor*.

2. Main Result

2.1. **Definitions and Notation.** In this paper, the term *graph* will be used to denote both a graph and a hypergraph. Similarly, the term *edge* will be used for a graph edge and a hypergraph edge.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph whose edges are properly colored, i.e. any two non-disjoint edges of $\mathcal{G}$ have distinct colors. A *rainbow matching* of $\mathcal{G}$ is a matching whose edges have pairwise different colors. For $E \in \mathcal{E}$, we let $c(E)$ denote the color of E. For any subset $\mathcal{H}$ of edges of $\mathcal{G}$, we let $C[\mathcal{H}] = \{c(E)\colon\ E \in \mathcal{H}\}$, and $F[\mathcal{H}] = C[\mathcal{E}] - C[\mathcal{H}]$. Thus $C[\mathcal{H}]$ is the set of colors of $\mathcal{H}$, and $F[\mathcal{H}]$ is the complementary set of free colors (i.e., the colors not appearing on edges in $\mathcal{H}$).

2.2. **Main result.** We first remark that clearly there is no rainbow 1-factor in a 1-factorization of $K_{2r}^{(r)}$ for any $r \geq 2$. For $n \geq 3$ and $r \geq 2$, we show that a 1-factorization of $K_{rn}^{(r)}$ always admits a rainbow 1-factor. We start by showing the existence of a rainbow 1-factor for the cases $r = 2$, $n = 3$ and $r = 2$, $n = 4$. Then we will prove the remaining cases in Theorems 2 and 3.

Lemma 1. *For any 1-factorization of $K_{2n} = K_{2n}^{(2)}$ there is a rainbow 1-factor when $n = 3$ or $n = 4$.*

Proof. Consider a 1-factorization of K_{2n}, and let $1, 2 \ldots, 2n-1$ be the colors used. If $n = 3$, select an edge of color 1, incident with vertices x and y. The remaining 4 colors are each assigned to one edge that is not incident to either x or y. Thus we can complete the matching with two differently colored and non-adjacent edges in the graph induced by the 4 vertices other than x and y.

If $n = 4$, starting with any triangle, it is possible to find in the 1-factorization of $G = K_8$ a rainbow K_4 (call it R), i.e., a K_4 whose edges have pairwise distinct colors. Let F be the graph induced by the vertices not in R. Since we have a 1-factorization of G, it follows (by a simple count) that for each color i present in R, there is at least one edge in F of color i. Since there are 6 edges in F and 6 different colors in R, it follows that F is also rainbow. Thus, we can pick two edges e_1 and e_2 from F and two other edges e_3 and e_4 from R so that they form a rainbow 1-factor of G. □

Although the case $r = 2$ was proved separately in [3], we give in Theorem 2 below a much shorter proof. The general result is then given in Theorem 3, where we provide an even simpler proof for the case $r > 2$.

Theorem 2. *For $n \geq 3$, any 1-factorization of K_{2n} includes a rainbow 1-factor.*

Proof. By Lemma 1, we can assume $n \geq 5$. Start with any 1-factorization, with the colors in the corresponding proper coloring named $1, 2, \ldots, 2n-1$. Let $\mathcal{M}$ be any maximal rainbow matching, and let k denote its cardinality. Suppose $k < n$. We will show that there must be a rainbow matching with

$k+1$ edges. Recall that $C[\mathcal{M}]$ denotes the set of colors of $\mathcal{M}$ and $F[\mathcal{M}]$ denotes the complementary set of free colors. Let s, t be two unmatched vertices. We may assume that $C[\mathcal{M}] = \{1, 2, \ldots, k\}$, and that $c(\{s,t\}) = 1$. We note that any edge incident with s whose color is in $F[\mathcal{M}]$ must be incident with an edge of $\mathcal{M}$, by maximality of $\mathcal{M}$.

Now consider all the (s,t)-paths of length three, whose first edge color α is in $F[\mathcal{M}]$, and second edge is in $\mathcal{M}$; we call these the *candidate* 3*-paths relative to* $\mathcal{M}$. We can assume that each of these has its third edge color either in $C[\mathcal{M}] - \{1\}$, or the third edge color is α again, for otherwise we could augment $\mathcal{M}$ to $k+1$ edges simply by deleting edge two of the path from it, and adding edges one and three. There are $2n-1-k$ of these candidate paths, and only $k-1$ colors in $C[\mathcal{M}] - \{1\}$, so it follows that at least $2(n-k) \geq 2$ of these paths have first and third edges the same color of $F[\mathcal{M}]$; we call such paths $\mathcal{M}$*-symmetric* (s,t)*-paths.*

Consider the $2n-k-1$ edges incident with t whose colors are in $F[\mathcal{M}]$. Each of these edges must be incident with an edge of $\mathcal{M}$, by maximality of $\mathcal{M}$; at most 2 of them, say the ones colored $k+1$ and $k+2$, are incident with the edge of $\mathcal{M}$ colored 1. Now let $L = \{k+3, k+4, \ldots, 2n-1\}$.

For each color $i \in L$, we define a slight variation of the $(\mathcal{M}, st)$ pair. If the edge of color i incident with vertex t is $e_t = \{t, z_i\}$, and z_i is incident with edge $e_i = \{z_i, t_i\}$ of $\mathcal{M}$, we let the corresponding matching be $\mathcal{M}_i = (\mathcal{M} - \{e_i\}) \cup \{e_t\}$; now t_i is unmatched (in $\mathcal{M}_i$), and we let our start/end vertex pair be s, t_i respectively. Note that $F[\mathcal{M}_i] = (F[\mathcal{M}] - \{i\}) \cup \{c(e_i)\}$. Also note that because i is neither $k+1$ nor $k+2$, $c(e_i) \neq 1$.

