SUBSPACE PARTITIONS OF $\mathbb{F}_q^n$ CONTAINING DIRECT SUMS

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Abstract. Let q be a prime power and n be a positive integer. A *subspace partition* of $V = \mathbb{F}_q^n$, the vector space of dimension n over $\mathbb{F}_q$, is a collection Π of subspaces of V such that each nonzero vector of V is contained in exactly one subspace in Π; the multiset of dimensions of subspaces in Π is then called a *Gaussian partition* of V. We say that Π *contains a direct sum* if there exist subspaces $W_1, \ldots, W_k \in \Pi$ such that $W_1 \oplus \cdots \oplus W_k = V$. In this paper, we study the problem of classifying the subspace partitions that contain a direct sum. In particular, given integers a_1 and a_2 with $n > a_1 > a_2 \geq 1$, our main theorem shows that if Π is a subspace partition of $\mathbb{F}_q^n$ with m_i subspaces of dimension a_i for $i = 1, 2$, then Π contains a direct sum when $a_1x_1 + a_2x_2 = n$ has a solution (x_1, x_2) for some integers $x_1, x_2 \geq 0$ and m_2 belongs to the union I of two natural intervals. The lower bound of I captures all subspace partitions with dimensions in $\{a_1, a_2\}$ that are currently known to exist. Moreover, we show the existence of infinite classes of subspace partitions without a direct sum when $m_2 \notin I$ or when the condition on the existence of a nonnegative integral solution (x_1, x_2) is not satisfied. We further conjecture that this theorem can be extended to any number of distinct dimensions, where the number of subspaces in each dimension has appropriate bounds. These results offer further evidence of the natural combinatorial relationship between Gaussian and integer partitions (when $q \to 1$) as well as subspace and set partitions.

Keywords: Subspace partition; vector space partition; direct-sum decomposition; spread; partial spread; integer partition.

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1. Introduction

Let $\mathbb{F}_q$ be the field with q elements, $\mathbb{F}_q^n$ denote the n-dimensional vector space over $\mathbb{F}_q$, and set $V = \mathbb{F}_q^n$. Moreover, let $E = \{e_1, \ldots, e_n\}$ be the standard ordered basis for V, identified with the set $\mathbf{n} = \{1, \ldots, n\}$. We sometimes write *$m$-subspace* to mean a vector subspace of dimension m. A *subspace partition*[1] of V is a collection Π of subspaces of V such that each nonzero vector of V is contained in exactly one subspace in Π. Given positive integers k, $a_1, \ldots, a_k$, and $u_1, \ldots, u_k$, we let $a_1^{u_1} \ldots a_k^{u_k}$ denote the multiset with u_i copies of a_i for $1 \leq i \leq k$. The *type* of a subspace partition Π of V is the multiset consisting of $\dim W$ for all $W \in \Pi$. In particular, a subspace partition of type a^u is called an *a-spread*, or, simply, a *spread*, and a subspace partition of type a^u1^v for some $v > 0$ is called a *partial a-spread*, or, a *partial spread*. Thus, Π has type $T(\Pi) = a_1^{u_1} \ldots a_k^{u_k}$ if it contains u_i subspaces of dimension

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[1]Also known as a "vector space partition" in the literature (see Heden [15] for a survey).

a_i for $1 \le i \le k$ and no other subspace. In this case, we have

$$\sum_{i=1}^{k} u_i(q^{a_i} - 1) = |V| - 1 = q^n - 1.$$

Because of this fact, we also call $T(\Pi)$ a *Gaussian partition* of V.

Let $\mathcal{P}(\mathbf{n})$ denote the lattice of set partitions of $\mathbf{n}$, and $\mathcal{P}(V)$ be the set of all subspace partitions of $V = \mathbb{F}_q^n$. We recall from [1] that $\mathcal{P}(V)$ forms a lattice. Consider two subspace partitions $\Pi = \{U_1, \ldots, U_k\}$ and $\Gamma = \{W_1, \ldots, W_t\}$ of V. We say Π is a *refinement* of Γ in $\mathcal{P}(V)$, and write $\Pi \preceq \Gamma$, if each subspace W_i is the union of a number of the subspaces U_j. We also say Γ is *coarser than* Π, and write $\Gamma \succeq \Pi$. The minimum element of this lattice is the subspace partition that contains all 1-subspaces of V and the maximum element is the subspace partition whose only subspace is V. The *meet* of two arbitrary partitions Π and Γ is the collection $\Pi \wedge \Gamma = \{U_i \cap W_j \,|\, 1 \le i \le k,\ \ 1 \le j \le t,\ \ U_i \cap W_j \ne \{0\}\}$, which is a subspace partition of V. The *join* of Π and Γ is defined to be

$$\Pi \vee \Gamma = \bigwedge_{\Omega \succeq \Pi,\, \Omega \succeq \Gamma} \Omega.$$

The problem of determining the size of the lattice $\mathcal{P}(V)$ is currently out of reach due to the fact that even the types of maximal partial spreads and their numbers are unknown quantities in general. We have instead studied some large subsets of the original lattice $\mathcal{P}(V)$, where only certain common constructions were permitted and other, irregular, constructions of subspace partitions were not. We will describe these "regular" subspace partitions in due course (Section 5). A particular example of a regular partition is a "basic" subspace partition (Section 4), which contains subspaces that are spanned by the subsets in a set partition of the basis E of V, and hence can be mapped to that set partition in $\mathcal{P}(\mathbf{n})$ in a natural way.

In this paper, we introduce the following definition, which will enable us to capture a natural subset of $\mathcal{P}(V)$ that merits further study in our view (see Section 2).

Definition 1. *We say that a subspace partition Π of V* contains a direct sum *if there exist subspaces $W_1, \ldots, W_k \in \Pi$ such that $W_1 \oplus \cdots \oplus W_k = V$.*

Do all subspace partitions contain direct sums? And to turn the question around, does there exist a subspace partition that cannot contain a direct sum due to the fact that no positive linear combination of the dimensions is equal to n? As we will see shortly, the answers are no and yes, respectively.

Our main result is as follows.

Theorem 1. *Let q be a prime power, n, a, b be integers such that $n > a > b > 0$, and Π be a subspace partition of $V = \mathbb{F}_q^n$ of type $a^u b^v$. Define $\mathcal{S}$ to be the set of all solutions (x, y) of the Diophantine equation $ax + by = n$ with $x, y \ge 0$. Let $y_0 = \min_{(x,y)\in\mathcal{S}} y$, $y_M = \max_{(x,y)\in\mathcal{S}} y$, $x_0 = (n - ay_0)/b$, $x_M = (n - ay_M)/b$, and consider the intervals*

$$I_1 = \left[\frac{q^{by_0} - 1}{q^b - 1}, \frac{q^{by_M} - 1}{q^b - 1}\right] \text{ and } I_2 = \left[\frac{q^n - q^{ax_0}}{q^b - 1}, \frac{q^n - q^{ax_M}}{q^b - 1}\right].$$

If $\mathcal{S} \ne \emptyset$ and $v \in I_1 \cup I_2$, then Π contains a direct sum. Conversely, if Π contains a direct sum, then $\mathcal{S} \ne \emptyset$.

We naturally ask whether: (i) there is an overlap between the intervals I_1 and I_2; (ii) regardless of an overlap, there are known subspace partitions of type $a^u b^v$ with $v \notin I_1 \cup I_2$

that admit or do not admit direct sums; and (iii) the bounds on v are tight regarding the existence of such subspace partitions. We will address these questions throughout the paper.

Remark 2. *In Section 6, we will discuss a construction of Beutelspacher* [6] *that yields subspace partitions of V of type $a^u b^v$ for which $\mathcal{S} = \emptyset$. Thus, these subspace partitions do not contain direct sums.*

Now assume that $\mathcal{S} \neq \emptyset$. Then, we have the following observations.

(1) *The lower bounds of the two intervals for v, the number of b-subspaces of Π, satisfy the inequality*
$$\frac{q^{by_0}-1}{q^b-1} \leq \frac{q^{ax_0}(q^{by_0}-1)}{q^b-1} = \frac{q^n-q^{ax_0}}{q^b-1}$$
for all $q \geq 2$. This is an equality, and the "intervals" are equal to the same integer $(q^{by_0}-1)/(q^b-1)$, when $x_0 = 0$ (thus, $x_M = 0$), which happens when we have the unique solution $\mathcal{S} = \{(0, n/b)\}$. The equality also holds in the case $y_0 = 0$, which means that both intervals start at 0, (and in light of the inequality below, $I_1 \subseteq I_2$). Moreover, if the minimum values x_M and y_0 are both zero, then this is equivalent to both a and b dividing n, and we have $I_1 = I_2 = [0, (q^n-1)/(q^b-1)]$.

