On Vector Space Partitions and Uniformly Resolvable Designs

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Abstract

Let $V_n(q)$ denote a vector space of dimension n over the field with q elements. A set $\mathcal{P}$ of subspaces of $V_n(q)$ is a *partition* of $V_n(q)$ if every nonzero vector in $V_n(q)$ is contained in exactly one subspace in $\mathcal{P}$. A *uniformly resolvable design* is a pairwise balanced design whose blocks can be resolved in such a way that all blocks in a given parallel class have the same size. A partition of $V_n(q)$ containing a_i subspaces of dimension n_i for $1 \leq i \leq k$ induces a uniformly resolvable design on q^n points with a_i parallel classes with block size q^{n_i}, $1 \leq i \leq k$, and also corresponds to a factorization of the complete graph K_{q^n} into a_i $K_{q^{n_i}}$-factors, $1 \leq i \leq k$. We present some sufficient and some necessary conditions for the existence of certain vector space partitions. For the partitions that are shown to exist, we give the corresponding uniformly resolvable designs. We also show that there exist uniformly resolvable designs on q^n points where corresponding partitions of $V_n(q)$ do not exist.

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1 Introduction

Let $V = V_n(q)$ denote the vector space of dimension n over the field $GF(q)$ with q elements. We say that a set $\mathcal{P} = \{V_i\}_{i=1}^k$ of subspaces of V is a *partition* of V if every nonzero element of V is in V_i for exactly one i. We will say that a partition $\mathcal{P}$ is of *type* $[(t_1, n_1), \ldots, (t_k, n_k)]$ if $\mathcal{P}$ consists of t_i subspaces of dimension n_i for $1 \leq i \leq k$, where $n_i \neq n_j$ for $i \neq j$.

Because an m-dimensional subspace contains exactly $q^m - 1$ nonzero elements, in order for a partition of V of type $[(t_1, n_1), \ldots, (t_k, n_k)]$ to exist, $t_1, \ldots, t_k$ must satisfy the Diophantine equation

$$\sum_{i=1}^{k} (q^{n_i} - 1)t_i = q^n - 1. \tag{1}$$

A second necessary condition comes from dimension considerations. If U and W are subspaces of V with $U \cap W = \{0\}$, we define the *direct sum* of U and W to be $U \oplus W = \{u + w \mid u \in U, w \in W\}$. Then the subspace $U \oplus W$ has dimension $\dim U + \dim W$. Therefore,

$$\text{if } V_1, V_2 \in \mathcal{P}, V_1 \neq V_2, \text{ then } \dim(V_1) + \dim(V_2) \leq \dim(V). \tag{2}$$

Bu [2] gives an additional necessary condition. Suppose $V_n(q)$ has a partition into the subspaces $W_1, \ldots, W_m$. Let U be a subspace of dimension $n-1$ of $V_n(q)$, and define $n'_i = \dim(U \cap W_i)$ for $1 \leq i \leq m$. Then n'_i is $\dim(W_i)$ or $\dim(W_i) - 1$ according as $W_i \subseteq U$ or not. Because $\{U \cap W_i\}_{i=1}^m$ is a partition of U, called the partition *induced* by U, the following also holds:

$$\sum_{1}^{m} (q^{n'_i} - 1) = q^{n-1} - 1. \tag{3}$$

Moreover, Bu presents a number of sufficient conditions for the existence of certain partitions of $V_n(q)$. The first of these is a well-known result.