As in the previous discussion, for each such i, there are $2n-1-k$ candidate 3-paths relative to $\mathcal{M}_i$, starting at s, ending at t_i , first edge color in $F[\mathcal{M}_i]$, second edge in $\mathcal{M}_i$. Again, we assume that at least two of these paths are symmetric. Else, as before, it is easy to find a rainbow matching with $k+1$ edges. Thus, listing the symmetric paths for $i = k+3, \ldots, 2n-1$, we get a total of at least $4n-2k-6$ paths in the list of symmetric candidate paths. However, because in each of these symmetric paths either the middle edge is in $\mathcal{M}$, or the path has the form sz_itt_i and therefore has the same first and third edges as sz_it_it, each of the symmetric candidate paths is uniquely determined by its first edge. Moreover, the starting/ending edge color α in these symmetric paths cannot be $c(e_i) = c(\{z_i, t_i\})$ (the only possible such path has vertex sequence stz_it_i, which is not symmetric because $c(e_i) \neq 1$), so α must be in $F[\mathcal{M}] = \{k+1, k+2, \ldots, 2n-1\}$. Therefore each of the possible starting colors $k+1, k+2, \ldots, 2n-1$ can only start one path in the list. It follows that $2n-1-k \geq 4n-2k-6$, so that $n \leq 4$, a contradiction.

We conclude then that there must be a rainbow matching with $k+1$ edges, so the result now follows. □

Theorem 3. *For $n \geq 3$ and $r \geq 2$, any 1-factorization of $K_{rn}^{(r)}$ includes a rainbow 1-factor.*

Proof. By Theorem 2, the result holds when $r = 2$. For $r > 2$, we can generalize the $r = 2$ proof in a natural way; however, taking advantage of the additional edges afforded by having $r > 2$, we give a more direct proof.

Arguing as in Lemma 1, it is easy to verify the result for $n = 3$. So we assume $n \geq 4$, and begin with a maximal rainbow matching $\mathcal{M}$ of cardinality k. We show that if $k < n$, then there is a rainbow matching with $k+1$ edges; the result then follows. It is easy to see that because $n \geq 4$, we must have $k \geq 3$.

Let the edges of $\mathcal{M}$ be labeled $m_1, m_2, \ldots, m_k$. Take any r vertices not included in an edge of $\mathcal{M}$, and let m_0 denote the edge formed by those vertices; because $\mathcal{M}$ is maximal, $c(m_0) \in C[\mathcal{M}]$. For any vertex v of m_i, $i = 0, \ldots, k$, we let v^* denote the complementary $r-1$ vertices of m_i. From now on, we write x^*y or yx^* to denote the union of the set x^* of size $r-1$ and $\{y\}$.

Let t be a fixed vertex of m_0. All edges of the form t^*v_i, where v_i is a vertex of m_i, $i > 0$, are mutually adjacent and so assigned mutually different colors, which also differ from $c(m_0)$. Therefore at most $k-1$ of these edges are assigned colors from $C[\mathcal{M}]$, so for some particular value of i from 1 to k, assume without loss of generality $i = 1$, all edges of the form t^*v, where v is a vertex in m_1, are assigned free colors.

To complete the proof, we will show the existence of a 6-cycle whose edges are of the form $m_1 = v^*v$, vt^*, $m_0 = t^*t$, tw_i^*, $m_i = w_i^*w_i$, w_iv^*, where v is a vertex in m_1, w_i is a vertex in m_i, with $i > 1$, and the three edges vt^*, tw_i^*, w_iv^* have been assigned 3 distinct free colors. Then adding these three edges to $\mathcal{M}$, and deleting m_1 and m_i from it, will give a rainbow matching with $k+1$ edges.

To see that such a cycle exists, first consider paths of five edges of the form $m_1 = v^*v$, vt^*, $m_0 = t^*t$, tw_i^*, $m_i = w_i^*w_i$. We wish to count the number of these paths in which the second and fourth edges are assigned two different, free colors. As determined above, for each vertex v from m_1, $c(vt^*)$ is free. There are $r(k-1)$ edges of the form tw_i^*, where w_i is a vertex of m_i, $1 < i \leq k$. These edges are all mutually adjacent and assigned different colors, so $a \leq r$ of them are assigned a color used on the second edge of the path. Also $b \leq k-1$ are assigned colors of $C[\mathcal{M}]$, yielding rb unacceptable paths. So the number of these paths with the second and fourth edges assigned different, free colors is at least $r^2(k-1) - a - rb$.

Finally, consider the last edge w_iv^* that completes the cycle; it is uniquely determined by the choices of w_i and v. Because v is a vertex in m_1 and $r > 2$, all such edges are mutually adjacent, so assigned different colors. As above, there are $r^2(k-1) - a - rb$ edges of the form w_iv^* that complete a 6-cycle in which the second and fourth edge colors are free and distinct. For w_iv^* we wish to avoid the k colors in $C[\mathcal{M}]$, the r colors assigned to the edges of the form vt^*, and the colors on the up to $r(k-1)$ edges of the form tw_i^* (there are $r(k-1) - a - b$ of these that have not already been discounted). Thus the total number of 6-cycles having the properties we desire is at least

$r^2(k-1)-a-rb-k-r-r(k-1)+a+b$. This quantity is a minimum when $b=k-1$, so it is at least $(r^2-2r)(k-1)-r-1$. It is straightforward to check this expression is positive for $r,k\geq 3$, so the result now follows. $\square$

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E-mail address: S. EL-ZANATI: SAAD@ILSTU.EDU
E-mail address: M. PLANTHOLT: MIKEP@ILSTU.EDU
E-mail address: P. SISSOKHO: PSISSOK@ILSTU.EDU
E-mail address: L. SPENCE: SPENCE@ILSTU.EDU