(2) *Similarly, the upper bounds satisfy*
$$\frac{q^{by_M}-1}{q^b-1} \leq \frac{q^{ax_M}(q^{by_M}-1)}{q^b-1} = \frac{q^n-q^{ax_M}}{q^b-1}$$
for all $q \geq 2$. We have an equality, and the intervals themselves are equal to the same integer, zero, if $y_M = 0$ (thus, $y_0 = 0$) and $\mathcal{S} = \{(n/a, 0)\}$. The other case of equality occurs and the intervals have the same right endpoint if $x_M = 0$ instead. Then we have $I_2 \subseteq I_1$.

(3) *When $\mathcal{S} = \{(x_0, y_0)\} = \{(x_M, y_M)\}$ with $x_0, y_0 > 0$, each "interval" consists of exactly one element, and we have*
$$\frac{q^{by_0}-1}{q^b-1} = \frac{q^{by_M}-1}{q^b-1} < \frac{q^n-q^{ax_0}}{q^b-1} = \frac{q^n-q^{ax_M}}{q^b-1}$$
for all $q \geq 2$.

(4) *The only possible instances of overlap between I_1 and I_2 are described in cases (1) and (2), where at least one of a or b must divide n. In every other case, including (3), it can be shown that I_1 is strictly on the left of I_2 for all $q \geq 2$. Consequently, the minimum of $I_1 \cup I_2$ is the minimum of I_1, and the maximum of $I_1 \cup I_2$ is the maximum of I_2.*

(5) *If $0 < v < (q^{by_0}-1)/(q^b-1)$, then it is not known in general whether there exist subspace partitions of V of type $a^u b^v$.*

As a justification of the upper bound $(q^n - q^{ax_M})/(q^b-1)$ on v in Theorem 1, we will exhibit in Section 8.2 an infinite class of subspace partitions of $V = \mathbb{F}_q^n$ of type $a^u b^v$ that do not contain direct sums and for which v is larger than this number. Moreover, this upper bound is necessary and sharp for $b \geq 4$.

The rest of the paper is organized as follows. In Section 2, we give some motivation for studying subspace partitions with direct sums. In Section 3, we introduce some technical background that will be used in subsequent sections. In Sections 4 and 5, we discuss basic subspace partitions and regular subspace partitions, respectively, and show that they both have direct sums. Section 6 is where we use Beutelspacher's work to exhibit an infinite class of subspace partitions with no direct sum. In Section 7, we prove our main theorem (Theorem 1) and justify our claim that the restrictions on v are natural, on both theoretical and practical grounds. Finally, in Section 8, we discuss the sharpness of the conditions imposed on v given in Theorem 1, and formulate a conjecture that generalizes Theorem 1.

2. Why study subspace partitions containing direct sums?

2.1. **Search for a combinatorial q-analogue of set partitions.** One of our main motivations for this line of research is to demonstrate the natural similarity between the lattice $\mathcal{P}(\mathbf{n})$ of set partitions of $\mathbf{n}$ and the lattice $\mathcal{P}(V)$ of all subspace partitions of $V = \mathbb{F}_q^n$. Proving general properties of the subspace and Gaussian partitions of V, with the goal of establishing them as the natural *combinatorial* q-analogues of the set partitions of $\mathbf{n}$ and integer partitions of n respectively, has been the theme of a series of papers [1, 2, 3]. In the course of this program, we have pointed out the difficulties in describing the full sets of subspace partitions and Gaussian partitions, but we have also been able to make some progress along those lines.

In order to state the first and most important analogy, we let $B_n = |\mathcal{P}(\mathbf{n})|$ be the nth Bell number and $B_n(q) = |\mathcal{P}(\mathbb{F}_q^n)|$ be the nth q-Bell number. Then

Theorem 2 (A. and S. [1])**.** $|\mathcal{P}(\mathbb{F}_q^n)| = B_n(q) \equiv B_n \pmod{q-1}$.

We will demonstrate that the set of subspace partitions containing direct sums are even more tractable and meaningful, and we do not lose any essential properties when working exclusively with such partitions.

2.2. **Motivation and evidence based on previous work.** Let us call any subspace of $V = \mathbb{F}_q^n$ spanned by a nonempty subset of the fixed basis E a *pure subspace*; call any subspace partition of V that consists of only pure subspaces and the remaining 1-subspaces of V a *pure subspace partition*; and say that any subspace partition that contains a number of pure subspaces whose direct sum is equal to V *contains a pure direct sum* (there is a mapping of the latter set of partitions onto $\mathcal{P}(\mathbf{n})$, where $e_i \in E$ goes to $i \in \mathbf{n}$). We maintain that the set $\mathcal{P}_D(V)$ of subspace partitions that contain direct sums is a very good candidate for a manageable q-analogue of $\mathcal{P}(\mathbf{n})$. First, it is easy to see why $\mathcal{P}_D(V)$ forms a lattice, just like $\mathcal{P}(\mathbf{n})$: it is a poset with a maximum element that is closed under the meet operation. Moreover, this lattice contains the pure direct sums that map to set partitions of E in a natural way, and Theorem 3 below shows that the number of subspace partitions in the lattice is congruent to the the size of $\mathcal{P}(\mathbf{n})$ modulo $(q-1)$; these are hallmarks of a very reasonable q-analogue of $\mathcal{P}(\mathbf{n})$.

In our first paper [1] in this series, we considered the action of the diagonal subgroup G of the general linear group $GL(n, q)$, with respect to the ordered basis E of $V = \mathbb{F}_q^n$, on the set of all subspace partitions of V. Note that the order of G is $(q-1)^n$. We were able to prove that if Π is a pure subspace partition of V, then it is fixed by G, and any other partition has a G-orbit of size divisible by $q-1$; this is the basis of the proof of Theorem 2. Going backwards from this result, we can argue that the number of subspace partitions of V that contain direct sums is congruent to B_n modulo $q-1$ as well. We recall that $GL(n, q)$ acts on subspace partitions of V in such a way that it preserves the *type* of each partition, which we have been calling a *Gaussian partition*.

Theorem 3. *Every $GL(n,q)$-orbit of $\mathcal{P}(\mathbb{F}_q^n)$ consists either entirely of subspace partitions containing direct sums or entirely of subspace partitions not containing direct sums. Moreover, each orbit of the former kind has subspace partitions containing pure direct sums. Finally, the number of subspace partitions of $V = \mathbb{F}_q^n$ containing direct sums is congruent to B_n modulo $q-1$. That is, we have*

$$|\mathcal{P}_D(\mathbb{F}_q^n)| \equiv B_n \pmod{q-1}.$$

Proof. Given any partition Π containing a direct sum $W_1 \oplus \cdots \oplus W_k = V$, and any choice of ordered bases $\mathcal{B}_1, \ldots, \mathcal{B}_k$ of the direct summands respectively, there exists an invertible linear operator g on V that sends the ordered basis $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ of V to any desired permutation of the ordered basis E. The same map sends Π to a partition Π' containing a pure direct sum, namely, $g(W_1) \oplus \cdots \oplus g(W_k) = V$. Now, as mentioned above, it follows from Propositions 2, 7, and 8 in [1] that the $GL(n,q)$-orbits of $\mathcal{P}(V)$ are uniquely subdivided into G-orbits, with size either equal to 1 (in case of pure subspace partitions) or a multiple of $q-1$ (for all the other subspace partitions). The former types reside in $\mathcal{P}_D(V)$, correspond to all set partitions of the basis E, and are directly responsible for the result $|\mathcal{P}_D(V)| \equiv B_n \pmod{q-1}$. □

To address the practicality of working solely with $\mathcal{P}_D(V)$, we would like to mention that the universally utilized regular construction we have described [2, 3] (also see Section 5) naturally gives rise to subspace partitions with many direct sums.

