PARTITIONS OF FINITE VECTOR SPACES INTO SUBSPACES

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ABSTRACT. Let $V_n(q)$ denote a vector space of dimension n over the field with q elements. A set $\mathcal{P}$ of subspaces of $V_n(q)$ is a *partition* of $V_n(q)$ if every nonzero element of $V_n(q)$ is contained in exactly one element of $\mathcal{P}$. Suppose there exists a partition of $V_n(q)$ into x_i subspaces of dimension n_i, $1 \leq i \leq k$. Then $x_1, \ldots, x_k$ satisfy the Diophantine equation $\sum_{i=1}^{k} (q^{n_i} - 1)x_i = q^n - 1$. However, not every solution of the Diophantine equation corresponds to a partition of $V_n(q)$. In this article, we show that there exists a partition of $V_n(2)$ into x subspaces of dimension 3 and y subspaces of dimension 2 if and only if $7x + 3y = 2^n - 1$ and $y \neq 1$. In doing so, we introduce techniques useful in constructing further partitions. We also show that partitions of $V_n(q)$ induce uniformly resolvable designs on q^n points.

1. INTRODUCTION AND NOTATION

Denote by $V = V_n(q)$ the vector space of dimension n over the field $GF(q)$ with q elements. For $W \subseteq V$, we call a set $\mathcal{P} = \{W_i\}_{i=1}^{\ell}$ of subspaces of V a *partition* of W if $\bigcup_1^{\ell} W_i \backslash \{0\} \subseteq W$ and every nonzero element of W is in W_i for exactly one i. We say that a partition $\mathcal{P}$ is of *type* $[(x_1, n_1), \ldots, (x_k, n_k)]$ if, for each j, $\mathcal{P}$ has exactly $\sum_{n_i=j} x_i$ subspaces of dimension j.

Because an m-dimensional subspace contains exactly $q^m - 1$ nonzero elements, in order for a partition of V of type $[(x_1, n_1), \ldots, (x_k, n_k)]$ to exist, $x_1, \ldots, x_k$ must satisfy the Diophantine equation

$$\sum_{i=1}^{k} (q^{n_i} - 1)x_i = q^n - 1. \tag{1}$$

A second necessary condition comes from dimension considerations. If U and W are subspaces of a vector space with $U \cap W = \{0\}$, the *direct sum* of U and W is defined to be $U \oplus W = \{u + w \mid u \in U, w \in W\}$. It is well-known that $U \oplus W$ has dimension $\dim U + \dim W$. Therefore,

$$\text{if } x_i \geq 2, \text{ then } n_i \leq n/2, \quad \text{and} \quad \text{if } i \neq j, \text{ then } n_i + n_j \leq n. \tag{2}$$

T. Bu [2] gives an additional necessary condition. Suppose $V_n(q)$ has a partition into the subspaces $W_1, \ldots, W_k$. Let U be a subspace of dimension $n-1$ of $V_n(q)$, and define $n_i' = \dim(U \cap W_i), i = 1, \ldots, k$. Then $n_i' = \dim(W_i)$ if $W_i \subseteq U$, and $n_i' = \dim(W_i) - 1$ if $W_i \not\subseteq U$. Because $\{U \cap W_i\}_{i=1}^k$ is a partition of U, we must also have

$$\sum_{i=1}^{k} (q^{n_i'} - 1) = q^{n-1} - 1. \tag{3}$$

Moreover, Bu proved the following two fundamental lemmas that we use in this article. (The first of these is well-known.)

Lemma 1.1 (Bu [2]). *Let n, r be positive integers such that r divides n. Then $V_n(q)$ can be partitioned into $\frac{q^n-1}{q^r-1}$ subspaces of dimension r.*

Lemma 1.2 (Bu [2]). *Let n, d be integers such that $1 \le d < n/2$. Then $V_n(q)$ can be partitioned into one subspace of dimension $n-d$ and q^{n-d} subspaces dimension d.*

Further work in this area was done by A. Beutelspacher and O. Heden. Given a subset $T \subseteq \{1, \ldots, n\}$, we say that a partition $\mathcal{P}$ of $V_n(q)$ is a *T-partition* if $\alpha \in T$ if and only if there is a $U \in \mathcal{P}$ with $\dim U = \alpha$. Let $T = \{t_1, \ldots, t_k\}$, where $t_1 < \cdots < t_k$. Beutelspacher proves in [1] that if $t_1 = 2$, then $V_{2t}(q)$ has a T–partition, and Heden in [6] reduces the hypothesis to $t_1 \ge 2$.