Lemma 1.1 (Bu [2]). *Let n, k be positive integers such that k divides n. Then $V_n(q)$ can be partitioned into $\frac{q^n-1}{q^k-1}$ subspaces of dimension k.*

Lemma 1.2 (Bu [2]). *Let n, d be integers such that $1 \leq d \leq n/2$. Then $V_n(q)$ can be partitioned into one subspace of dimension $n - d$ and q^{n-d} subspaces of dimension d.*

Lemma 1.3 (Bu [2]). *Let $k > 0$ and $s > 1$ be integers and $n = ks - 1$. Then $V_n(q)$ can be partitioned into $q^{(k-1)s}$ subspaces of dimension $s - 1$ and $(q^{(k-1)s} - 1)/(q^s - 1)$ subspaces of dimension s.*

Further work related to partitions was done by A. Beutelspacher and O. Heden. Given a subset $T \subseteq \{1, \ldots, n\}$, we say a partition $\mathcal{P}$ of $V_n(q)$ is a *T-partition* if $\alpha \in T$ if and only if there is a $U \in \mathcal{P}$ with $\dim U = \alpha$. Let $T = \{t_1, \ldots, t_k = t\}$, with $t_1 < \cdots < t_k$. Beutelspacher proves in [1] that if $t_1 = 2$, then $V_{2t}(q)$ has a T-partition, and Heden in [7] reduces the hypothesis to $t_1 \geq 2$. In [8], Heden proves a number of other interesting results. In particular, he characterizes completely the partitions of $V_n(2)$ of type $[(x_1, 1), (x_2, 2), (x_3, 3), (1, n-3)]$ for all $n \geq 9$.

A related question is whether the nonzero elements of a finite abelian group G can be partitioned into disjoint subsets S_i such that the sum of the elements in each S_i is zero. In [11], Tannenbaum settles this question for $G = \mathbb{Z}_2^n$ and $n > 1$.

We note that the partition problem relates to the study of translation planes and to the problem of finding optimal partial spreads and has applications to byte error control codes. For further information, we direct the interested reader to the article by Clark and Dunning [3] and the references therein.

In this article we generalize Lemma 1.1 to construct all possible partitions of $V_n(q)$ of type $[(x, r), (y, t)]$, where rt divides n. For each positive integer n and each prime power q, we construct one partition of $V_{2n}(q)$ of type $[(x, n), (y, n-1)]$.

We present some previously unknown necessary conditions for a solution of the Diophantine equation (1) to correspond to a partition. We show that a partition $\mathcal{P}$ of $V_n(q)$ of type $[(t_1, n_1), \ldots, (t_k, n_k)]$ naturally induces a factorization of the complete graph K_{q^n} into t_i $K_{q^{n_i}}$-factors for $1 \leq i \leq k$, i.e., a $(q^n, \{q^{n_1}, \ldots, q^{n_k}\}, \{t_1, \ldots, t_k\})$-uniformly resolvable design. We also show that there exist uniformly resolvable designs on q^n points where corresponding partitions of $V_n(q)$ do not exist.

In two previous articles we exhibited additional results on the partition problem. In [5], when $q = 2$ and $n = \dim(V) \leq 7$, we found all solutions to the Diophantine equation (1) with $n_1 > \cdots > n_k$ for which there exists a partition of V (Heden [8] had settled the $n = 6$ case in 1986). In [6], we showed that if $n > 2$, and x and y are nonnegative integers satisfying $x(2^3 - 1) + y(2^2 - 1) = 2^n - 1$, then there exists a partition of $V_n(2)$ of type

$[(x,3),(y,2)]$ if and only if $y \neq 1$.

2 Some New Partitions

By generalizing the construction in Bu's Lemma 1.1 we can construct all possible partitions of type $[(x,r),(y,t)]$ when $rt = n$. Note that if d and e are positive integers, then $q^d - 1$ divides $q^e - 1$ if and only if d divides e. Also $\gcd(q^d - 1, q^e - 1) = q^{\gcd(d,e)} - 1$.

Lemma 2.1. *Let r and t be positive integers with $rt = n$, and let x and y be nonnegative integers such that*

$$x(q^r - 1) + y(q^t - 1) = q^n - 1. \tag{4}$$

Then there exists a partition of $V_n(q)$ into x subspaces of dimension r and y subspaces of dimension t.