2.3. **Research question inspired by subspace partitions containing direct sums.** The *classification problem for subspace partitions* consists of finding necessary and sufficient conditions for the existence of subspace partitions of a given type. This problem has received considerable attention (e.g., see [6, 9, 12, 13, 14, 16]). There are only a few known necessary conditions for the existence of subspace partitions. In particular, Heden's *Tail Theorem* [14, Theorem 1] (also see the discussion in Section 8.1) proves a lower bound on the number of subspaces of smallest dimension in an arbitrary subspace partition of $\mathbb{F}_q^n$. As mentioned earlier, the lower bound from the interval I_1 in Theorem 1 is satisfied by all currently known subspace partitions with subspaces of two different dimensions. As far as we know, this is also the case for the known subspace partitions with more than two different dimensions. This motivates the following question.

Question 3. *Let Π be a subspace partition Π of $V = \mathbb{F}_q^n$ with subspaces of dimensions $d_1, \ldots, d_m$, where $d_1 < \ldots < d_m$. Let s be the minimum value of x_1 over all nonnegative solutions $(x_1, \ldots, x_m)$ of the Diophantine equation $d_1x_1 + \ldots + d_mx_m = n$. Is it true that the number, $N_1(\Pi)$, of subspaces of dimension d_1 in Π satisfies the inequality*

$$N_1(\Pi) \geq \frac{q^{d_1 s} - 1}{q^{d_1} - 1}?$$

If the answer to this question is affirmative, then for $d_1 \geq 2$ and $s \geq 1$, it would provide a lower bound on $N_1(\Pi)$ that is generally stronger than current best, which is given by Heden's Tail Theorem. This in turn will be a useful tool in the general classification problem for subspace partitions, since it will give a new necessary condition for their existence. Thus, the classification problem may benefit from the study of subspace partitions with direct sums.

And finally, proving or disproving Conjecture 25 that forms a stronger form of Theorem 1 and is intimately related to the question above would increase our understanding of the classification problem.

3. Gaussian coefficients, cyclotomic polynomials, and the packing condition

3.1. **Gaussian coefficients.** Let q be a variable and n, k be nonnegative integers with $n \geq k$. The *Gaussian (binomial) coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q$$

is a polynomial in $\mathbb{Z}[q]$ that is a q-analogue of the binomial coefficient $\binom{n}{k}$, which counts the number of k-dimensional subspaces of $\mathbb{F}_q^n$ when q is a prime power. It can be defined in terms of the *q-number*

$$[n]_q \stackrel{def}{=} \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-2} + q^{n-1}$$

and the *q-factorial*

$$[n]_q! \stackrel{def}{=} [n]_q [n-1]_q \cdots [2]_q [1]_q,$$

via the formula

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \stackrel{def}{=} \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

We note that

$$\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k} \quad \text{and} \quad \lim_{q \to 1} [n]_q = n.$$

Remark 4. *We will make use of the identity*

$$\frac{[xy]_q}{[y]_q} = [x]_{q^y}.$$

3.2. **Cyclotomic polynomials.** The *kth cyclotomic polynomial* $\Phi_k(q) \in \mathbb{C}[q]$ is defined by

$$\Phi_k(q) = \prod (q - \zeta_m),$$

where ζ_m are the primitive kth roots of unity, that is, $\zeta_m = e^{2\pi i m/k}$, and $\gcd(m, k) = 1$.

It is well known that $\Phi_k(q)$ is irreducible with integer coefficients, and we have $\deg(\Phi_k) = \varphi(k)$, the Euler totient function evaluated at k. The first four cyclotomic polynomials are

$$\Phi_1(q) = q - 1, \;\; \Phi_2(q) = q + 1, \;\; \Phi_3(q) = q^2 + q + 1, \;\; \Phi_4(q) = q^2 + 1, \; \ldots.$$

The polynomial $q^n - 1 \in \mathbb{Z}[q]$, which is the product of $(q - \zeta)$ for *all* nth roots ζ of 1, is a product of cyclotomic polynomials:

$$q^n - 1 = \prod_{k|n} \Phi_k(q).$$

In particular, $\Phi_1(q) = q - 1$ is always a factor, and we have

$$[n]_q = \prod_{k|n,\, k>1} \Phi_k(q)$$

(the empty product is 1). Furthermore, $\Phi_i \neq \Phi_n$ for $i \neq n$. Using these properties, we infer that

(1) $a \mid n$ in $\mathbb{Z} \iff (q^a - 1) \mid (q^n - 1)$ in $\mathbb{Z}[q] \iff [a]_q \mid [n]_q$ in $\mathbb{Z}[q]$;
(2) $\gcd(q^a - 1, q^b - 1) = q^{\gcd(a,b)} - 1$ and $\gcd([a]_q, [b]_q) = [\gcd(a, b)]_q$ in $\mathbb{Z}[q]$;
(3) $\Phi_k(q) > 0$ for $q \geq 2$ and all $k \geq 1$ due to irreducibility and, hence, the lack of real roots when $k \geq 3$.

3.3. **More on q-numbers.** The q-numbers are monic polynomials in $\mathbb{Z}[q]$, which is a unique factorization domain, but not a Euclidean domain or a Bézout domain [10]. However, due to the following property, it is possible to compute the greatest common divisor of $[a]_q$ and $[b]_q$ for ($a > b > 0$ and b not dividing a) via the Euclidean algorithm, and in turn obtain a Bézout identity with coefficients involving $\mathbb{Z}$-linear combinations of polynomials of type

$$q^u [\Phi_{i_1}(q)]^{v_1} \cdots [\Phi_{i_k}(q)]^{v_k}. \tag{1}$$

Lemma 5. *Let $n > a > 0$ be integers such that n is not divisible by a, and let m, r be the unique quotient and remainder respectively, as dictated by the Division Algorithm when n is divided by a:*

$$n = am + r, \quad 0 < r < a.$$

Then we have

$$[n]_q = q^r \frac{[am]_q}{[a]_q} [a]_q + [r]_q = q^r [m]_{q^a} [a]_q + [r]_q.$$

Note the use of Remark 4 in the lemma and the next proposition. We see that the Euclidean algorithm works for q-numbers in a similar way as it does for natural numbers. This lemma in fact follows from a more general and easily verifiable identity:

Proposition 6. *Let $a_1, \ldots, a_h, x_1, \ldots, x_h, n$ be positive integers, with $h \geq 2$, and*

$$a_1 x_1 + \cdots + a_h x_h = n. \tag{2}$$

Then for any prime power q, we have

$$[a_1 x_1]_q + q^{a_1 x_1} [a_2 x_2]_q + \cdots + q^{a_1 x_1 + \cdots + a_{h-1} x_{h-1}} [a_h x_h]_q = [n]_q,$$

and $[n]_q$ is the following $\mathbb{Z}[q]$-linear combination of the $[a_i]_q$ with coefficients of type shown in (1)*:*

$$[x_1]_{q^{a_1}} [a_1]_q + q^{a_1 x_1} [x_2]_{q^{a_2}} [a_2]_q + \cdots + q^{a_1 x_1 + \cdots + a_{h-1} x_{h-1}} [x_h]_{q^{a_h}} [a_h]_q = [n]_q.$$

Note that the last identity in Proposition 6 is reduced to the identity in (2) when we let $q \to 1$.

3.4. **The packing condition.** Let $T(\Pi) = a_1^{u_1} \cdots a_h^{u_h}$, with $a_1 > \cdots > a_h$ and $u_i > 0$, be the type of a subspace partition Π of $V = \mathbb{F}_q^n$ for a specific q, i.e., a Gaussian partition of V. Then $(u_1, \ldots, u_h)$ necessarily satisfies the linear Diophantine equation

$$(q^{a_1} - 1) u_1 + \cdots + (q^{a_h} - 1) u_h = q^n - 1, \tag{3}$$

known as the *packing condition*, because it represents the distribution of the nonzero vectors of V into the subspaces of Π. Dividing throughout by $q - 1$, we obtain the identity

$$[a_1]_q u_1 + \cdots + [a_h]_q u_h = [n]_q. \tag{4}$$

In the literature (e.g., see the survey [15]), solutions of (4) are usually studied for specific values of q, in order to produce examples and counterexamples for the existence of subspace partitions of a certain type. However, there are certain solutions $(u_1, \ldots, u_h)$ where all u_i are polynomials in q, which, moreover, take only positive values for $q \geq 1$, and we will call these h-tuples the *positive solutions.* In particular, the *basic subspace partitions* that we will define in Section 4 will have types that correspond to positive polynomial solutions of the packing conditions, each involving products and quotients of q and q-integers (or, of q and cyclotomic polynomials). The obvious advantage of a positive solution is that every $u_i(q)$ can be reduced modulo $q-1$ (or, by substituting $q=1$) to one coordinate x_i of a positive integral solution of the Diophantine equation

$$a_1x_1 + \cdots + a_hx_h = n,$$

which represents an ordinary partition of the integer n. When generic subspace partitions whose types are represented by positive solutions $(u_1(q), \ldots, u_h(q))$ exist, we can ask the question of whether they contain direct sums whose summands consist of $x_i = u_i(1)$ subspaces of dimension i for $1 \leq i \leq h$.