We note that the partition problem has applications in combinatorial designs. Suppose, for example, that $\mathcal{P}$ is a partition of $V_{2n}(q)$ into q^n+1 subspaces of dimension n. If $\mathcal{A}$ denotes the subspaces in $\mathcal{P}$ along with all their cosets, then $(V_{2n}(q), \mathcal{A})$ is an affine plane of order q^n (which in turn is a K_{q^n}–factorization of the complete graph $K_{q^{2n}}$). More generally, a partition of $V_n(q)$ of type $[(r_1, n_1), \ldots, (r_k, n_k)]$ naturally induces a factorization of K_{q^n} into r_i $K_{q^{n_i}}$–factors for $1 \le i \le t$, which is in turn a $(q^n, \{q^{n_1}, \ldots, q^{n_k}\}, \{r_1, \ldots, r_k\})$–uniformly resolvable design (see Section 5). Moreover, the partition problem is related to the problem of finding optimal partial spreads and has applications to byte error control codes. For further information on these applications, we direct the reader to the articles by Herzog and Schönheim [7], Lindström [8], and Clark and Dunning [3].

In this article, we introduce some general techniques for constructing partitions of finite vector spaces and of direct sums of subspaces. Our main result is that for $n \ge 2$, $V_n(2)$ can be partitioned into x subspaces of dimension 3 and y subspaces of dimension 2 if and only if $7x + 3y = 2^n - 1$ and $y \neq 1$. A consequence of this result is that there exists a $(2^n, \{8,4\}, \{x,y\})$–uniformly resolvable design for all nonnegative integers x and $y \neq 1$ satisfying $7x + 3y = 2^n - 1$.

In an earlier article [5], we determined all solutions of the Diophantine equation (1) with $q = 2$ and $n \leq 7$ for which there exists a partition of $V_n(2)$.

2. Another Necessary Condition

Lemma 2.1. *Let $V = V_n(q)$ and $t < n$ be a positive integer. Then there does not exist a partition $\{V_1, \ldots, V_k\}$ of V such that $\dim V_1 = t$ and $\dim V_j > t$ for all $j > 1$.*

Proof. Assume such a partition exists. Choose a projection π of V onto V_1, and let $W = \ker(\pi)$. Then $V = V_1 \oplus W$. Furthermore, for any $j > 1$, $\dim \pi(V_j) \leq t < \dim V_j$, so $V_j \cap W \neq \{0\}$. Note that $V_2 \cap W, \ldots, V_k \cap W$ forms a partition of W.

Let $n_j = \dim V_j$ and $m_j = \dim(V_j \cap W)$. Then

$$\textstyle\sum_{j=2}^{k}(q^{n_j} - 1) = (q^n - 1) - (q^t - 1) = q^n - q^t,$$
$$\textstyle\sum_{j=2}^{k}(q^{m_j} - 1) = q^{n-t} - 1,$$

and so

$$\sum_{j=2}^{k}(q^{n_j} - q^{m_j}) = q^n - q^t - q^{n-t} + 1.$$

Since the left side is 0 modulo q and the right side is 1 modulo q, we have a contradiction. □

Thus there does not exist a partition of $V_8(2)$ of type $[(36,3),(1,2)]$ even though the necessary conditions in (1) and (2) are satisfied in this case. We note that there is a solution of $7x + 3y = 2^n - 1$ with $y = 1$ if and only if $n \equiv 2 \pmod 3$.

3. Partitions of Direct Sums

In this section, we develop some constructive methods for partitioning the direct sums of subspaces. In Section 4, we use these techniques to determine which partitions of $V_n(2)$ of type $[(x,3),(y,2)]$ exist.

Theorem 3.1. *Let V be a finite-dimensional vector space over the finite field with q elements. Suppose that U and W are subspaces of V such that $V = U \oplus W$. Assume that $\dim W = s$ and that U has a subspace partition $\{U_1, \ldots, U_t\}$, where $\dim U_i = d_i \leq s$ for $1 \leq i \leq t$. Then for each i and $\gamma \in W \backslash \{0\}$, we can define a d_i-dimensional subspace $U_{i\gamma}$ of V such that U, W, and the subspaces $U_{i\gamma}$ form a partition of V.*

Proof. We can think of W as the finite field with q^s elements. Let $\{w_1, \ldots, w_s\}$ be a basis for W, and let $\{u_{i1}, \ldots, u_{id_i}\}$ be a basis for U_i for $1 \le i \le t$. For each $1 \le i \le t$ and each $\gamma \in W^* = W \backslash \{0\}$, define $U_{i\gamma}$ to be the span of $S_{i\gamma} = \{u_{i1} + \gamma w_1, \ldots, u_{id_i} + \gamma w_{d_i}\}$. Here each term γw_j is the product of two elements of the field W.

