ON THE TYPE(S) OF MINIMUM SIZE SUBSPACE PARTITIONS

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Abstract. Let $V = V(kt + r, q)$ be a vector space of dimension $kt + r$ over the finite field with q elements. Let $\sigma_q(kt + r, t)$ denote the minimum size of a subspace partition $\mathcal{P}$ of V in which t is the largest dimension of a subspace. We denote by n_{d_i} the number of subspaces of dimension d_i that occur in $\mathcal{P}$ and we say $[d_1^{n_{d_1}}, \ldots, d_m^{n_{d_m}}]$ is the *type* of $\mathcal{P}$. In this paper, we show that a partition of minimum size has a unique partition type if $t + r$ is an even integer. We also consider the case when $t + r$ is an odd integer, but only give partial results since this case is indeed more intricate.

1. Introduction

Let $V = V(n, q)$ denote a vector space of dimension $n = kt + r$ over the finite field with q elements. A *subspace partition* $\mathcal{P}$ of V, also known as a *vector space partition*, is a collection of nonzero subspaces of V such that each *point*, that is, 1-dimensional subspace, of V is in exactly one subspace of $\mathcal{P}$. We denote by n_{d_i} the number of subspaces of dimension d_i that occur in $\mathcal{P}$ and we say $[d_1^{n_{d_1}}, \ldots, d_m^{n_{d_m}}]$ is the *type* of $\mathcal{P}$, where $d_1 < \ldots < d_m$ and $n_i > 0$ for $1 \leq i \leq m$. The *size* of a subspace partition $\mathcal{P}$ is the number of subspaces in $\mathcal{P}$. Let $\sigma_q(n, t)$ denote the *minimum size* of a subspace partition of V in which the largest subspace has dimension t.

Generalizing a theorem in [8], the following theorem was proved by the authors of the present paper in [4].

Theorem 1. *Let $n, k, t,$ and r be integers such that $1 \leq r < t$, $k \geq 1$, and $n = kt + r$. Then*

$$\sigma_q(n, t) = q^t + 1 \quad \text{for } n < 2t ,$$

and

$$\sigma_q(n, t) = q^{t+r} \sum_{i=0}^{k-2} q^{it} + q^{\lceil \frac{t+r}{2} \rceil} + 1 \text{ for } n \geq 2t .$$

The question studied here is whether or not a subspace partition of minimum size, that is, attaining the lower bound given in Theorem 1, is of a type which just depends on n and t. We found that this is true in many cases and, in particular, when $t + r$ is an even integer. The case when $t+r$ is odd turned out to be more intricate and we obtain just partial answers in this case.

Let $\ell = q^r \sum_{i=0}^{k-2} q^{it}$. Our main results are thus the following theorems:

Theorem 2. *Let n, k, t, and r be integers such that $1 \le r < t$, $k \ge 2$, $t+r = 2s$ for some integer s, and $n = kt+r$. Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of size $\sigma_q(n,t)$ and with maximum subspace dimension t. Then $\mathcal{P}$ has type $[s^{n_s}, t^{n_t}]$, where*

$$n_s = q^s + 1, \quad \text{and} \quad n_t = \ell q^t.$$

Theorem 3. *Let n, k, t, and r be integers such that $r = t-1$, $k \ge 2$, and $n = kt+r$. Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of size $\sigma_q(n,t)$ and with maximum subspace dimension t. Then $\mathcal{P}$ has type $[(t-1)^{n_{t-1}}, t^{n_t}]$, where*

$$n_{t-1} = q^t, \quad \text{and} \quad n_t = \ell q^t + 1.$$

When $r < t-1$ we obtain the following result:

Theorem 4. *Let n, k, t, and r be integers such that $1 \le r < t-1$, $k \ge 2$, and $t+r = 2s-1$ for some integer s. Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of size $\sigma_q(n,t)$ and with maximum subspace dimension t. If the number of subspaces of dimension t is $n_t = \ell q^t$, then $\mathcal{P}$ has type $[(s-1)^{n_{s-1}}, s^1, t^{n_t}]$, where*

$$n_{s-1} = q^s, \quad \text{and} \quad n_t = \ell q^t.$$

It must be remarked that subspace partitions of types as indicated in the three previous theorems indeed exist and are well known, see Section 2.1 for a construction of them.