Proof. We can identify $V_i(q)$ with the field of order q^i. Let $V_i^*(q)$ denote the cyclic multiplicative group of nonzero elements of $V_i(q)$. Note that $V_n^*(q)$ has a unique cyclic subgroup of each order dividing $q^n - 1$. Consider the subgroup $H = V_r^*(q)V_t^*(q)$ of $G = V_n^*(q)$. Let the coset decomposition of G relative to H be $h_1H \cup \ldots \cup h_kH$, where $k = |G|/|H|$. Then we have

$$|H| = \frac{|V_r^*(q)| \cdot |V_t^*(q)|}{|V_d^*(q)|} = \frac{(q^r - 1)(q^t - 1)}{q^d - 1},$$

where $d = \gcd(r,t)$.

Set $a = |H|/(q^r - 1) = (q^t - 1)/(q^d - 1)$ and $b = |H|/(q^t - 1) = (q^r - 1)/(q^d - 1)$. Since $\gcd(q^r - 1, q^t - 1) = q^d - 1$, a and b are relatively prime. Dividing (4) by $|H|$ yields

$$x\frac{q^r - 1}{|H|} + y\frac{q^t - 1}{|H|} = \frac{x}{a} + \frac{y}{b} = \frac{|G|}{|H|} = k.$$

Then $bx + ay = abk$, from which it follows that a divides x and b divides y. Consequently, $x' = x/a$ and $y' = y/b$ are integers and $x' + y' = k$.

Now $V_r^*(q)$ and $V_t^*(q)$ are subgroups of H, hence we can decompose $h_1H, \ldots, h_{x'}H$ into cosets relative to $\mathrm{GF}^*(q^r)$ and $h_{x'+1}H, \ldots, h_kH$ into cosets relative to $\mathrm{GF}^*(q^t)$. Moreover, there are $x'a = x$ (respectively $y'b = y$) cosets relative to $V_r^*(q)$ (respectively $V_t^*(q)$), and, as is noted in [2], each of them induces a subspace of $V_n(q)$ of dimension r (respectively t). This concludes the proof of the lemma. □

Theorem 2.2. *Let n, r, and t be positive integers such that rt divides n. Then for all pairs of nonnegative integers x, y such that $x(q^r-1)+y(q^t-1) = q^n-1$, there exists a partition of $V_n(q)$ into x subspaces of dimension r and y subspaces of dimension t.*

Proof. From Lemma 1.1, $V_n(q)$ has a partition $\mathcal{P}$ into subspaces of dimension rt. By Lemma 2.1, each of the rt-dimensional subspaces of $\mathcal{P}$ can be further partitioned into any possible combination of subspaces of dimension r and t. This in turn implies (e.g., using induction) the existence of all partitions of $V_n(q)$ into subspaces of dimension r and t. □

2.1 Partitions of $V_{2n}(q)$ into subspaces of dimensions n and $n-1$

In this section, we prove results about partitions of $V_{2n}(q)$ into subspaces of dimensions n and k, $1 \le k < n$.

Lemma 2.3. *$V_{2n}(q)$ can be partitioned into $q+1$ subspaces of dimension n and $q^{n+1}-q$ subspaces of dimension $n-1$.*

Proof. Let $V' = V_{2n+1}(q)$. By Lemma 1.2, substituting $2n+1$ for n and n for d, there exists a partition of V' into subspaces $W_1, \ldots, W_r$, where $r = q^{n+1}+1$, $\dim W_1 = n+1$, and $\dim W_j = n$ for $j > 1$.

Let V be a subspace of V' of dimension $2n$ such that $W_1 \not\subseteq V$, and let $V_j = W_j \cap V$, $1 \le j \le r$. Then each V_j has dimension $n-1$ or n. Let x and y of the subspaces V_j have dimension $n-1$ and n, respectively. Then

$$\begin{array}{rcrcl} x & + & y & = & q^{n+1}+1, \\ (q^{n-1}-1)x & + & (q^n-1)y & = & q^{2n}-1, \end{array}$$

which implies that $x = q^{n+1}-q$ and $y = q+1$. □

For the rest of this section we restrict ourselves to the case $q = 2$. We will show that if $1 \le k < n$, then $V_{2n}(2)$ can be partitioned into $2^n - 2^k + 2$ subspaces of dimension n and $2^n - 1$ subspaces of dimension k. This allows us to completely settle the problem of partitioning $V_{2n}(2)$ into subspaces of dimensions n and $n-1$. However, the general problem of partitioning $V_{2n}(q)$ into subspaces of dimensions n and $n-1$ remains open for $q > 2$.