Since $\mathbb{Z}[q]$ is a unique factorization domain, where we can find Bézout coefficients for Gaussian numbers, the following is also true:

Remark 7. *All solutions $(u_1, \ldots, u_h) \in \mathbb{Z}[q]^h$ of the packing condition given by* (4)*, i.e.,*

$$[a_1]_q u_1 + \cdots + [a_h]_q u_h = [n]_q,$$

can be found by processes similar to known methods for ordinary linear Diophantine equations (*e.g., see Bond* [8])*, where the arbitrary parameters are also polynomials in $\mathbb{Z}[q]$.*

4. Basic subspace partitions and direct sums

4.1. **General Notions.** We start with the following construction due to Beutelspacher [7].

Proposition 8 (Beutelspacher [7])**.** *Let U and W be subspaces of $V = \mathbb{F}_q^n$ such that $V = W \oplus U$, and $d = \dim U \leq \dim W = n-d$. Let $\{w_1, \ldots, w_{n-d}\}$ be a basis of W, and $\{u_1, \ldots, u_d\}$ be a basis of U. Moreover, we identify W with the field $\mathbb{F}_{q^{n-d}}$. For every element $\gamma \in W$, define a subspace U_γ of V by*

$$U_\gamma = \operatorname{span}(\{u_1 + \gamma w_1, \ldots, u_d + \gamma w_d\}),$$

where $U_0 = U$. Then $\dim U_\gamma = d$, $U_\gamma \cap U_{\gamma'} = \{0\}$ for $\gamma \neq \gamma'$, and the collection

$$\{W\} \cup \{U_\gamma : \gamma \in W\}$$

of subspaces forms a subspace partition of V of type $(n-d)^1 d^{q^{n-d}}$.

Definition 9. *We call a subspace partition Π of V* basic *if it is obtained from a set partition $\{E_1, \ldots, E_k\}$ of the chosen basis E, subject to the following conditions:*

1. *Let $|E_i| = a_i$. Then for $i < j$, we have $a_i \geq a_j$.*
2. *If $k = 1$, then $\Pi = \{V\}$. Otherwise, Π is obtained by applying the construction in Proposition 8 to V with $W = \langle E_1 \cup \cdots \cup E_{k-1} \rangle$ and $U = \langle E_k \rangle$, then to $\langle E_1 \cup \cdots \cup E_{k-1} \rangle$ with $W = \langle E_1 \cup \cdots \cup E_{k-2} \rangle$ and $U = \langle E_{k-1} \rangle$, etc., down to $\langle E_1 \cup E_2 \rangle$ with $W = \langle E_1 \rangle$ and $U = \langle E_2 \rangle$.*

Clearly, Π contains the pure direct sum $\langle E_1 \rangle \oplus \cdots \oplus \langle E_k \rangle$.

Note that due to the different choices of identification of W with $\mathbb{F}_{q^s}$ in Proposition 8, there may be multiple basic subspace partitions associated with a set partition of E. However, all of these basic subspace partitions are in the same orbit of $\mathrm{GL}(n, q)$.

Proposition 10. (a) *The basic Gaussian partition of $\mathbb{F}_q^n$ that corresponds to the integer partition $a_1 \cdots a_k$ of n, with $a_1 \geq \cdots \geq a_k$, is given by*

$$T = a_1^1 a_2^{q^{a_1}} a_3^{q^{a_1+a_2}} \cdots a_k^{q^{a_1+\cdots+a_{k-1}}}.$$

(b) *For a Gaussian partition written as in part* (a)*, the exponent q^t of any dimension a_i reflects the sum $t = a_1 + \cdots + a_{i-1}$ of the parts of the corresponding integer partition that come before a_i (the empty sum is zero).*

(c) *If we require the dimensions a_i to be distinct, then the basic Gaussian partition corresponding to the integer partition $a_1^{x_1} \cdots a_h^{x_h}$ of n, with $a_1 > \cdots > a_h \geq 1$, is given by*

$$T = a_1^{1+q^{a_1}+\cdots+q^{(x_1-1)a_1}} a_2^{q^{x_1a_1}\left(1+q^{a_2}+\cdots+q^{(x_2-1)a_2}\right)} \cdots a_h^{q^{x_1a_1+\cdots+x_{h-1}a_{h-1}}\left(1+q^{a_h}+\cdots+q^{(x_h-1)a_h}\right)}.$$

Corollary 11. *The packing condition corresponding to the type T described in part (c) of Proposition 10 is*

$$[a_1]_q \cdot [x_1]_{q^{a_1}} + [a_2]_q \cdot q^{x_1a_1} [x_2]_{q^{a_2}} + \cdots + [a_h]_q \cdot q^{x_1a_1+\cdots+x_{h-1}a_{h-1}} [x_h]_{q^{a_h}} = [n]_q.$$

This is exactly the linear combination given by Proposition 6, where ordering of the dimensions was not important.

4.2. **Basic solutions of the packing condition for subspace partitions of type a^ub^v.** Remark 7 and Corollary 11 reduce to the familiar result below when $h = 2$:

Theorem 4. *Let (x_0, y_0) be a nonnegative integer solution of the Diophantine equation $ax + by = n$ with $a > b > 0$ such that x_0 is maximal, $d = \gcd(a, b)$, $a = a'd$, $b = b'd$, and $c = \lfloor x_0/b' \rfloor$. Then all nonnegative solutions (x, y) of the equation are given by*

$$x = x_0 - ib' \quad \text{and} \quad y = y_0 + ia', \text{ with } 0 \leq i \leq c.$$

The corresponding basic subspace partitions have types $a^{u_i(q)}b^{v_i(q)}$, where

$$u_0(q) = \frac{[ax_0]_q}{[a]_q} = [x_0]_{q^a}, \quad v_0(q) = q^{ax_0}\frac{[by_0]_q}{[b]_q} = q^{ax_0}[y_0]_{q^b},$$

$$u_i(q) = [x_0 - ib']_{q^a} = u_0(q) - k_i(q)\frac{[b]_q}{[d]_q} = [x_0]_{q^a} - k_i(q)[b']_{q^d},$$

$$v_i(q) = q^{a(x_0-ib')}[y_0 + ia']_{q^b} = v_0(q) + k_i(q)\frac{[a]_q}{[d]_q} = q^{ax_0}[y_0]_{q^b} + k_i(q)[a']_{q^d},$$

and the $k_i(q)$ are monic positive polynomials that are products of q and cyclotomic polynomials:

$$k_i(q) = q^{a(x_0-ib')}\frac{[ia'b']_{q^d}}{[a']_{q^d}[b']_{q^d}}.$$

Moreover, we have

$$\lim_{q\to 1} k_i(q) = i,$$

reducing (u_i, v_i) to (x_i, y_i) and the packing condition $[a]_q u_i + [b]_q v_i = [n]_q$ to $ax_i + by_i = n$ as $q \to 1$.

Note that Remark 4 has been used to achieve brevity in Theorem 4. Also note that $x_M = x_c = x_0 - cb'$ and $y_M = y_c = y_0 + ca'$ when we compare the notation of Theorem 4 to that of Theorem 1.

5. Regular subspace partitions and direct sums

In this section, we introduce a class of subspace partitions, which can be obtained through a recursive process, and show that they contain direct sums in a strong sense.