When $t+r$ is odd and $n_t < \ell q^t$, we were able to derive only two new non-trivial necessary conditions:

Theorem 5. *Let n, k, t, and r be integers such that $1 \le r < t-1$, $k \ge 2$, and $t+r = 2s-1$ for some integer s. Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of size $\sigma_q(n,t)$.*

(1) *If $\mathcal{P}$ has type $[a^{n_a}, t^{n_t}]$, then $a = t-1$,*

$$n_a = q^{t-1} + q^{t-2} + 1, \quad \text{and} \ n_t = \ell q^t - q^{t-2}.$$

(2) *If $\mathcal{P}$ has type $[a^{n_a}, b^{n_b}, t^{n_t}]$, then $a = s-1$, $b = s$,*

$$n_a = q^s - \delta q \frac{q^{t-s}-1}{q-1}, \ n_b = \delta \frac{q^{t-s+1}-1}{q-1} + 1, \quad \text{and} \ n_t = \ell q^t - \delta,$$

for some integer δ such that

$$0 \le \delta \le \frac{(q^{s-2}-1)(q-1)}{q^{t-s}-1}.$$

In search for a reasonable conjecture when $t+r$ is odd, and also for the sake of exploring new methods, we first did a computer search in the particular case when $q=2$, $t=5$, $k=2$ and $r=2$, thereby using the Simplex Algorithm on the known necessary linear constraints for existence of subspace partitions. These linear constraints were found by Lehmann and Heden in [7]. This search showed that in these cases the type of a minimum size subspace partition is unique. This experience led us to make the following conjecture:

Conjecture 1. *Let $n, k, t,$ and r be integers such that $1 \leq r < t-1$, $k \geq 2$, and $t+r = 2s-1$ for some integer s. Every minimum size subspace partition $\mathcal{P}$ of $V(n,q)$, with t as the highest dimension in $\mathcal{P}$, is of type*

$$[(t-1)^{q^{t-1}+q^{t-2}+1}, t^{\ell q^t - q^{t-2}}] \text{ or } [(s-1)^{q^s}, s^1, t^{\ell q^t}].$$

As we found the linear programming approach fruitful, we tried to use it in the more general situation, when $t>3, r=t-3$, and q is any prime power. Unfortunately, this led to rather complicated expressions that were tedious to evaluate. However, by using that approach we were able to prove Conjecture 1 for the special case when $k=2$. The details in that proof might be published elsewhere.

2. Some preliminary results

2.1. **The ideal partition.** The following lemma due to Herzog and Schönheim [6] and independently Beutelspacher [1] and Bu [2], ensures the existence of a partition $\mathcal{P}_0$ of V of minimum size.

Lemma 1. *Let n and d be integers such that $1 \leq d \leq n/2$. Then $V(n,q)$ admits a partition with one subspace of dimension $n-d$ and q^{n-d} subspaces of dimension d.*

Let $n = kt+r$ and $s = \lceil (t+r)/2 \rceil$, where $k \geq 2$ and $1 \leq r < t-1$. By a recursive application of this lemma we find a subspace partition $\mathcal{P}_0$ of $V(n,q)$ that consists of

$$b_0 = q^s + 1$$

subspaces of dimension less than or equal to s and

$$a_0 = \ell q^t = q^{t+r} + q^{2t+r} + \ldots + q^{(k-1)t+r} = q^{t+r}\frac{q^{(k-1)t}-1}{q^t-1}$$

spaces of dimension t. When $t+r$ is even, we get ℓq^t subspaces of dimension t and q^s subspaces of dimension s. When $t+r$ is odd, we get ℓq^t subspaces of dimension t, one subspace of dimension s, and q^s subspaces of dimension $s-1$.

The total number of subspaces in $\mathcal{P}_0$ is then

$$\sigma_q(n,t) = a_0 + b_0 = \ell q^t + q^s + 1.$$

We refer to $\mathcal{P}_0$ as the *ideal* partition and often compare our results to this particular partition.