If K is a field we denote by $K^\times$ the nonzero elements of K. It is easy to check that if K has 2^n elements, then the map $\sigma : a \to a^2$ is an automorphism of K fixing exactly the elements 0 and 1 of K.

Lemma 2.4. *Let K be the field with 2^n elements, and suppose σ is an automorphism of K fixing exactly the elements 0 and 1. Let $r, s, t, u \in K^\times$ be such that $(\sigma(r)s, rs) = (\sigma(t)u, tu)$. Then $(r, s) = (t, u)$.*

Proof. Let $(\sigma(r)s, rs) = (\sigma(t)u, tu)$. Then we have $rt^{-1} = s^{-1}u = \sigma(r)\sigma(t^{-1}) = \sigma(rt^{-1})$. Then by assumption $rt^{-1} = 1$, so $r = t$, and $s = u$. □

Note that $V_n(2)$ can be considered to be the field K with 2^n elements. This is done in the following theorem.

Theorem 2.5. *Let $V = V_{2n}(2)$. Since V is the direct sum of two subspaces of dimension n, we can identify V with $\{(a, b) : a, b \in K\}$, where K is the field with 2^n elements. Suppose σ is an automorphism of K fixing exactly the elements 0 and 1. Let $U_0 = W_0 = \{(a, 0) : a \in K\}$ and $U_\infty = W_\infty = \{(0, b) : b \in K\}$. For every $c \in K^\times$ let $U_c = \{(\sigma(c)b, cb) : b \in K\}$ and $W_c = \{(\sigma(a)c, ac) : a \in K\}$. Then the sets $\mathcal{P} = \{U_c : c \in K^\times\} \cup \{U_0, U_\infty\}$ and $\mathcal{Q} = \{W_b : b \in K^\times\} \cup \{W_0, W_\infty\}$ form partitions of V such that for any $c, b \in K^\times$, we have $U_c \cap W_b = \{(0, 0), (\sigma(c)b, cb)\}$.*

Proof. To prove $\mathcal{P}$ is a partition, we first note that $U_0 \cap U_\infty = U_0 \cap U_c = U_\infty \cap U_c = \{(0, 0)\}$ for all $c \in K^\times$. Next, we note that it is clear that U_c is a subspace for all $c \in K^\times$. Now let $c, d \in K^\times$. Then if $(x, y) \in U_c \cap U_d$, there exist $a, b \in K$ such that $(\sigma(c)a, ca) = (x, y) = (\sigma(d)b, db)$. By Lemma 2.4 if a and b are nonzero we have $c = d$. Hence for $c \neq d$, $U_c \cap U_d = \{(0, 0)\}$. Finally, we get that $|\bigcup_{c \in K^\times} U_c \cup U_0 \cup U_\infty| = (2^n + 1)(2^n - 1) + 1 = 2^{2n} = |V|$, hence every vector in V is in a subspace in $\mathcal{P}$.

Similarly, to prove $\mathcal{Q}$ is a partition, note that $W_0 \cap W_\infty = W_0 \cap W_c = W_\infty \cap W_c = \{(0, 0)\}$ for all $c \in K^\times$. To show W_c is a subspace ($c \in K^\times$), we note that if $(\sigma(a)c, ac), (\sigma(a')c, a'c) \in W_c$, then $(\sigma(a)c, ac) + (\sigma(a')c, a'c) = ((\sigma(a) + \sigma(a'))c, (a + a')c) = (\sigma(a + a')c, (a + a')c) \in W_c$. Therefore, W_c is a subspace. Now let $a, b \in K^\times$. Then if $(x, y) \in W_a \cap W_b$, there exist $c, d \in K^\times$ such that $(\sigma(c)a, ca) = (x, y) = (\sigma(d)b, db)$. By Lemma 2.4, we have $a = b$. Hence for $a \neq b$, $W_a \cap W_b = \{(0, 0)\}$. Finally, we get that $|\bigcup_{c \in K^\times} W_c \cup W_0 \cup W_\infty| = (2^n + 1)(2^n - 1) + 1 = 2^{2n} = |V|$, hence every vector in V is in a subspace in $\mathcal{Q}$.