We claim that if $(i, \gamma) \neq (i', \gamma')$ then $U_{i\gamma} \cap U_{i'\gamma'} = \{0\}$. For suppose that $y \in U_{i\gamma} \cap U_{i'\gamma'}$. Then $\sum_{j=1}^{d_i} a_j(u_{ij} + \gamma w_j) = y = \sum_{j=1}^{d_{i'}} b_j(u_{i'j} + \gamma' w_j)$, where the a_j and b_j are scalars in F. Hence

$$\sum_{j=1}^{d_i} a_j u_{ij} - \sum_{j=1}^{d_{i'}} b_j u_{i'j} = \sum_{j=1}^{d_{i'}} b_j \gamma' w_j - \sum_{j=1}^{d_i} a_j \gamma w_j \in U \cap W = \{0\}.$$

If $i \neq i'$, then the set $\{u_{i1}, \ldots, u_{id_i}, u_{i'1}, \ldots, u_{i'd_{i'}}\}$ is linearly independent, and so $a_j = 0$ for $1 \le j \le d_i$. Thus $y = 0$.

If $i = i'$, then we have $\sum_{j=1}^{d_i}(a_j - b_j)u_{ij} = 0$ which implies that $a_j = b_j$ for all j. Hence $(\gamma' - \gamma)\sum_{j=1}^{d_i} a_j w_j = 0$. If $\gamma \neq \gamma'$, we conclude that $\sum_{j=1}^{d_i} a_j w_j = 0$. Again $a_j = 0$ for all j, and so $y = 0$.

For any (i, γ), we can prove that $U \cap U_{i\gamma} = \{0\} = W \cap U_{i\gamma}$ in a similar way. It is also easy to show that the set $S_{i\gamma}$ is linearly independent; so $U_{i\gamma}$ has dimension $|S_{i\gamma}| = d_i$.

Finally, to show that the set of subspaces

$$\{U, W\} \cup \{U_{i\gamma} \mid 1 \le i \le t, \gamma \in W^*\}$$

forms a partition, we need to show every $v \in V$ is contained in one of these subspaces. But for $v \in V$, there exist $u \in U$ and $w \in W$ such that $v = u + w$. If $v \notin U \cup W$, then $u \neq 0$, $w \neq 0$, and there exists i such that $u \in U_i$. Let $u = \sum_{j=1}^{d_i} a_j u_{ij}$. Then $\beta = \sum_{j=1}^{d_i} a_j w_j \neq 0$. It is straightforward to check that $v = \sum_{j=1}^{d_i} a_j(u_{ij} + \gamma w_j) \in U_{i\gamma}$, where $\gamma = \beta^{-1} w$. □

If U and W are subspaces such that $U \cap W = \{0\}$, we define $U \boxplus W$ to be $(U \oplus W) \backslash (U \cup W)$. We use this notation to state a useful corollary to Theorem 3.1.

Corollary 3.2. *Let A and B be subspaces of $V_n(q)$ with $A \cap B = \{0\}$ such that the dimension of A is no more than that of B. Then there exist $k = |B \backslash \{0\}|$ subspaces $A_1, \ldots, A_k$ with the same dimensions as A that partition $A \boxplus B$.*

Proof. Apply Theorem 3.1 with $t = 1$, $U = U_1 = A$, and $W = B$. □

We note that Bu's Lemma 1.2 is a special case of Corollary 3.2. (Take A and B to be subspaces of dimensions d and $n-d$, respectively, with intersection $\{0\}$.)

The next theorem illustrates the usefulness of the $\boxplus$ notation.

Theorem 3.3. *Let A and B be subspaces of a vector space with $A \cap B = \{0\}$, and let $A_1, \ldots, A_k$ be a partition of A into subspaces. Then $A \boxplus B$ has the (set) partition $\{A_1 \boxplus B, \ldots, A_k \boxplus B\}$.*

Proof. If $v \in A_i \boxplus B$, then there exist nonzero vectors $a \in A_i$ and $b \in B$ such that $v = a + b$. Thus $v \in A \boxplus B$.

Conversely, if $v \in A \boxplus B$, then $v = a + b$ for unique nonzero vectors $a \in A$ and $b \in B$. Hence $a \in A_i$ for a unique i, and $v \in A_i \boxplus B$. This shows that the sets $A_i \boxplus B$ are disjoint and cover $A \boxplus B$. □

4. Partitions of $V_n(2)$ into subspaces of dimensions 3 and 2

In this section, we prove our main theorem (Theorem 4.4). We start with three lemmas.