2.2. **Some fundamental lemmas.** We will often use the *packing condition. It* gives a set theoretic necessary condition for the existence of a subspace partition $\mathcal{P}$:

$$|V(n,q)| - 1 = \sum_{U \in \mathcal{P}} (|U| - 1).$$

Throughout this paper we will let $\mathcal{H}$ denote the set of all hyperplanes of $V(n,q)$. For any hyperplane $H \in \mathcal{H}$, let $[b_1^H \dots b_m^H]$ be the *induced type* of H with respect to the partition $\mathcal{P}$, where b_i^H denotes the number of subspaces of dimension d_i in $\mathcal{P}$ that are completely contained in H. Lehmann and Heden observed in [7] that the following relation is useful in the study of subspace partitions:

$$|\mathcal{P}| = 1 + \sum_{i=1}^{m} b_i^H q^{d_i} \ . \tag{1}$$

This relation is called the *second packing condition* and is used in the proof of the next lemma.

Lemma 2. *Let $n, k, t,$ and r be integers such that $k \geq 2, 1 \leq r < t-1$, and $n = kt + r$. Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of size $\sigma_q(n,t)$ and with maximum subspace dimension t. Then*

$$n_t \leq \ell q^t.$$

Note that if $r = t - 1$, the ideal partition contains $n_t = \ell q^t + 1$ subspaces of dimension t.

Proof. Suppose to the contrary that $n_t = \ell q^t + \delta$, for some integer $\delta \geq 1$. By counting pairs (H, W), where W is a subspace of dimension t in $\mathcal{P}$ that is contained in the hyperplane H, we obtain

$$\sum_{H \in \mathcal{H}} b_t^H = n_t \frac{q^{n-t} - 1}{q - 1} = (\ell q^t + \delta) \frac{q^{n-t} - 1}{q - 1}.$$

Since $\ell = (q^{n-t} - q^r)/(q^t - 1)$, the average value of the above sum is

$$b_{\text{ave}} = \frac{\sum_{H \in \mathcal{H}} b_t^H}{|\mathcal{H}|} = \frac{(\ell q^t + \delta)(q^{n-t} - 1)}{q^n - 1} = \ell + \frac{\delta(q^{n-t} - 1) - (q^{n-t} - q^r)}{q^n - 1}.$$

As $\delta \geq 1$ and $q^r > 1$, the expression above is strictly larger than ℓ. Hence, there exists a hyperplane H^* that contains $b_t^{H^*} \geq \ell + 1$ subspaces of dimension t. Thus, it follows from Equation (1) that

$$|\mathcal{P}| \geq 1 + b_t^{H^*} q^t \geq 1 + (\ell + 1) q^t > \ell q^t + q^s + 1,$$

where the last inequality holds since $t > s$. This is a contradiction and thus $\delta < 1$. $\square$

Lemma 3. *Let n, k, t, and r be integers such that $k \geq 2, 1 \leq r < t-1$, and $n = kt + r$. Assume that $t + r$ is an odd integer and let $\mathcal{P}$ be a minimum size subspace partition of $V(n, q)$ consisting of subspaces of dimension t, subspaces of dimension $s = (t + r + 1)/2$, and subspaces of dimension less than s.*

If the number of subspaces of dimension t is $n_t = \ell q^t - \delta$, where $\delta \geq 0$, then the number of subspaces of dimension s is at least equal to

$$b = 1 + \delta \frac{q^{t-s+1} - 1}{q - 1}. \tag{2}$$

Proof. We consider the worst case scenario. By counting the number of vectors in subspaces of dimension t, that must be substituted into vectors of subspaces of dimension s when deleting δ subspaces of dimension t from a partition of the same type as the ideal partition, we get that the number of spaces of dimension s will be at least equal to

$$1 + \delta + \delta \frac{(q^t - 1) - (q^s - 1)}{(q^s - 1) - (q^{s-1} - 1)} = 1 + \delta + \delta q \frac{q^{t-s} - 1}{q - 1}.$$

□

The next result, Lemma 5, is also fundamental in our presentation. The proof of it uses Lemma 4 which was originally proved by Năstase and Sissokho [8] (also see [4, 5]).