Definition 12. *A subspace partition of $V = \mathbb{F}_q^n$ is called* regular *if it can be recursively obtained by applying a construction of the kind described below a finite number, m, of times: Initially, set the trivial subspace partition $\Pi_0 = \{V\}$ of V. At any stage i, where $1 \le i < m$, select a subspace $W \in \Pi_{i-1}$ with $\dim W = t$ and a positive integer d such that $t \ge 2d$, and then replace W with a subspace partition $\mathcal{L}$ of type $(t-d)^1 d^{q^{t-d}}$ to obtain a subspace partition $\Pi_i = (\Pi_{i-1} \setminus \{W\}) \cup \mathcal{L}$ of V.*

Remark 13. *Note that Definition 12 is justified by the fact that the partitions $\mathcal{L}$ of type $T = (t-d)^1 d^{q^{t-d}}$ exist. For instance, one can use Proposition 8. Thus, basic subspace partitions (see Definition 9) are regular. However, a subspace partition of type T need not be obtained via Proposition 8, which produced very specific subspace partitions of type T. Indeed, Bu* [9, Lemma 4] *gives a different construction for obtaining a subspace partition of type T by intersecting a $(t-d)$-spread Γ of $\mathbb{F}_q^{2(t-d)}$ with some $(t-d)$-subspace U that belongs to Γ. Moreover, if $t = 2d$, then $T = d^{q^d+1}$ and $\mathcal{L}$ is a d-spread, and there are several different constructions of spreads (e.g., see* [4, 19] *and* [11, Section 1]*).*

Proposition 14. *For positive integers t and d with $t \ge 2d$, let Π be an arbitrary subspace partition of $W = \mathbb{F}_q^t$ of type $(t-d)^1 d^{q^{t-d}}$. Then for any subspace $X \in \Pi$, there exists a subspace $Y \in \Pi$ such that $W = X \oplus Y$.*

Proof. If $\dim X = t-d$, then let Y be any other subspace in Π. Since Π has type $(t-d)^1 d^{q^{t-d}}$, we must have $\dim Y = d$. Also, as $X \cap Y = \{0\}$ and $\dim X + \dim Y = \dim W$, we have $W = X \oplus Y$. On the other hand, if $\dim X = d$, then let Y be the only subspace in Π such that $\dim Y = t-d$. Thus, $W = X \oplus Y$ as argued above. $\square$

Note that the Gaussian partition corresponding to the subspace partition in Proposition 14 satisfies the packing condition

$$[t-d]_q \cdot 1 + [d]_q \cdot q^{t-d} = [t]_q.$$

Theorem 5. *Every regular subspace partition Π of $V = \mathbb{F}_q^n$ contains a direct sum. Moreover, every subspace of Π occurs as a summand in some direct sum contained in Π.*

Proof. By Definition 12, we may proceed by induction on the number, m, of applications of the construction that is used to arrive at Π. For $m = 1$, starting with $\Pi_0 = \{V\}$ and applying the construction to $W = V$, we obtain a subspace partition Π_1 that contains a direct sum with summand X_1 for any $X_1 \in \Pi_1$ by Proposition 14.

Now let Π_k denote a subspace partition of V obtained after $m = k$ applications of the construction in Definition 12 for some $k \ge 1$. Furthermore, assume that we can always find a direct sum contained in Π_k with summand X_k for any $X_k \in \Pi_k$. Consider another application of the construction to some arbitrary subspace $W \in \Pi_k$, yielding a subspace partition $\mathcal{L}_{k+1}$ of W. Thus, $\Pi_{k+1} = (\Pi_k \setminus \{W\}) \cup \mathcal{L}_{k+1}$ is the resulting subspace partition of V. It remains to show that Π_{k+1} contains a direct sum with summand X_{k+1} for any $X_{k+1} \in \Pi_{k+1}$. There are two possibilities for a given subspace X_{k+1}. (a) If X_{k+1} belongs to $\Pi_{k+1} \setminus \mathcal{L}_{k+1} = \Pi_k \setminus \{W\}$, then it sits inside a direct sum D_k equal to V contained within Π_k by the induction hypothesis, so we consider the remaining summands of D_k. If these summands are also in $(\Pi_k \setminus \{W\}) \subset \Pi_{k+1}$, then Π_{k+1} contains the direct sum D_k, and we are done. The only other possibility is that D_k also contains the summand W. Then, we use the base case of the induction hypothesis to write $W = X \oplus Y$, for some subspaces $X, Y \in \mathcal{L}_{k+1}$, and replace W with these two summands in D_k. This gives us a direct sum D_{k+1} in Π_{k+1}, which includes the summand X_{k+1}. (b) Suppose that $X_{k+1} \in \mathcal{L}_{k+1}$. There exists a direct sum D_k contained in Π_k that has W as a summand by the induction hypothesis, and W can be written as $W = X_{k+1} \oplus Y$ for some $Y \in \mathcal{L}_{k+1}$ by the base case ($m = 1$). Once again, we replace the summand W in D_k with $X_{k+1} \oplus Y$ and obtain a direct sum D_{k+1} in Π_{k+1} with summand X_{k+1}. This proves the induction step and completes the proof of the theorem. □

6. Subspace Partitions that do not contain a direct sum

In this section only we are using Beutelspacher's convention: the *type* of a subspace partition Π is the set of *distinct* dimensions of the subspaces in Π.

By Theorem 1, we know that direct sums exist in a subspace partition of $\mathbb{F}_q^n$ with two distinct dimensions a and b when there exists an integer partition of n with distinct parts a and b, subject to some natural constraints on v, the number of subspaces of dimension b. Let $\mathcal{S}$ once again denote the set of all nonnegative integer solutions of $ax + by = n$. In this section, we will use the work of Beutelspacher [6] to show that even when $\mathcal{S} = \emptyset$, there may still exist a subspace partition of $\mathbb{F}_q^n$, which necessarily excludes a direct sum due to the impossibility of finding dimensions that add up to n.

In his work on the necessary and sufficient conditions for the existence of certain subspace partitions, Beutelspacher [6] constructed subspace partitions that contain exactly two distinct dimensions, which can be obtained by "regular construction" and "intersection by a hyperplane" operations. He used the notion of the *Frobenius number* $g(A)$ of a set A of positive integers with $\gcd(A) = 1$, which is the largest integer that cannot be written as a linear combination of elements of A with nonnegative integer coefficients. For example, if $\gcd(a, b) = 1$, then $g(a, b) = ab - a - b$. Thus, $g(a, a-1) = a^2 - 3a + 1$, meaning that any subspace partition Π_a of $\mathbb{F}_q^{a^2-3a+1}$ of type $\{a, a-1\}$ cannot possibly contain a direct sum. The existence of the subspace partitions Π_a follows from Lemma 15 and Lemma 17, both of which are due to Beutelspacher [6].

Lemma 15 (Beutelspacher [6, Lemma 4])**.** *Let k, r, a be integers with $0 \le k \le r \le a - 1$ and $a \ge 3$. Suppose that there exists a subspace partition Π in $V = \mathbb{F}_q^{2a+2r-k-1}$ of type*

$$\{a + r - k,\ a + r - k - 1\}$$

with $|\Pi| = q^{a+r} + 1$. Then there is a hyperplane $V' = \mathbb{F}_q^{2a+2r-k-2}$ of V, in which Π induces a partition Π' of type

$$\{a + r - k - 1,\ a + r - k - 2\}.$$

Remark 16. (1) *The induced partition Π' must have the same total number of subspaces as Π, since the smallest dimension of a subspace in Π is at least 2.*

(2) *Solving the packing condition*

$$[a + r - k]_q u + [a + r - k - 1]_q \underbrace{(q^{a+r} + 1 - u)}_{v} = [2a + 2r - k - 1]_q,$$

we obtain

$$u = [k + 1]_q \text{ and } v = q^{a+r} + 1 - [k + 1]_q.$$

(3) *Replacing k by $k+1$, we determine the solution of the induced packing condition*

$$[a+r-k-1]_q u + [a+r-k-2]_q \underbrace{(q^{a+r}+1-u)}_{v} = [2a+2r-k-2]_q$$

to be

$$u = [k+2]_q \text{ and } v = q^{a+r}+1-[k+2]_q.$$

The next lemma is also due to Beutelspacher [6], and it follows from Lemma 15.

Lemma 17 (Beutelspacher [6, Example 1])**.** *Let a and r be integers such that $a \geq 3$ and $0 \leq r \leq a-1$. Then,*
(i) $\mathbb{F}_q^{2a-1+r}$ admits a subspace partition into $u = [r+1]_q$ subspaces of dimension a and $q^{a+r}+1-[r+1]_q$ subspaces of dimension $a-1$.
(ii) $\mathbb{F}_q^n$ admits a subspace partition of type $\{a, a-1\}$ if and only if $n \geq 2a-1$ (this is trivially true for $a=2$ as well).