Lemma 4.1. *Let U and W be subspaces of $V_n(2)$ of dimension 3 such that $U \cap W = \{0\}$. Then $U \boxplus W$ can be partitioned into both*

(i) 4 *subspaces of dimension* 3 *and* 7 *subspaces of dimension* 2*, and*
(ii) 1 *subspace of dimension* 3 *and* 14 *subspaces of dimension* 2*.*

Proof. As proved in [5], $V_6(2)$ has a partition of type $[(6,3),(7,2)]$ (respectively, $[(3,3),(14,2)]$). Take U and W to be distinct subspaces of dimension 3 in such a partition. Then the remaining subspaces give a partition of $U \boxplus W$ into 4 subspaces (respectively, 1 subspace) of dimension 3 and 7 (respectively, 14) subspaces of dimension 2. □

Lemma 4.2. *Let L and W be subspaces of $V_{n+3}(2)$ of dimensions n and 3, respectively, such that $L \cap W = \{0\}$. If L has a partition of type $[(x,r),(y,3),(z,2)]$ with $r \geq 3$, then $L \boxplus W$ has a partition of type $[(2^n - 3j - 1, 3), (7j, 2)]$ for all integers j with $z \leq j \leq z + 2y$.*

Proof. In some partition of L of type $[(x,r),(y,3),(z,2)]$, let the subspaces of dimension r be $A_1, \ldots, A_x$, of dimension 3 be $B_1, \ldots, B_y$, and of dimension 2 be $C_1, \ldots, C_z$. Note that the sets $A_i \boxplus W$ $(1 \leq i \leq x)$, $B_i \boxplus W$ $(1 \leq i \leq y)$, and $C_i \boxplus W$ $(1 \leq i \leq z)$, form a (set) partition of $L \boxplus W$ by Theorem 3.3.

For $z \leq j \leq z + 2y$, define integers k and ℓ by $j - z = 2k + \ell$, $0 \leq \ell < 2$. Note that $k \geq 0$ and $k + l \leq y$.

Use Corollary 3.2 to partition each of the x sets $A_i \boxplus W$ into $2^r - 1$ subspaces of dimension 3.

For $i \leq k$, use Lemma 4.1 to partition each set $B_i \boxplus W$ into 1 subspace of dimension 3 and 14 subspaces of dimension 2. If $\ell = 1$, use the same lemma to partition $B_{k+1} \boxplus W$ into 4 subspaces of dimension 3 and 7 subspaces of dimension 2. Then use Corollary 3.2 to partition any sets $B_i \boxplus W$ with $i > k + \ell$ into 7 subspaces of dimension 3.

Finally, use Corollary 3.2 to partition each of the z sets $C_i \boxplus W$ into 7 subspaces of dimension 2.

Thus we have a partition of $L \boxplus W$ into subspaces of dimensions 3 and 2, with

$$k \cdot 14 + \ell \cdot 7 + z \cdot 7 = 7(2k + \ell + z) = 7j$$

subspaces of dimension 2. Since $L \boxplus W$ has $2^{n+3} - 1 - (2^n - 1) - 7 = 7(2^n - 1)$ elements, the number of subspaces of dimension 3 must be $\frac{1}{7}(7(2^n - 1) - 7j \cdot 3) = 2^n - 3j - 1$. □

Lemma 4.3. *For any $r \geq 0$, let m_r be 0, 5, or 1 according to whether r is congruent to 0, 1, or 2 modulo 3, respectively. Let δ be the least residue of r modulo 2 and M_r be the integer $(2^r - 1 - 3m_r)/7$. Then*
(i) $2M_r + \lfloor M_r/3 \rfloor + m_r = \lfloor 2^r/3 \rfloor$
(ii) $\lfloor M_r/3 \rfloor + \lfloor 2^r/3 \rfloor + \delta = \lfloor M_{r+3}/3 \rfloor$.

Proof. Let R be the least residue of r modulo 6. The order of 2 modulo 21 is 6, so that $2^r \equiv 2^R \pmod{21}$. Let $2^r = 21k + 2^R$. Note that

$$\lfloor 2^r/3 \rfloor = 7k + \lfloor 2^R/3 \rfloor = 7k + \langle 0, 0, 1, 2, 5, 10 \rangle, \tag{4}$$

where, by $\langle a_0, a_1, a_2, a_3, a_4, a_5 \rangle$, we mean a_R.

Now

$$2^R - (3m_r + 1) = \langle 1, 2, 4, 8, 16, 32 \rangle - \langle 1, 16, 4, 1, 16, 4 \rangle = \langle 0, -14, 0, 7, 0, 28 \rangle,$$

so that

$$M_r = \frac{21k + 2^R - 1 - 3m_r}{7} = 3k + \langle 0, -2, 0, 1, 0, 4 \rangle.$$

Then $\lfloor M_r/3 \rfloor = k + \langle 0, -1, 0, 0, 0, 1 \rangle$.