Lemma 4. *Let n, k, t, and r be integers such that $k \geq 2, 1 \leq r < t$, and $n = kt + r$. Let $\mathcal{P}$ be a subspace partition of $V(n, q)$ with no subspace of dimension higher than t. Assume furthermore that $\mathcal{P}$ contains a subspace of dimension t and a subspace of dimension d, with $0 \leq d < t$. Then*

$$|\mathcal{P}| \geq \ell q^t + q^d + 1.$$

Lemma 5. *Let n, k, t, and r be integers such that $k \geq 2, 1 \leq r < t - 1$, and $n = kt + r$. Let $\mathcal{P}$ be a subspace partition of $V(n, q)$ of size $\sigma_q(n, t)$ and with maximum subspace dimension t. Then the second largest dimension of a subspace in $\mathcal{P}$ is $s = \lceil (t + r)/2 \rceil$.*

Proof. Let a denote the dimension of the second largest dimension that appear among the members in $\mathcal{P}$. If $a > s$, then it follows from Lemma 4 that

$$|\mathcal{P}| \geq \ell q^t + q^a + 1 > \ell q^t + q^s + 1 = \sigma_q(n, t),$$

which is a contradiction. So we may assume that $a \leq s$. We now show that $a \leq s - 1$ cannot hold. Indeed suppose, $a \leq s - 1$. Since $s \neq t$, it follows from Lemma 2 that $n_t \leq \ell q^t$. Since $0 < a \leq s - 1 < t$, the integer $|\mathcal{P}|$ is minimized when $n_t \leq \ell q^t$ is as large as possible. Thus, by selecting $n_t = \ell q^t$, counting vectors, and using the fact that

$$(q^n - 1) - \ell q^t (q^t - 1) = q^{2s-1} - 1. \tag{3}$$

we obtain

$$\begin{aligned} |\mathcal{P}| &\geq \ell q^t + \frac{(q^n-1)-\ell q^t(q^t-1)}{q^a-1} \\ &\geq \ell q^t + \frac{q^{2s-1}-1}{q^{s-1}-1} \\ &> \ell q^t + q^s + 1, \end{aligned}$$

which is a contradiction. This proves the lemma. □

If there are just three distinct dimensions in a subspace partition $\mathcal{P}$ of minimum size, then we can determine the smallest dimension that is present. To prove this result, we will use the following theorem due to Heden [3].

Theorem 6 (Tail Condition). *Let $\mathcal{P}$ be a partition of $V(n,q)$ of type $[d_1^{n_1} \dots d_m^{n_m}]$, where $d_1 < \dots < d_m$ and $n_i > 0$ are integers for all $1 \leq i \leq m$. Then*

- (i) *if $q^{d_2-d_1}$ does not divide n_1 and if $d_2 < 2d_1$, then $n_1 \geq q^{d_1}+1$.*
- (ii) *if $q^{d_2-d_1}$ does not divide n_1 and $d_2 \geq 2d_1$, then either $n_1 = (q^{d_2}-1)/(q^{d_1}-1)$ or $n_1 > 2q^{d_2-d_1}$.*
- (iii) *if $q^{d_2-d_1}$ divides n_1 and $d_2 < 2d_1$, then $n_1 \geq q^{d_2} - q^{d_1} + q^{d_2-d_1}$.*
- (iv) *if $q^{d_2-d_1}$ divides n_1 and $d_2 \geq 2d_1$, then $n_1 \geq q^{d_2}$.*

We can now prove the following lemma.

Lemma 6. *Let $a, k, r, s,$ and t be positive integers such that $n = kt+r$, $k \geq 2, 1 \leq r < t-1$, $1 \leq a < s$, and $t+r = 2s-1$. If $\mathcal{P}$ is a partition of $V = V(n,q)$ of type $[a^{n_a}, s^{n_s}, t^{n_t}]$ and of size $\sigma_q(n,t)$, then $a = s-1$. Furthermore, if the number of subspaces of dimension t is $n_t = \ell q^t - \delta$, where $\delta \geq 0$, then $\delta \leq q^r - 1$.*

Proof. Let $\delta \geq 0$ be an integer and assume that $\mathcal{P}$ has $\ell q^t - \delta$ members of dimension t, n_s members of dimension s, and n_a members of dimension a. By counting the number of vectors in $V(n,q)$, we obtain

$$(\ell q^t - \delta)(q^t-1) + n_s(q^s-1) + n_a(q^a-1) = q^n - 1. \tag{4}$$

Since $\ell q^t + q^s + 1 = |\mathcal{P}| = (\ell q^t - \delta) + n_s + n_a$ and $(q^n-1) - \ell q^t(q^t-1) = q^{2s-1}-1$, Equation (4) implies

$$n_a(q^s - q^a) + \delta(q^t - q^s) = q^{2s} - q^{2s-1}. \tag{5}$$

For $\delta = 0$, Equation (5) implies that $q^{s-a}-1$ divides $q-1$. Thus $s - a = 1$. So we assume in the following that $\delta > 0$.