Finally, let $(0,0) \neq (x,y) \in U_c \cap W_b$ for some $b, c \in K^\times$. Then there exist $b', c' \in K^\times$ such that $(\sigma(c)b', cb') = (\sigma(c')b, c'b)$. Now, by Lemma 2.4, we have $b' = b$ and $c' = c$. Therefore, $U_c \cap W_b = \{(0,0), (\sigma(c)b, cb)\}$ as claimed. □

We now use the above theorem to reconfigure subspaces in a partition. Let $k \leq n$ and let $Y \subseteq K$ be a subspace of dimension k. Then we can consider the set of subspaces

$$\mathcal{R} = \{U_0, U_\infty\} \cup \{U_c : c \in K \setminus Y\} \cup \{W_{Y,b} : b \in K^\times\}$$

where $W_{Y,b} = \{(\sigma(a)b, ab) : a \in Y\}$ for every $b \in K^\times$ and σ is as in Theorem 2.5. As in its proof, it follows that $W_{Y,b}$ is a subspace since Y is a subspace and σ is an automorphism of K. Thus we get the following proposition.

Proposition 2.6. *For any positive integer n and for $1 \leq k \leq n$, the set $\mathcal{R}$ of subspaces given above forms a partition of $V_{2n}(2)$ of type $[(2^n - 2^k + 2, n), (2^n - 1, k)]$.*

Proof. We have already shown that the elements of $\mathcal{R}$ are subspaces of V and that $U_0 \cap U_\infty = U_0 \cap U_c = U_\infty \cap U_c = U_c \cap U_{c'} = \{(0,0)\}$ whenever $c \neq c'$. Also, it follows from the fact that $\mathcal{Q}$ is a partition that $U_0 \cap W_{Y,b} = U_\infty \cap W_{Y,b} = W_{Y,b} \cap W_{Y,b'} = \{(0,0)\}$ whenever $b \neq b'$. Furthermore, since for each $b \in K^\times$ we have $W_{Y,b} \subseteq \cup_{0 \neq y \in Y} U_y$, we have $W_{Y,b} \cap U_c = \{(0,0)\}$ for each $c \notin Y$. Therefore the subspaces have trivial pairwise intersections. Finally, in the union of these subspaces are $(2^n - 2^k + 2)(2^n - 1) + (2^n - 1)(2^k - 1) + 1 = 2^{2n} = |V|$ elements. Hence $\mathcal{R}$ is a partition, as claimed. □

The above result is proved by Heden (Theorem 2 in [8]) when $\gcd(k,n) = 1$. Our next result is also covered by Theorem 2 in [8]. For completeness, we include our own short proof here.

Theorem 2.7. *Let $n \geq 2$, x, and y be nonnegative integers. Then $V_{2n}(2)$ can be partitioned into x subspaces of dimension n and y subspaces of dimension $n-1$ if and only if $x(2^n - 1) + y(2^{n-1} - 1) = 2^{2n} - 1$.*

Proof. The cases $n = 2$ and $n = 3$ are treated in [5]. For $n > 3$ there are three nonnegative solutions to the Diophantine equation $x(2^n - 1) + y(2^{n-1} - 1) = 2^{2n} - 1$, namely, $S_1 := \{x = 2^n + 1, y = 0\}$, $S_2 := \{x = 2^n - 2^{n-1} + 2, y = 2^n - 1\}$, and $S_3 := \{x = 3, y = 2^{n+1} - 2\}$. By Lemma 1.1, there exists a partition of $V = V_{2n}(2)$ of type $[(2^n + 1, n)]$, which corresponds to the solution S_1. Similarly, using Proposition 2.6 with $k = n-1$ yields a partition of V of type $[(2^n - 2^{n-1} + 2, n), (2^n - 1, n-1)]$, which corresponds to the solution S_2. Finally, applying Lemma 2.3 with $q = 2$ yields a partition of V of type $[(3, n), (2^{n+1} - 2, n-1)]$, which corresponds to the solution S_3. This concludes the proof. □