Finally

$$\begin{aligned} &2M_r + \lfloor M_r/3 \rfloor + m_r \\ &\quad = 6k + \langle 0, -4, 0, 2, 0, 8 \rangle + k + \langle 0, -1, 0, 0, 0, 1 \rangle + \langle 0, 5, 1, 0, 5, 1 \rangle \\ &\quad = 7k + \langle 0, 0, 1, 2, 5, 10 \rangle. \end{aligned}$$

In light of (4), this proves (*i*).

For (ii), we have

$$\left\lfloor \frac{M_{r+3}}{3} \right\rfloor = \left\lfloor \frac{2^{r+3}-1-3m_r}{21} \right\rfloor = \left\lfloor \frac{8(21k+2^R)-1-3m_r}{21} \right\rfloor$$
$$= 8k + \left\lfloor \frac{2^{R+3}-1-3m_r}{21} \right\rfloor = 8k + \langle 0,0,1,3,5,12\rangle,$$

while from our previous computations we have

$$\left\lfloor \frac{M_r}{3} \right\rfloor + \left\lfloor \frac{2^r}{3} \right\rfloor + \delta$$
$$= k + \langle 0,-1,0,0,0,1\rangle + 7k + \langle 0,0,1,2,5,10\rangle + \langle 0,1,0,1,0,1\rangle$$
$$= 8k + \langle 0,0,1,3,5,12\rangle$$

also. □

We now prove our main theorem.

Theorem 4.4. *Let $n > 2$, x, and y be nonnegative integers. There exists a partition of $V_n(2)$ of type $[(x,3),(y,2)]$ if and only if $7x+3y = 2^n-1$ and $y \neq 1$.*

Proof. Clearly (x,y) must satisfy the Diophantine equation $7x+3y = 2^n-1$. Moreover, by Lemma 2.1, there is no partition of $V_n(2)$ of type $[(x,3),(1,2)]$.

Our proof that there is a partition when $y \neq 1$ will be by induction on n. For $3 \leq n \leq 7$, the theorem is proved in [5], and for $n = 8$, see the Appendix. Hence we may assume that $n \geq 9$ and that the theorem holds for all n' with $3 \leq n' < n$.

For $r \geq 2$, it can be checked that the nonnegative solutions of the Diophantine equation $7x+3y = 2^r-1$ are given by $x = M_r - 3i$ and $y = m_r + 7i$, where $0 \leq i \leq \lfloor M_r/3 \rfloor$. Here the integers M_r and m_r are as given in the statement of Lemma 4.3.

Let L and W be subspaces of $V_n(2)$ of dimensions $n-3 \geq 6$ and 3, respectively, such that $L \cap W = \{0\}$. By Lemma 1.2, L has a partition of type

$$P = [(1,n-6),(2^{n-6},3)].$$

Also, by the induction hypothesis, L has partitions of type

$$Q_i = [(M_{n-3}-3i,3),(m_{n-3}+7i,2)]$$

for all $0 \leq i \leq \lfloor M_{n-3}/3 \rfloor$ such that $m_{n-3}+7i \neq 1$. Note that $m_{n-3}+7i = 1$ only when $i = 0$ and $n \equiv 2 \pmod 3$. To summarize,

(5) L has a partition of type $[(M_{n-3}-3i,3),(m_{n-3}+7i,2)]$
for $0 \leq i \leq \lfloor M_{n-3}/3 \rfloor$, except when $i = 0$ and $n \equiv 2$ mod 3.

By Lemma 4.2, the partition of type P implies a partition of $L \boxplus W$ of type

$$T_j = [(2^{n-3} - 3j - 1, 3), (7j, 2)]$$

for all j such that $0 \le j \le 2 \cdot 2^{n-6} = 2^{n-5}$, and by the same lemma each partition of L of type Q_i implies a partition of $L \boxplus W$ of type T_j for all j such that

$$(6) \quad m_{n-3} + 7i \le j \le m_{n-3} + 7i + 2(M_{n-3} - 3i) = 2M_{n-3} + m_{n-3} + i.$$