We also note from Equation (5) that q^{s-a} divides n_a. If $s \geq 2a$, then it follows from Theorem 6(iv) that $n_a \geq q^s$. This would contradict the fact that $|\mathcal{P}| = \ell q^t + q^s + 1$ since $\delta > 0$ and by Lemma 3, we have $n_s \geq 1 + \delta + \delta \frac{q^{t-s}-1}{q-1} > 1 + \delta$. Moreover, if $s < 2a$, then Theorem 6(iii) implies that $n_a \geq q^s - q^a + q^{s-a}$. Thus, Equation (5) yields

$$
\begin{aligned}
n_a = & \frac{q^{2s} - q^{2s-1} - \delta(q^t - q^s)}{q^s - q^a} \geq q^s - q^a + q^{s-a} \\
\Rightarrow \; & q^{2s} - q^{2s-1} - \delta(q^t - q^s) \geq (q^s - q^a + q^{s-a})(q^s - q^a) \\
\Rightarrow \; & -q^{s+a}(q^{s-1-a} - 2) - \delta(q^t - q^s) - q^s(q^{s-a} - 1) - q^{2a} \geq 0. \quad (6)
\end{aligned}
$$

If $s - 1 - a \geq 1$, then $q^{s-1-a} - 2 \geq 0$ and the expression on the left of Inequality (6) is negative. This would yield a contradiction. Hence, $s - 1 - a \leq 0$ and thus $a = s - 1$ since $a < s$.

By using the relations $a = s - 1$ and $t + r = 2s - 1$ in Inequality (6), we obtain

$$\delta(q^t - q^s) \leq q^{t+r} - q^{t+r-1} - q^{s+1} + q^s. \quad (7)$$

The left side is a linear increasing function of δ. For $\delta = q^r$, the left side is strictly larger than the right side, as $s \leq t - 1$. This proves the lemma. □

2.3. **The structure of the set of points outside subspaces of dimension t in a minimum size subspace partition.** In this subsection, let $\mathcal{H}$ denote the set of all hyperplanes in $V = V(n, q)$ and let $\mathcal{A}$ denote the family of subspaces of dimension t in a subspace partition $\mathcal{P}$ of V. Let x_i denote the number of hyperplanes in V that contain exactly i members of $\mathcal{A}$.

Lemma 7. *Let n, k, t, and r be integers such that $k \geq 2, 1 \leq r < t - 1$, and $n = kt + r$. Let $\mathcal{P}$ be a subspace partition of $V(n, q)$ of size $\sigma_q(n, t)$ and with maximum subspace dimension t. Assume that $\mathcal{P}$ contains exactly $n_t = \ell q^t$ members of dimension t. If $x_i \neq 0$, then*

$$\ell - q^r \leq i \leq \ell. \quad (8)$$

Proof. The points of a subspace U of V not belonging to a hyperplane $H \in \mathcal{H}$ are called the *black points to H in U*, and are denoted by $B_H(U)$. If U is a subspace of H then $B_H(U)$ is the empty set. Elementary linear algebra arguments give that if U is not a subspace of H then

$$|B_H(U)| = q^{\dim(U) - 1}.$$

Let $\mathcal{B} = \mathcal{P} \setminus \mathcal{A}$ denote the set of members of $\mathcal{P}$ that do not have dimension t. Then, $|\mathcal{B}| = q^s + 1$. If H is a hyperplane that contains all members of $\mathcal{B}$ then the points of V not belonging to H are distributed among the members of $\mathcal{A}$. So if i members of $\mathcal{A}$ are contained in H we get the equation

$$(n_t - i)q^{t-1} = q^{kt+r-1}.$$

This proves the left inequality in Equation (8).