Example 18. *Let $a \geq 5$ be an integer and $n = a^2-3a+1$. Then $\mathbb{F}_q^n$ admits a subspace partition of type $\{a, a-1\}$ that does not contain a direct sum.*

Proof. Since $a \geq 5$, we have $n = a^2-3a+1 \geq 2a-1$. Thus, it follows from Lemma 17 that $\mathbb{F}_q^n$ has a subspace partition Π_a of type $\{a, a-1\}$. Also, as $n = a^2-3a+1 = g(a, a-1)$, n cannot be written as a linear combination of a and $a-1$ with nonnegative coefficients. Therefore, the subspace partition Π_a of $\mathbb{F}_q^n$ cannot contain a direct sum. □

Even with no direct sum contained in Π_a, there is a connection to our main result. We write

$$n = a^2-3a+1 = 2a-1+\underbrace{(a-5)}_{k}(a-1)+\underbrace{(a-3)}_{r},$$

so Remark 16 and the proof of Lemma 17 show that initially we have $[r+1]_q = [a-2]_q$ a-subspaces, which may be multiplied by a power of q during the last stage. As $q \to 1$, we have $u \to x = a-2$, so that we can solve for the limit y of v as $q \to 1$ from

$$a(a-2)+(a-1)y = a^2-3a+1 \Longrightarrow y = -1.$$

That is, the packing condition

$$[a]_q u + [a-1]_q v = [n]_q$$

is reduced to the *signed integer partition* (see Andrews [5])

$$a^{a-2}(a-1)^{-1}$$

of $n = a^2-3a+1$.

7. Proof of Theorem 1

In this section, we prove our main theorem, Theorem 1. To that end, we will prove Lemma 19 (see Section 7.1) and Lemma 21 (see Section 7.2).

7.1. **Subspace partitions of type $a^u b^v$ between two extreme basic solutions.** Our goal is to show that if the Diophantine equation $ax + by = n$ has a nonempty set $\mathcal{S}$ of nonnegative solutions for given $a > b > 0$, then any subspace partition of $\mathbb{F}_q^n$ of type $a^u b^v$ must contain a direct sum, provided that v is in the second interval (given in Theorem 1)

$$I_2 = \left[\frac{q^n - q^{ax_0}}{q^b - 1}, \frac{q^n - q^{ax_M}}{q^b - 1}\right] = \left[q^{ax_0}[y_0]_{q^b}, q^{ax_M}[y_M]_{q^b}\right],$$

where $y_0 = \min_{(x,y)\in\mathcal{S}} y$, $y_M = \max_{(x,y)\in\mathcal{S}} y$, $x_0 = (n - ay_0)/b$, and $x_M = (n - ay_M)/b$. Note that there is also a corresponding interval J_2 that contains u, which is the counterpart of v in the partition type $a^u b^v$. Let us use the notation of Theorem 4, which is where the intervals I_2 and J_2 originate from. Given basic Gaussian partitions $a^{u_i(q)} b^{v_i(q)}$, we will let the expressions $u_i(q)$ and $v_i(q)$ act as endpoints of subintervals of J_2 and I_2, respectively. More precisely, we will consider those (u, v) for which $u_i(q) \le u \le u_{i-1}(q)$ and $v_{i-1}(q) \le v \le v_i(q)$. If there exists a subspace partition Π of type $a^u b^v$ where u and v fall into one of these pairs of subintervals, then we will show that Π must contain a direct sum. Since (i) the endpoints are polynomials in $\mathbb{Z}[q]$ with nonnegative coefficients, (ii) $r, s \in \mathbb{Z}$ with $0 \le r \le s$ implies that $0 \le [r]_q \le [s]_q$ for all $q \ge 1$ in $\mathbb{Z}$, and (iii) expansions in base $q \ge 2$ are unique, we are justified in using these inequalities.

Lemma 19. *Let $a, b, n, d, a', b', c, x_0, y_0$ be as in Theorem 4, and assume that Π is a subspace partition of $V = \mathbb{F}_q^n$ of type $a^u b^v$. For a fixed integer i, $1 \le i \le c$, let u and v satisfy the conditions*

$$[x_0 - ib']_{q^a} = u_i(q) \le u \le u_{i-1}(q) = [x_0 - (i-1)b']_{q^a}$$

and

$$q^{a(x_0-(i-1)b')}[y_0 + (i-1)a']_{q^b} = v_{i-1}(q) \le v \le v_i(q) = q^{a(x_0-ib')}[y_0 + ia']_{q^b}.$$

Then Π contains a direct sum of the form

$$U_1 \oplus \cdots \oplus U_{y_0+ia'} \oplus W_1 \oplus \cdots \oplus W_{x_0-ib'},$$

where the U_j's and W_j's are b-subspaces and a-subspaces of Π respectively.

Proof. Let $W_1, \ldots, W_u$ be the list of all a-subspaces of the subspace partition Π. By reordering the W_i's if necessary, we may assume that

$$\widehat{W} = W_1 \oplus \cdots \oplus W_t$$

is a maximal direct sum by a-subspaces of Π for some t. This implies that $\dim(W_i \cap \widehat{W}) \ge 1$ for $t < i \le u$. Since each 1-subspace of V belongs to exactly one subspace in Π, the number N of 1-subspaces X of V such that $X \subseteq \widehat{W}$ and $X \not\subseteq W_1 \cup \cdots \cup W_t$ must satisfy

$$N = [at]_q - t[a]_q \ge (u-t)[1]_q = u - t \iff [at]_q + t \ge u + t[a]_q. \tag{5}$$

Now, we claim that the number t of the direct summands of dimension a satisfies $t \ge x_0 - ib'$. Suppose not; that is, $t \le x_0 - ib' - 1$. Then by uniqueness of base-q expansions, we have

$$\begin{aligned} at \le a(x_0 - ib' - 1) &\iff 1 + q + \cdots + q^{at-1} < 1 + q^a + \cdots + q^{a(x_0-ib'-1)} \\ &\implies [at]_q < [x_0 - ib']_{q^a}. \end{aligned} \tag{6}$$

Since $t < t[a]_q$ and $u \ge u_i(q) = [x_0 - ib']_{q^a}$, it follows from (6) that

$$[at]_q + t < u + t[a]_q. \tag{7}$$

Since (5) contradicts (7), we must have $t \geq x_0 - ib'$. Next, fix a direct sum with $x_0 - ib'$ summands W_i, say $W_1 \oplus \cdots \oplus W_{x_0-ib'}$, and maximize the direct sum

$$W_1 \oplus \cdots \oplus W_{x_0-ib'} \oplus U_1 \oplus \cdots \oplus U_s$$

by adding as many b-subspaces U_j of Π as possible. We claim that $s = y_0 + ia'$. Suppose not; then $s \leq y_0 + ia' - 1$, and by arguing as for the direct sum $\widehat{W}$ above, it suffices to show that

$$[a(x_0 - ib') + bs]_q - (x_0 - ib')[a]_q - s[b]_q < v - s,$$

or simply,

$$\begin{aligned} & [a(x_0 - ib') + bs]_q < v_{i-1}(q) = q^{a(x_0-(i-1)b')}[y_0 + (i-1)a']_{q^b} \\ \iff & 1 + q + \cdots + q^{a(x_0-ib')+bs-1} < q^{a(x_0-(i-1)b')}(1 + q^b + \cdots + q^{b(y_0+(i-1)a'-1)}). \end{aligned}$$

This last statement will be true if

$$\begin{aligned} & a(x_0 - ib') + bs - 1 < b(y_0 + (i-1)a' - 1) + a(x_0 - (i-1)b') \\ \iff & ax_0 - iab' + bs \leq by_0 + ia'b - a'b - b + ax_0 - iab' + ab' \\ & (\text{where } ab' = a'db' = a'b) \\ \iff & bs \leq by_0 + bia' - b \\ \iff & bs \leq b(y_0 + ia' - 1) \\ \iff & s \leq y_0 + ia' - 1. \end{aligned}$$

□

Corollary 20 (Spreads). *Every a-spread of $\mathbb{F}_q^n$ contains a direct sum.*

7.2. **The other kind of subspace partitions of type $a^u b^v$.** We now consider the case when v is in the first interval (see Theorem 1)

$$I_1 = \left[\frac{q^{by_0} - 1}{q^b - 1}, \frac{q^{by_M} - 1}{q^b - 1}\right] = \left[[y_0]_{q^b}, [y_M]_{q^b}\right], \tag{8}$$

where $x_0 = \max_{(x,y)\in\mathcal{S}} x$, $x_M = \min_{(x,y)\in\mathcal{S}} x$, $y_0 = (n - ax_0)/b$, and $y_M = (n - ax_M)/b$ (once again, there is an interval J_1 for the matching values of u). Then, by the identity $by_0 + ax_0 = n$ and Proposition 6, the solution (u_0, v_0) of the packing condition $[a]_q u + [b]_q v = [n]_q$ can be obtained as follows (note the reversal of order):

$$[y_0]_{q^b}[b]_q + q^{by_0}[x_0]_{q^a}[a]_q = [n]_q.$$

We will consider a sequence of intervals where v increases from $v_0 = [y_0]_{q^b}$ to $[y_0 + a']_{q^b}$, then from $[y_0 + a']_{q^b}$ to $[y_0 + 2a']_{q^b}$, etc., until we reach $v_c = [y_0 + ca']_{q^b} = [y_M]_{q^b}$ for $c = \lfloor x_0/b' \rfloor$, to prove that subspace partitions of type $a^u b^v$ in each interval contain direct sums.