3 Some nonexistence results

Here we show that not all solutions $t_1, \ldots, t_k$ of the Diophantine equation $\sum_{i=1}^{k} t_i(q^{n_i} - 1) = q^n - 1$ that satisfy condition (2) correspond to partitions of $V_n(q)$. Although this fact was already shown by Bu [2] and Heden [8], our approach here is different and yields a new necessary condition.

Lemma 3.1. *Let $\mathcal{P}$ be a partition of $V_n(q)$ in which the dimension of the subspace with minimal dimension is m.*

(a) The number of subspaces in $\mathcal{P}$ is congruent to 1 modulo q^m.

(b) If $\mathcal{P}$ contains more than one subspace, then the number of subspaces of dimension m is at least $q+1$ if $m = 1$, and exceeds 1 in any case.

Proof. (a) Suppose that $\mathcal{P}$ contains a_i subspaces of dimension i for $i = 1, 2, \ldots, k$, and let

$$s = a_k + a_{k-1} + \cdots + a_1.$$

Add this equation to the equation

$$a_k(q^k - 1) + a_{k-1}(q^{k-1} - 1) + \cdots + a_1(q - 1) = q^n - 1$$

to obtain

$$a_k q^k + a_{k-1} q^{k-1} + \cdots + a_1 q = q^n - 1 + s. \tag{5}$$

Since $a_i = 0$ for $i < m$, each term on the left side of this equation is divisible by q^m. Hence the right side must be congruent to 0 modulo q^m, and so (a) follows.

(b) Suppose $\mathcal{P}$ contains the subspace V of dimension m. Choose a subspace W of dimension $n-1$ such that W does not contain V. Let $\mathcal{P}'$ be the partition of W consisting of all sets $W \cap U$ with $U \in \mathcal{P}$ that are not $\{0\}$, and let $s' = a'_k + \ldots + a'_1$, where a'_i denotes the number of subspaces in $\mathcal{P}'$ of dimension i ($1 \leq i \leq k$).

If $m = 1$, let b_1 be the number of subspaces of dimension 1 in $\mathcal{P}$ not contained in W. Then $s' = s - b_1$, and so q divides b_1 by part (a). If $a_1 = 1$, then $b_1 = 1$, a contradiction. Thus $a_1 > 1$, and we can choose W with the additional restriction that W contains a subspace of $\mathcal{P}$ of dimension 1. Thus $a_1 > b_1 \geq q$.

If $m > 1$ and $a_m = 1$, then the equation for $\mathcal{P}'$ analogous to equation (5) is

$$a'_k q^k + a'_{k-1} q^{k-1} + \cdots + a'_1 q = q^{n-1} - 1 + s'.$$

In this case, $s' = s$, $a'_{m-1} = 1$ and $a'_i = 0$ for $i < m - 1$; so the preceding equation becomes

$$a'_k q^k + a'_{k-1} q^{k-1} + \cdots + a'_m q^m + q^{m-1} = q^{n-1} - 1 + s.$$

Since q^m divides every term on the left side except the last, this again contradicts (a). □

Thus by Lemma 3.1(b), there does not exist a partition of $V_5(2)$ into 10 subspaces of dimension 2 and 1 subspace of dimension 1, even though $10(2^2 - 1) + 1(2^1 - 1) = 2^5 - 1$.

Theorem 3.2. *Let m be the smallest dimension of any subspace in a partition of $V_n(q)$, where $m < n$. Then the number of subspaces of dimension m in the partition is at least $mq + 1$.*

Proof. The proof will be by induction on m. If $m = 1$, the result follows from Lemma 3.1(b). Suppose we know that if the dimension of a smallest subspace in a partition is $m - 1$, then there are at least $(m - 1)q + 1$ such subspaces, where $m \geq 2$.