The other extremal situation appears when no member of $\mathcal{B}$ is contained in the hyperplane H. Let U_j, for $1 \leq j \leq q^s + 1$, denote the members of $\mathcal{B}$. By counting the number of points of the subspaces in $\mathcal{B}$, we get

$$\sum_{j=1}^{|\mathcal{B}|} \frac{q^{\dim(U_j)}-1}{q-1} = \frac{q^{t+r}-1}{q-1}.$$

Thus, if $W \not\subseteq H$ for all $W \in \mathcal{B}$, then the total number of black points to H in the subspaces of $\mathcal{B}$ is equal to

$$\sum_{j=1}^{|\mathcal{B}|} q^{\dim(U_j)-1} = \frac{1}{q}\Big(q^{t+r}-1+\sum_{j=1}^{|\mathcal{B}|}1\Big) = q^{t+r-1}+q^{s-1}.$$

So if i members of $\mathcal{A}$ are contained in H, then we obtain in this extremal case the equation

$$(n_t - i)q^{t-1} + q^{t+r-1} + q^{s-1} = q^{kt+r-1}.$$

This relation can be simplified to

$$i = \ell + q^{s-t},$$

which is impossible since i and ℓ are integers, and $s < t$ implies that $0 < q^{s-t} < 1$. Hence, we conclude that H must contain at least one member of $\mathcal{B}$ and that inequality $i \leq \ell$ in (8) holds. This concludes the proof of the lemma. □

Proposition 1. *Let n, k, t, and r be integers such that $k \geq 2, 1 \leq r < t-1$, and $n = kt + r$. Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of size $\sigma_q(n,t)$ and with maximum subspace dimension t and with $n_t = \ell q^t$. Then the set of points in V that do not belong to members in $\mathcal{P}$ of dimension t constitutes a subspace $W \subseteq V$ of dimension $t+r$.*

Proof. Trivially, but what will be used below, $x_i \geq 0$ for all i.

From Lemma 7, we know that

$$x_i \neq 0 \qquad \Longrightarrow \qquad c = \ell - q^r \leq i \leq \ell.$$

As each member of $\mathcal{A}$ is contained in exactly $(q^{(k-1)t+r}-1)/(q-1)$ hyperplanes, we get by double counting incidences (H, U), for $H \in \mathcal{H}$ with $U \subseteq H$, that

$$\sum_{i=c}^{\ell} i x_i = n_t \cdot \frac{q^{(k-1)t+r}-1}{q-1} = C. \tag{9}$$

Any two members of $\mathcal{A}$ are contained in $(q^{(k-2)t+r}-1)/(q-1)$ hyperplanes. Thus, by double counting incidences, we get

$$\sum_{i=c}^{\ell} \binom{i}{2} x_i = \binom{n_t}{2} \frac{q^{(k-2)t+r}-1}{q-1} = D. \tag{10}$$

Furthermore, by counting the number of hyperplanes in V we get that

$$\sum_{i=c}^{\ell} x_i = \frac{q^{kt+r}-1}{q-1} = E. \tag{11}$$

Observe that the constants C, D and E are independent of the particular choice of subspace partition of minimum size that contains a set $\mathcal{A}$ as assumed in the proposition. This is especially true for the ideal partition $\mathcal{P}_0$, a fact that will soon be used.

We obtain from the Equations (9), (10) and (11) that

$$\sum_{i=c}^{\ell} x_i(i-c)(i-\ell) = 2D + C - (c+\ell)C + c\ell E. \tag{12}$$

We will soon use the following most trivial facts

$$(i-c)(i-\ell)\begin{cases} =0 & \text{if } i=c, \\ <0 & \text{if } c<i<\ell, \\ =0 & \text{if } i=\ell. \end{cases} \tag{13}$$

In order to show that the right side of Equation (12) is equal to zero we consider the ideal partition $\mathcal{P}_0$. From the construction of the ideal partition $\mathcal{P}_0$, it follows that the points in the q^s+1 subspaces of dimension less than or equal to $s = \lceil (t+r)/2 \rceil$ in $\mathcal{P}_0$ constitute a subspace W of dimension $t+r$. Any hyperplane $H \in \mathcal{H}$ either contains W or intersects W in $(q^{\dim(W)-1}-1)/(q-1)$ points. These are the two extremal cases in the proof of Lemma 7. So for the ideal partition $x_i = 0$ for $c < i < \ell$. Then, it follows from Equation (13) that the left side of Equation (12) is equal to zero.