We have a partition of $L \boxplus W$ of type T_j for $0 \le j \le 2^{n-5}$, and also, taking $i = 1$ in (6), for $m_{n-3} + 7 \le j \le 2M_{n-3} + m_{n-3} + 1$. Since $2^{n-5} \ge 16 > m_{n-3} + 7$, partitions of type T_j exist for $0 \le j \le 2M_{n-3} + m_{n-3} + 1$, and taking $i = 2, \ldots, \lfloor M_{n-3}/3 \rfloor$ in (6) extends this to $0 \le j \le 2M_{n-3} + m_{n-3} + \lfloor M_{n-3}/3 \rfloor = \lfloor 2^{n-3}/3 \rfloor$, where the last equality comes from Lemma 4.3. To summarize,

$$(7) \quad L \boxplus W \text{ has a partition of type } [(2^{n-3} - 3j - 1, 3), (7j, 2)] \text{ for } 0 \le j \le \lfloor 2^{n-3}/3 \rfloor.$$

Since $L \boxplus W = (L \oplus W) \backslash (L \cup W)$, by (5) and (7) the space $V_n(2) = L \oplus W$ has a partition of type

$$[(2^{n-3} - 3j - 1 + M_{n-3} - 3i + 1, 3), (7j + m_{n-3} + 7i, 2)]$$

for the appropriate values of i and j. Note that

$$2^{n-3} + M_{n-3} = \frac{7 \cdot 2^{n-3} + 2^{n-3} - 1 - 3m_{n-3}}{7} = \frac{2^n - 1 - 3m_n}{7} = M_n.$$

Thus $V_n(2)$ has a partition of type $[(M_n - 3(i+j), 3), (m_n + 7(i+j), 2)]$ for

$$\gamma \le i + j \le \left\lfloor \frac{M_{n-3}}{3} \right\rfloor + \left\lfloor \frac{2^{n-3}}{3} \right\rfloor = \left\lfloor \frac{M_n}{3} \right\rfloor - \delta,$$

where the last equality again comes from Lemma 4.3. Here δ is the least residue of $n - 3$ modulo 2, and γ is 1 if $n \equiv 2 \pmod 3$ and 0 otherwise. Having $\gamma = 1$ when $n \equiv 2 \pmod 3$ is acceptable in proving the induction step, since then $i + j = 0$ would correspond to a partition with exactly $y = 1$ subspace of dimension 2.

The desired upper limit of $i + j = \lfloor M_n/3 \rfloor$ is achieved unless $n - 3$ is odd. But in this case n is even, and so 3 divides $2^n - 1 - 3m_n = 7M_n$. Then $\lfloor M_n/3 \rfloor = M_n/3$, and a partition with $i + j = M_n/3$ would consist entirely of subspaces with dimension 2. Such a partition exists by Lemma 1.1. □

Theorem 4.4 leads to the following corollary.

Corollary 4.5. *For* $n > 6$ *and all* $0 \le j \le \lfloor 2^{n-3}/3 \rfloor$, *there exists a partition of* $V_n(2)$ *of type* $[(1, n-3), (2^{n-3} - 3j, 3), (7j, 2)]$ *except when* $n = 7$ *and* $j = 1$.

Proof. We use the notation in the proof of Theorem 4.4. For $n = 7$, the theorem is proved in [5], and for $n = 8$, see Theorem 6.1 in the Appendix. Hence we may assume $n \ge 9$.

Let L and W be subspaces of $V_n(2)$ of dimensions $n-3 \ge 6$ and 3, respectively, such that $L \cap W = \{0\}$. As in the proof of Theorem 4.4, we see that

$$L \boxplus W \text{ has a partition of type } [(2^{n-3} - 3j - 1, 3), (7j, 2)] \quad \text{for } 0 \le j \le \lfloor 2^{n-3}/3 \rfloor. \tag{8}$$

Since $L \boxplus W = (L \oplus W) \backslash (L \cup W)$, it follows from (8) that $V_n(2) = L \oplus W$ has a subspace partition of type

$$\begin{aligned}\big[(1, \dim L), (1, \dim W), (2^{n-3} - 3j - 1, 3), (7j, 2)\big] \\ = [(1, n-3), (2^{n-3} - 3j, 3), (7j, 2)]\end{aligned}$$

for $0 \le j \le \lfloor 2^{n-3}/3 \rfloor$. □

5. Application to Uniformly Resolvable Designs

In this brief section, we exhibit some new uniformly resolvable designs which arise naturally from vector space partitions. We begin with some definitions.

A *design* is a pair $(X, \mathcal{B})$, where X is a finite set of elements called *points* and $\mathcal{B}$ is a collection of nonempty subsets of X called *blocks*. Suppose $v \ge 2$ and L is a set of integers ≥ 2. A (v, L)*–pairwise balanced design* (abbreviated (v, L)–PBD) is a design $(X, \mathcal{B})$ such that $|X| = v$, $|B| \in L$ for all $B \in \mathcal{B}$, and every pair of distinct points is contained in exactly one block. It is easy to see that a (v, L)–PBD is a decomposition of the complete graph K_v into complete subgraphs with orders in L.