Table 1

(x_i, y_i)	$u_i(q)$	$v_i(q)$	$k_i(q)$
$(x_0 - ib', y_0 + ia')$	$q^{b(y_0+ia')}[x_0 - ib']_{q^a}$	$[y_0 + ia']_{q^b}$	$q^{by_0} \frac{[ia'b']_{q^d}}{[a']_{q^d}[b']_{q^d}}$

Table 1 displays the endpoints of the intervals corresponding to the integer partitions of n into at most two distinct parts of sizes a and b: all of the solutions $(u_i(q), v_i(q))$ are obtained directly from Proposition 6, where the order in which we construct the solution is exactly the opposite of that for basic subspace partitions. Note that, unlike the situation in Section 4, these solutions do not necessarily correspond to actual subspace partitions of type $a^{u_i(q)}b^{v_i(q)}$. However, there might exist subspace partitions of type $a^u b^v$ with $u_{i+1}(q) \leq u \leq u_i(q)$ and $v_i(q) \leq v \leq v_{i+1}(q)$. While basic subspace partitions have the advantage of regular constructibility, the reverse-ordering stretches the solutions of the packing condition to their extremal values, which are, in some cases, known to be existing Gaussian partitions. We have also included in the table the polynomial $k_i(q)$ that serves as the link between the particular solution (u_0, v_0) and the ith solution (u_i, v_i) for $0 \leq i \leq c$:

$$u_i(q) = u_0 - k_i(q)\frac{[b]_q}{[d]_q} \text{ and } v_i(q) = v_0 + k_i(q)\frac{[a]_q}{[d]_q}.$$

As in the basic case, we find that $(u_i, v_i) \to (x_i, y_i)$, and $k_i(q) \to i$ as $q \to 1$. The proof of Lemma 21 is essentially the same as the proof of Lemma 19, except that we are dealing with slightly different endpoints for the intervals. Thus, we only give a sketch of this proof and refer to the proof of Lemma 19 for the omitted parts.

Lemma 21. *Let n, a, b be positive integers such that the Diophantine equation $ax + by = n$ has nonnegative solutions, and (x_0, y_0) be the solution where x_0 is maximal and y_0 is minimal; set $d = \gcd(a, b)$, so that there exist positive integers a', b', n' satisfying $a = a'd$, $b = b'd$, and $n = n'd$. Moreover, let $c = \lfloor x_0/b' \rfloor$ and fix an integer i, $0 \leq i \leq c-1$. If Π is a subspace partition of $V = \mathbb{F}_q^n$ of type $a^u b^v$, with*

$$q^{b(y_0+(i+1)a')}[x_0 - (i+1)b']_{q^a} = u_{i+1}(q) \leq u \leq u_i(q) = q^{b(y_0+ia')}[x_0 - ib']_{q^a}$$

and

$$[y_0 + ia']_{q^b} = v_i(q) \leq v \leq v_{i+1}(q) = [y_0 + (i+1)a']_{q^b},$$

then Π contains a direct sum of the form

$$V = U_1 \oplus \cdots \oplus U_{y_0+ia'} \oplus W_1 \oplus \cdots \oplus W_{x_0-ib'},$$

where the U_j's and W_j's are b-subspaces and a-subspaces of Π respectively.

Proof. Let $U_1, \ldots, U_v$ be the list of all b-subspaces of the subspace partition Π. By reordering the U_i's if necessary, we may assume that

$$\widehat{U} = U_1 \oplus \cdots \oplus U_s$$

is a maximal direct sum by b-subspaces of Π. By arguing as in the first part of the proof of Lemma 19, we infer that $s \geq y_0 + ia'$. Now fix a direct sum with $y_0 + ia'$ summands U_i, say $U_1 \oplus \cdots \oplus U_{y_0+ia'}$, and maximize the direct sum

$$U_1 \oplus \cdots \oplus U_{y_0+ia'} \oplus W_1 \oplus \cdots \oplus W_t$$

by adding as many a-subspaces W_j of Π as possible. We claim that $t = x_0 - ib'$. Suppose not; that is, $t \leq x_0 - ib' - 1$. Then by arguing as in the second part of the proof of Lemma 19, we infer that $t = x_0 - ib'$. □

Proof of Theorem 1. Since its converse direction is trivial, Theorem 1 follows from Lemmas 19 and 21. □

We conclude this section with the following corollary, which follows from Theorem 1 by setting $b = 1$.

Corollary 22 (Partial spreads)**.** *Any subspace partition of $\mathbb{F}_q^n$ of type $a^u 1^v$ contains a direct sum.*

Proof. Write $n = am + r$, where m and r are integers such that $m \geq 1$ and $0 \leq r < a$. Since $b = 1$, it follows that $y_0 = \min_{(x,y)\in\mathcal{S}} y = r$ and $y_M = \max_{(x,y)\in\mathcal{S}} y = n$. Let Π be an arbitrary subspace partition of $\mathbb{F}_q^n$ of type $a^u 1^v$. Since $v \geq [a]_q > [r]_q = [y_0]_{q^b}$ by Heden's theorem [14, Theorem 1], it follows that

$$[y_0]_{q^b} \leq v \leq [n]_q = [y_M]_{q^b}.$$

Thus, Π contains a direct sum by Theorem 1. □

8. Why these special bounds on v?

In this section, we give some justification of our choices for the lower and upper bounds on the number v of subspaces of dimension b in our main theorem (Theorem 1).

First and foremost, the bounds for v in Theorem 1 are combinatorially natural, in that they showcase the intimate relationship between integer and Gaussian partitions: for every (nonnegative) integer partition of n with parts a and b, there is a range of Gaussian partitions containing a direct sum whose summands have dimensions matching multiplicities of the integer partition, which can be accomplished in two different ways! The boundaries are polynomials in $\mathbb{Z}[q]$ and are positive for all $q \geq 1$ (or identically zero in case of a spread), and each pair (u_i, v_i) of q-polynomial values for $1 \leq i \leq c$ is related to the minimal solution (u_0, v_0) via a positive polynomial $k_i(q)$ that goes to i as $q \to 1$, which mimics the relationship of the corresponding solution (x_i, y_i) of $ax + by = n$ to (x_0, y_0). Note that although we only need prime-power values of q in an actual partition, the fact that we have *positive polynomial* (or identically zero) endpoint solutions $(u_i(q), v_i(q))$, as well connecting polynomials $k_i(q)$, makes it possible for us to substitute $q = 1$ and realize the connection to a –nonnegative– integer partition.

8.1. The lower bound.

As mentioned in Remark 2, there are currently no known subspace partitions of $V = \mathbb{F}_q^n$ of type $a^u b^v$ for which $v < [y_0]_{q^b}$. Hence, the lower bound $v \geq [y_0]_{q^b}$, where $[y_0]_{q^b}$ is the minimum of the endpoints of the intervals I_1 and I_2 in Theorem 1, is a reasonable one in practice as well. For $b = 1$, this lower bound becomes $v \geq [y_0]_q$, where y_0 is the remainder of n on division by a, and x_0 is the quotient. Thus, the "next" solution (u, v) of the packing condition $[a]_q u + [1]_q v = [n]_q$ for smaller v satisfies

$$v = v_0 - [a]_q = [y_0]_q - [a]_q < 0$$

for all $q \geq 1$, since $0 \leq y_0 < a$. (Also see Eisfeld and Storme [11] for this $b = 1$ case.) Even when $b > 1$, the next solution

$$v = [y_0]_{q^b} - [a']_{q^d}$$

(where $d = \gcd(a, b)$, $a = a'd$, and $b' = bd$) is a polynomial function of q that is negative for some values of the parameters q, n, a, and b, where $a > b$ and $q \geq 1$. More precisely, we have

$$\begin{aligned} v = [y_0]_{q^b} - [a']_{q^d} < 0 &\iff \frac{q^{y_0 b} - 1}{q^b - 1} - \frac{q^a - 1}{q^d - 1} < 0 \\ &\iff \frac{-q^{a+b} + q^{y_0 b + d} - q^{y_0 b} + q^a + q^b - q^d}{(q^b - 1)(q^d - 1)} < 0, \end{aligned}$$

which holds if $y_0 = 1$ or if $a > b(y_0 - 1) + d$ and $q \geq 2$. Moreover, a theorem of Heden [14, Theorem 1] shows that for any partition of V type $a^u b^v$ with $a > b$, we have $v \geq q^b + 1$ if $a < 2b$ and $v \geq (q^a - 1)/(q^b - 1)$ if $a \geq 2b$. These bounds are tight in some cases (see Heden [14]).