Consider a partition of $V_n(q)$ where a smallest subspace has dimension $m \geq 2$ and $m < n$. We know that there are at least two subspaces of dimension m in the partition. We know also that we can find a subspace W of $V_n(q)$ with dimension $n-1$ that contains one of these subspaces of dimension m but not the other. For each i, let there be a_i subspaces of dimension i in our original partition, and let there be b_i subspaces of dimension i in the partition of W that it induces.

Now we have

$$\sum a_i(q^i - 1) = q^n - 1 \quad \text{and} \quad \sum b_i(q^i - 1) = q^{n-1} - 1.$$

The smallest dimension of a subspace in the induced partition is $m-1$, and $0 < b_{m-1} < a_m$ by our choice of W. Also $\sum a_i = \sum b_i$ because $m \geq 2$. Then subtracting the second displayed equation from the first gives

$$\sum (a_i - b_i) q^i = q^n - q^{n-1}.$$

Since $a_{m-1} = 0$ this implies that q divides b_{m-1}. Now the induction hypotheses says that $b_{m-1} \geq (m-1)q + 1$. But q divides b_{m-1}, so $b_{m-1} \geq mq$. Since $a_m > b_{m-1}$, we have $a_m \geq mq + 1$. □

By the previous theorem no partition of $V_{12}(2)$ into 23 subspaces of dimension 6, 84 of dimension 5, and 6 of dimension 3 can exist, even though $23(2^6-1)+84(2^5-1)+6(2^3-1) = 2^{12}-1$ and all previously known necessary conditions for the existence of such a partition are satisfied.

It follows from [5] that the bound in Theorem 3.2 is sharp for $q = 2$ and $m = 1$. We suspect, however, that a sharper bound exists in other cases.

4 Uniformly resolvable designs from vector space partitions

The finite vector space partition problem has a natural application in constructions of uniformly resolvable designs. We begin with some definitions.

A *design* is a pair $(X, \mathcal{A})$, where X is a set of elements called *points*, and $\mathcal{A}$ is a collection of nonempty subsets of X called *blocks*. Suppose $v \geq 2$, $\lambda \geq 1$, and $L \subseteq \{n \in \mathbb{Z} : n \geq 2\}$. A (v, L, λ)-*pairwise balanced design* (abbreviated (v, L, λ)-PBD) is a design $(X, \mathcal{A})$ where: (1) $|X| = v$, (2) $|A| \in L$ for all $A \in \mathcal{A}$, and (3) every pair of distinct points is contained in exactly λ blocks. We will restrict our interest to designs where $\lambda = 1$ and we will denote a $(v, L, 1)$-PBD simply as a (v, L)-PBD. It is easy to see that a (v, L)-PBD is equivalent to a decomposition of the complete graph K_v into complete subgraphs with orders in L.

Suppose $(X, \mathcal{A})$ is a (v, L)-PBD. A *parallel class* in $(X, \mathcal{A})$ is a subset of disjoint blocks from $\mathcal{A}$ whose union is X. A partition of $\mathcal{A}$ into r parallel classes is called a *resolution*, and $(X, \mathcal{A})$ is said to be a *resolvable* PBD if $\mathcal{A}$ has at least one resolution.

A parallel class in a (v, L)-PBD is *uniform* if every block in the parallel class is of the same size. Let $L = \{\ell_1, \ell_2, \dots, \ell_r\}$ be an ordered set of integers

≥ 2 and let $R = \{t_1, t_2, \ldots, t_r\}$ be an ordered multiset of positive integers. A *uniformly resolvable design*, denoted (v, L, R)-URD, is a resolvable (v, L)-PBD with t_i parallel classes with blocks of size ℓ_i for $1 \leq i \leq r$. It is easy to see that a $(v, \{\ell_1, \ldots, \ell_r\}, \{t_1, \ldots, t_r\})$-URD is equivalent to a factorization of K_v into t_i K_{ℓ_i}-factors for $1 \leq i \leq r$. Thus, for a (v, L, R)-URD to exist, one can check that the following must hold.