Thus, we obtain from Equation (13) that for any partition $\mathcal{P}$,

$$\sum_{i=c+1}^{\ell-1} x_i(i-c)(i-\ell) = 0.$$

As $x_i \geq 0$, we may thus conclude from the equation above and Equation (13) that

$$c < i < \ell \qquad \Longrightarrow \qquad x_i = 0.$$

Hence, we can now use Equation (9) and Equation (11) (or refer to the ideal subspace partition, which must have the same solution x_c and x_ℓ to these two equations) to calculate x_c (and x_ℓ). We then get that

$$x_c = \frac{q^{(k-1)t}-1}{q-1}. \tag{14}$$

Let $\mathcal{H}_0$ denote the set of all hyperplanes that intersect $q^{(k-1)t+r}$ members of $\mathcal{A}$ so $x_c = |\mathcal{H}_0|$. Let W denote the intersection of all these hyperplanes and let S denote the set of points not contained in any member of $\mathcal{A}$.

From the argument used in the proof of Lemma 7, we deduce that

$$S \subseteq \bigcap_{H \in \mathcal{H}_0} H = W, \tag{15}$$

and from Equation (14), we obtain

$$\dim(W) = n - (k-1)t = t + r.$$

Moreover, the number of points of S is equal to

$$|S| = \frac{q^n - 1}{q-1} - \ell q^t \frac{q^t - 1}{q-1} = \frac{q^{t+r} - 1}{q-1} = |P_W|.$$

Thus, it follows from Equation (15) that the set of points in S will constitute the subspace W. □

3. Proofs of the results

3.1. **The case $t + r$ is even.** In this case the following lemma is true:

Lemma 8. *Let n, k, t, and r be integers such that $k \geq 2, 1 \leq r < t-1$, and $n = kt + r$. Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of minimum size $\sigma_q(n,t)$ and having the largest subspace dimension t. If $t + r = 2s$ is even, then*

$$n_t \geq \ell q^t.$$

Proof. From Lemma 5 we know that $s = (t+r)/2$ denotes the second largest dimension among the dimensions that appear in $\mathcal{P}$. Assume that $\mathcal{P}$ has $n_t = \ell q^t - \delta$ subspaces of dimension t, with $\delta \geq 1$. Then the number, N, of points covered by the subspaces in $\mathcal{P}$ satisfies

$$\begin{aligned} N &\leq (\ell q^t - \delta)\frac{q^t - 1}{q-1} + (q^s + 1 + \delta)\frac{q^s - 1}{q-1} \\ &= \frac{q^{kt+r} - 1}{q-1} - \delta \frac{q^t - q^s}{q-1} = N - \delta \frac{q^t - q^s}{q-1}. \end{aligned} \tag{16}$$

Since we assumed $\delta \geq 1$, we have a contradiction. Thus $\delta \leq 0$, and the proof is complete. □

We now prove Theorem 2.

Proof. From Lemma 8 and Lemma 2 we deduce that a minimum size subspace partition $\mathcal{P}$ of V in the case $t + r$ is even has a subfamily $\mathcal{A}$ consisting of

$$n_t = \ell q^t = q^{t+r}\frac{q^{(k-1)t} - 1}{q^t - 1}$$

spaces of dimension t. By Proposition 1, the set of points in the subspaces in the complement family $\mathcal{A}' = \mathcal{P} \setminus \mathcal{A}$ constitute a $(t+r)$-dimensional subspace W of V.

There is just one type of subspace partition of W into $q^{(t+r)/2}+1$ subspaces, namely, a subspace partition that solely consists of subspaces of dimension $(t+r)/2$. □

3.2. **The case $r = t-1$.** In this case, we show that a subspace partition $\mathcal{P}$ of V of minimum size is unique.

When $r = t-1$, the ideal subspace partition consists of $n_t = \ell q^t + 1$ subspaces of dimension t and q^t subspaces of dimension $t-1$. Assume there is another subspace partition of size $\sigma_q(n,t)$ consisting of subspaces of dimension t and $a < t-1$. Let n_a denote the number of subspaces of dimension a and $n_t = \ell q^t + 1 + x$ be the number of subspaces of dimension t. Thus

$$n_a + x = q^s = q^t. \tag{17}$$

From the packing condition we can conclude that $x \geq 0$. If we enumerate vectors, thereby comparing with the ideal partition, we get that

$$q^t(q^{t-1}-1) = x(q^t-1) + n_a(q^a-1).$$

Combining these two equations we obtain

$$n_a(q^t - q^a) = (q-1)q^{2t-1}.$$

There is just one solution to this equation, $a = t-1$ and $n_a = q^t$. This proves Theorem 3.