Suppose $(X, \mathcal{B})$ is a (v, L)–PBD. A *parallel class* in $(X, \mathcal{B})$ is a subset of disjoint blocks from $\mathcal{B}$ whose union is X. A partition of $\mathcal{B}$ into parallel classes is called a *resolution*, and $(X, \mathcal{B})$ is said to be a *resolvable* PBD if $\mathcal{B}$ has at least one resolution.

A parallel class in a (v, L)–PBD is *uniform* if all its blocks have the same size. A *uniformly resolvable design*, (v, L, R)–URD, is a resolvable (v, L)–PBD such that (1) all its parallel classes are uniform, (2) R is a multiset with $|R| = |L|$, and (3) for each $\ell \in L$ there is a nonnegative integer $r_\ell \in R$ such that there are exactly r_ℓ parallel classes of size ℓ. If $v = \ell^n$, $L = \{\ell^{n_1}, \dots, \ell^{n_t}\}$, and $R = \{r_1, \dots, r_t\}$, then a (v, L, R)-URD is a factorization of K_{ℓ^n} into r_i $K_{\ell^{n_i}}$–factors for $1 \leq i \leq t$. In this case, we must have (see [4]):

(a) $\ell^n - 1 = \sum_{i=1}^{t} r_i(\ell^{n_i} - 1)$,
(b) if $r_i \geq 2$, then $n_i \leq n/2$, and
(c) if $i \neq j$, then $n_i + n_j \leq n$.

If U is a subset of $V_n(q)$, we denote by $K(U)$ the complete graph whose vertices are labeled with elements of U. If U and W are subsets of $V_n(q)$ with $0 \notin W$, we define $G(U, W)$ to be the subgraph of $K(V_n(q))$ with edge set $\{\{u, u + w\} : u \in U, w \in W\}$. It is easy to see that if W is a subspace of $V_n(q)$ of dimension n_i, then $K(V_n(q), W \setminus \{0\})$ is a $K_{q^{n_i}}$–factor of K_{q^n}. Moreover, if W_1 and W_2 are subspaces of $V_n(q)$ such that $W_1 \cap W_2 = \{0\}$, then the factors they induce are disjoint. Thus a partition $\mathcal{P}$ of $V_n(q)$ of type $[(r_1, n_1), \ldots, (r_t, n_t)]$ induces a factorization of K_{q^n} into r_i $K_{q^{n_i}}$–factors for $1 \leq i \leq t$. Equivalently, if we let $\mathcal{B}$ denote the subspaces in $\mathcal{P}$ together with all their cosets, then $(V_n(q), \mathcal{B})$ is a $(q^n, \{q^{n_1}, \ldots, q^{n_t}\}, \{r_1, \ldots, r_t\})$–URD. Thus, we have the following result on URDs.

Theorem 5.1. *If $n > 2$ and x and $y \neq 1$ are nonnegative integers such that $7x+3y = 2^n - 1$, then there exists a $(2^n, \{8, 4\}, \{x, y\})$–URD. Moreover, if n is not congruent to 2 modulo 3, then these are all the (v, L, R)–URDs with $v = 2^n$ and $L = \{8, 4\}$.*

In terms of graph factorizations, Theorem 5.1 says that K_{2^n} can be factored into x K_8–factors and y K_4–factors for all pairs of nonnegative integers x, y such that $7x + 3y = 2^n - 1$ and $y \neq 1$.

It is important to note that the condition $y \neq 1$ in Theorem 5.1 may not be necessary for the existence of a $(2^n, \{8, 4\}, \{x, y\})$–URD. This is because Lemma 2.1 may not have a counterpart for URDs. For example, there does exist a $(32, \{4, 2\}, \{10, 1\})$–URD even though there is no partition of $V_5(2)$ into 10 subspaces of dimension 2 and one subspace of dimension one. A $(32, \{4, 2\}, \{10, 1\})$–URD is better known as a 4-RGDD of type 2^{16} (a resolvable group divisible design with block size four and 16 groups of size two) and was first found by H. Shen [9] in 1996.

Moreover, if a $(2^{n_0}, \{8, 4\}, \{x_0, 1\})$–URD exists for some $n_0 \equiv 2 \pmod 3$, then a $(2^n, \{8, 4\}, \{x, 1\})$–URD can be constructed recursively for all $n \geq n_0$ with $n \equiv 2 \pmod 3$ and $7x + 3 = 2^n - 1$. Thus it would suffice to construct a $(256, \{8, 4\}, \{36, 1\})$–URD (i.e., an 8-RGDD of type 4^{64}) in order to make the condition $7x + 3y = 2^n - 1$ necessary and sufficient for the existence of a $(2^n, \{8, 4\}, \{x, y\})$–URD.