Lemma 4.1 (see [4]). *The following conditions are necessary for the existence of a $(v, \{\ell_1, \ldots, \ell_r\}, \{t_1, \ldots, t_r\})$-URD.*

1. $v \equiv 0 \pmod{\ell_i}$ *for each* ℓ_i*;*
2. $v - 1 = \sum_{i=1}^{r} t_i(\ell_i - 1)$*;*
3. *if* $t_i > 1$*, then* $v \geq \ell_i^2$*; and*
4. *if* $i \neq j$*, then* $v \geq \ell_i \ell_j$*.*

Note that if $v = q^n$ and $\ell_i = q^{n_i}$, then condition 2 in Lemma 4.1 is the same as the Diophantine equation (1). Similarly, conditions 3 and 4 above correspond to condition (2) in the partition problem.

For further information on URDs, we direct the reader to [4] and the references therein. We single out however a particularly nice result due to Rees [9] who showed that there exists a $(6n, \{3, 2\}, \{r, s\})$-URD if and only if $2r + s = 6n - 1$ and $(n, r, s) \notin \{(1, 2, 1), (2, 5, 1)\}$.

If W is a subset of $V_n(q)$, we denote the complete graph with vertices labeled with elements of W by $K(W)$. If W and X are subsets of $V_n(q)$ with $0 \notin X$, we define $G(W, X)$ to be the subgraph of $K(V_n(q))$ with edge set $\{\{w, w+x\} : w \in W, x \in X\}$. It is easy to see that if X is a subspace of $V_n(q)$ of dimension n_i, then $G(V_n(q), X \setminus \{0\})$ is a $K_{q^{n_i}}$-factor of K_{q^n}. Moreover, if X_1 and X_2 are disjoint subspaces, then the factors they induce are also disjoint. Thus a partition $\mathcal{P}$ of $V_n(q)$ of type $[(t_1, n_1), \ldots, (t_k, n_k)]$ induces a factorization of K_{q^n} into t_i $K_{q^{n_i}}$-factors for $1 \leq i \leq k$. Equivalently, if we let $\mathcal{A}$ denote the subspaces in $\mathcal{P}$, along with all their cosets, then, $(V_n(q), \mathcal{A})$ is a $(q^n, \{q^{n_1}, \ldots, q^{n_k}\}, \{t_1, \ldots, t_k\})$-URD. Thus we have the following result on URDs as a corollary to Theorem 2.2.

Corollary 4.2. *Let n, r, and t be positive integers such that rt divides n. Let q be a prime power and let x and y be nonnegative integers. Then there exists a $(q^n, \{q^r, q^t\}, \{x, y\})$-URD if and only if $x(q^r - 1) + y(q^t - 1) = q^n - 1$.*

In terms of graph factorizations, Corollary 4.2 says that $K_{q^{rts}}$ can be factored into x K_{q^r}-factors and y K_{q^t}-factors if and only if $x(q^r-1)+y(q^t-1)=q^{rts}-1$.

Similarly, we have the following result on URDs as a corollary to Theorem 2.7.

Corollary 4.3. *Let n be a positive integer. Let x and y be nonnegative integers. Then there exists a $(2^{2n},\{2^n,2^{n-1}\},\{x,y\})$-URD if and only if $x(2^n-1)+y(2^{n-1}-1)=2^{2n}-1$.*

As a final note, we observe that some of the necessary conditions for the existence of certain partitions of $V_n(q)$ (for example Lemma 3.1(b) and Theorem 3.2) may not have counterparts for URDs. For example, there does exist a $(32,\{4,2\},\{10,1\})$–URD even though there is no partition of $V_5(2)$ into 10 subspaces of dimension 2 and one subspace of dimension one. A $(32,\{4,2\},\{10,1\})$–URD is better known as a 4-RGDD of type 2^{16} (a resolvable group divisible design with block size four and 16 groups of size two) and was first found by H. Shen [10] in 1996.

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