3.3. **The case $t+r$ odd and $t+r \leq 2t-3$.** By Lemma 2, we know that $n_t \leq \ell q^t$. Let s be as above, that is $s = (t+r+1)/2$.

If $n_t = \ell q^t$, we can argue as in Proposition 1. Specifically, let W be defined as in Proposition 1. No subspace partition of W can contain two subspaces of dimension s, as $\dim(W) = 2s-1$. Hence a minimum size subspace partition of W consists of one subspace of dimension s and the remaining subspaces of dimension $s-1$. This proves Theorem 4.

Now we consider two distinct cases, when either two or three distinct dimensions occur in $\mathcal{P}$, and we derive some new necessary conditions. When there are just two distinct dimensions appearing in $\mathcal{P}$, then by Lemma 5 these dimensions must be t and s. Thus, by the packing condition and the assumption that the subspace partition $\mathcal{P}$ has the minimum size $\sigma_q(n,t)$, we get the following system of equations:

$$\begin{cases} n_s & + & n_t & = & \sigma_q(n,t) \\ n_s(q^s-1) & + & n_t(q^t-1) & = & q^n - 1 \end{cases} \tag{18}$$

This system has only one integer solution, which is the one given in Theorem 5.

If there are three distinct dimensions t, s, and a appearing in $\mathcal{P}$, then the packing condition gives the following system of equations:

$$\begin{cases} n_a & + & n_s & + & n_t & = & \sigma_q(n,t), \\ n_a(q^a-1) & + & n_s(q^s-1) & + & n_t(q^t-1) & = & q^n - 1. \end{cases}$$

By Lemma 6, we know that $a = s-1$. Let $n_t = \ell q^t - \delta$, $n_s = 1 + \delta + x$, and $n_{s-1} = q^s - x$. From the system of equations above, or Equation (7) in the proof of Lemma 6, we have that

$$x(q^s - q^{s-1}) = \delta(q^t - q^s),$$

which we transform into

$$x(q-1) = \delta(q^{t-s+1} - q).$$

Solving for x in this equation gives the remaining part of Theorem 5.

4. Remarks

Corollary 1. *Let n, k, r, t be integers such that $k \geq 2$, $n = kt + r$ and $1 \leq r < t$. Let $\mathcal{P}$ be a partition of $V(n, q)$ containing a partial t-spread of maximum size and let a be the second largest dimension of a subspace in $\mathcal{P}$. Then $|\mathcal{P}| > \sigma_q(n, t)$, in the following cases:*

(1) *$t + r$ is even*
(2) *$t + r$ is odd and $r \neq t - 1$.*

In other words, $\mathcal{P}$ does not have minimum size unless possibly $n = (k + 1)t - 1$.

Proof. Let $\mathcal{P}$ be a partition of $V(n, q)$ of size $|\mathcal{P}| = \sigma_q(n, t) = \ell q^t + q^s + 1$. First, if $t + r$ is even, then by Theorem 2, $n_t = \ell q^t$. Next, assume that $t + r$ is odd and $r \neq t - 1$. Then $s \neq t$, where $t + r = 2s - 1$. Now, it follows from Lemma 2 that $n_t \leq \ell q^t$. Hence, the result holds. □

The methods used in the proof of Theorem 5 cannot be extended to rule out the existence of a subspace partition $\mathcal{P}_x$ of type $[3^{13}4^{28}]$ in $V(9, 2)$. This subspace partition has size 41 which is equal to the minimum size of a subspace partition in $V(9, 2)$ with subspaces of maximum subspace dimension 4. The ideal partition of $V(9, 2)$ has type $[2^8 3^1 4^{32}]$ and is of size 41. Although we tried very hard (using a computer search) to construct such a partition, we have not succeeded yet. However, we still believe that the subspace partition $\mathcal{P}_x$ exists.

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