Using a recent result of Năstase et al. [17], it is also possible to show that if $b = 2$, then the condition $v \geq [y_0]_{q^b}$ of Theorem 1 can be dropped whenever $a > [r]_q$, where $r \equiv n \pmod{a}$ and $0 \leq r < a$. For more details, see [18], which studies the problem of finding the maximum/minimum number of a-subspaces (equivalently, the minimum/maximum number of b-subspaces) in a subspace partition of $\mathbb{F}_q^n$ of type $a^u b^v$, where $a > b > 1$.

8.2. **The upper bound.** In this subsection, we shall use Lemma 17 in Section 6 to construct an infinite class of subspace partitions of $V = \mathbb{F}_q^n$ of type $a^u b^v$ such that $v > \max(I_1 \cup I_2)$ and which do not contain direct sums.

Lemma 23. *Let q be a prime power, n, a, t, d be integers such that $t \geq 0$, $b = a - 1 > d > 2$, and $n = da + tb^2$. Let $\mathcal{S}$ be the set of all solutions (x, y) of the Diophantine equation $ax + by = n$ with $x, y \geq 0$. Then, for c as in Theorem 4, the following properties hold:*

(1)
$$\mathcal{S} = \left\{ (d + ib, tb - ia) : \ i \in \mathbb{Z} \text{ and } 0 \leq i \leq \left\lfloor \frac{tb}{a} \right\rfloor \right\}.$$
$$x_0 = \max_{(x,y)\in\mathcal{S}} x = d + \left\lfloor \frac{tb}{a} \right\rfloor b \text{ and } y_0 = \min_{(x,y)\in\mathcal{S}} y = tb - \left\lfloor \frac{tb}{a} \right\rfloor a.$$
$$x_M = \min_{(x,y)\in\mathcal{S}} x = d \text{ and } y_M = \max_{(x,y)\in\mathcal{S}} y = tb.$$
$$c = \left\lfloor \frac{x_0}{b} \right\rfloor = \left\lfloor \frac{tb}{a} \right\rfloor.$$

(2) *There exists a subspace partition $\Pi_{t,d}$ of $V = \mathbb{F}_q^n$ of type $a^u b^v$, where $b = a - 1$, $u = [d]_q$, and $v = q^{b+d}[tb + d - 1]_{q^b} + 1 - [d]_q$. Moreover, $\Pi_{t,d}$ does not admit a direct sum.*

Remark 24. *Note that the parameter c that appears in Lemma 23 can be arbitrarily large. In particular, $c = 0$ if $0 \leq t \leq 1$ and $c > 0$ if $t > 1$.*

Proof of Lemma 23. For the proof of (1), note that $(x, y) = (d, tb) \in \mathcal{S}$, and all solutions of $ax + by = n$ are of the form
$$(d + ib, tb - ia), \quad i \in \mathbb{Z},$$
since $\gcd(a, b) = \gcd(a, a - 1) = 1$. Moreover, $x_M = d$, as $d - b < 0$, which makes $y_M = tb$. These values correspond to the smallest value 0 of i. Then the largest value c (as in Theorem 4) of i must be the one that satisfies
$$\begin{aligned} &tb - ca \geq 0 \text{ and } tb - (c + 1)a < 0 \\ \iff\ & c \leq \frac{tb}{a} \text{ and } c + 1 > \frac{tb}{a} \\ \iff\ & c = \left\lfloor \frac{tb}{a} \right\rfloor = \left\lfloor \frac{x_0}{b} \right\rfloor \quad \text{(by Theorem 4)}. \end{aligned}$$
Hence, we have $x_0 = d + cb$ and $y_0 = tb - ca$.

For the proof of (2), we set $m = 2a + d - 2 = 2b + d$, and write

$$n = d(b+1) + tb^2 = \underbrace{(2b+d)}_{m} + (tb + d - 2)b.$$

Since $m = 2b + d > b$, we can apply Proposition 8 recursively to obtain a subspace partition of $V = \mathbb{F}_q^n$ of type $m^1 b^{(q^n - q^m)/(q^b-1)}$. On the other hand, it follows from part (i) of Lemma 17 (applied with $r = d-1$) that $\mathbb{F}_q^m = \mathbb{F}_q^{2a-1+(d-1)}$ admits a subspace partition into $u = [d]_q$ subspaces of dimension a and $q^{b+d} + 1 - [d]_q$ subspaces of dimension b. Thus, $V = \mathbb{F}_q^n$ admits a subspace partition $\Pi_{t,d}$ of type $a^u b^v$, where $u = [d]_q$, and

$$v = \frac{q^n - q^m}{q^b - 1} + q^{b+d} + 1 - u = q^{b+d}[tb + d - 1]_{q^b} + 1 - [d]_q.$$

Moreover, the above construction guarantees that the a-subspaces of $\Pi_{t,d}$ belong to a subspace of V of dimension $m = 2b + d$.

Now assume that $\Pi_{t,d}$ admits a direct sum. Since $x \geq x_M = d$ for any $(x, y) \in \mathcal{S}$, the number of a-subspaces in the direct sum must span a subspace of dimension $xa \geq da$. When $d > 2$, this is impossible, as the a-subspaces of $\Pi_{t,d}$ belong to a subspace of V of dimension $2b + d$ (as observed in the preceding paragraph) and $2b + d < da \leq xa$. $\square$

Unlike the case with the lower bound, we have seen that subspace partitions of type $a^u b^v$ may exist when $\mathcal{S} \neq \emptyset$ and v is greater than the larger of the two upper bounds (we shall verify the position of v shortly). Since we have $b \geq 4$ and

$$[x_M]_{q^b} = [d]_{q^b} > [d]_q = u,$$

it follows that

$$v > q^{ax_M}[y_M]_{q^b},$$

which is the maximum of $I_1 \cup I_2$ in Theorem 1 by Remark 2. Hence, the subspace partitions $\Pi_{t,d}$ of $\mathbb{F}_q^n$ (see Lemma 23) fall outside the region of Theorem 1, and they do not have direct sums, either. Since the basic subspace partitions given by Theorem 4 satisfy $v \leq q^{ax_M}[y_M]_{q^b}$, it follows from Lemma 23 and the above discussion that for each $b \geq 4$, the upper bound $q^{ax_M}[y_M]_{q^b}$ is necessary and sharp. If $b = 1$, then it follows from Corollary 22 that the bounds on v in Theorem 1 can be dropped. For $b \in \{2, 3\}$, we do not know whether those bounds can be dropped or not.

8.3. **Generalization.** Considering the availability of exact formulas for the solutions of linear Diophantine equations in many variables that generalize the two-variable case (again, see Bond [8]), and in light of Remark 2, we make the following conjecture.

Conjecture 25. *Let q be a prime power, n, h, $a_1, \ldots, a_h$ be positive integers with $h \geq 2$, and Π be a subspace partition of $V = \mathbb{F}_q^n$ of type $a_1^{u_1} \cdots a_h^{u_h}$. Define $\mathcal{S}$ to be the set of all solutions $(x_1, \ldots, x_h)$ of the Diophantine equation $a_1x_1 + \cdots + a_hx_h = n$ with $x_i \geq 0$ for $1 \leq i \leq h$. If $\mathcal{S} \neq \emptyset$ and $u_1, \ldots, u_{h-1}$ have positive q-polynomial bounds determined by the unions of intervals arising from different applications of Proposition 6, then Π contains a direct sum.*

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