6. Appendix: Partitions of $V_8(2)$ into subspaces of dimensions 5, 3, and 2.

In this appendix, we show that the main theorem is true for $n = 8$ by proving the following stronger result.

Theorem 6.1. *Suppose that x, y, and z are nonnegative integers that satisfy $31x + 7y + 3z = 255$. Then $V_8(2)$ has a partition of type $[(x, 5), (y, 3), (z, 2)]$ unless such a partition is precluded by condition (2) or Lemma 2.1.*

6.1. **Notation and Generalities.** We start with some notation and the basic setup used in the proof of Theorem 6.1.

Let V be a subspace of $V_n(2)$ with basis v_1, v_2, v_3, and define v_i for $i > 0$ recursively by

$$v_{i+3} = v_i + v_{i+1}. \tag{9}$$

Then it is easily checked that $v_i = v_j$ if and only if $i \equiv j \pmod 7$. It follows that any 7 consecutive elements of the sequence $\{v_i\}$ constitute $V\backslash\{0\}$, and any 3 consecutive elements are linearly independent.

Note that from (9) we have $v_{i+1} + v_{i+3} = v_i$, or

$$v_i + v_{i+2} = v_{i+6} \tag{10}$$

for all i. Likewise

$$v_i + v_{i+3} = v_{i+1} \tag{11}$$

for all i. Equations (9), (10), and (11) can be used to add any v_i and v_j, since any two integers differ by at most 3 modulo 7.

Now choose subspaces X, U, and W of $V_8(2)$ of dimensions 2, 3, and 3, respectively, such that $V_8(2) = X \oplus U \oplus W$. Let U and W have bases $\{u_1, u_2, u_3\}$ and $\{w_1, w_2, w_3\}$, respectively, and define u_i and w_i for $i > 0$ by the recurrence relations

$$u_{i+3} = u_i + u_{i+1} \quad \text{and} \quad w_{i+3} = w_i + w_{i+1}.$$

Set $a_{ij} = u_j + w_{i+j}$ for all i and j. For fixed i, the sequence a_{ij} also satisfies (9), and since a_{i1}, a_{i2}, a_{i3} are linearly independent, any 7 consecutive terms $a_{ij}, \ldots, a_{i,j+6}$ are the nonzero elements of a 3-dimensional subspace of $U \boxplus W$. Call the set of these 7 nonzero elements A_i.

Note that if $v \in A_i \cap A_{i'}$, then $v = u_j + w_{i+j} = u_{j'} + w_{i'+j'}$. Thus $u_j - u_{j'} = w_{i'+j'} - w_{i+j} = 0$, and so $i \equiv i'$ and $j \equiv j' \pmod 7$. We see that $A_1 \cup \{0\}, \ldots, A_7 \cup \{0\}$ form a subspace partition of $(U \boxplus W) \cup \{0\}$.

6.2. **Proof of Theorem 6.1.** We use the preceding notation and definitions to define 7-element sequences U^j, W^j, and A_i^j by

$$U^j = (u_j, \dots u_{j+6}), \quad W^j = (w_j, \dots w_{j+6}),$$

$$\text{and} \quad A_i^j = U^j + W^{i+j} = (a_{ij}, \dots, a_{i,j+6}),$$

where addition is pointwise. Note that the set of entries in A_i^j is A_i. Furthermore, $U^i + U^{i+1} = U^{i+3}$, and so the sequences U^i and W^i also satisfy (9), (10), and (11).

It is not hard to see that $\{X \oplus U, X + W^*, X + (U \boxplus W)\}$ is a (set) partition of $X \oplus U \oplus W$, where $W^* = W \backslash \{0\}$. Let $X = \{0, x_1, x_2, x_3\}$, and recall that $U \boxplus W$ has the partition $\{A_0, \dots, A_6\}$. Thus the sets

$$X \oplus U, \quad W^*, \quad x_j + W^* \ (1 \le j \le 3), \text{ and}$$

$$A_i \ (0 \le i \le 6), \quad x_j + A_i \ (1 \le j \le 3, \ 0 \le i \le 6),$$

form a (set) partition of $V_8(2)$ in which $X \oplus U$ is a 5-dimensional subspace and each of the other 32 sets consists of 7 nonzero elements.

Now define $C_1 = x_3 + W^1$, $C_2 = x_1 + A_0^4$, $C_3 = x_2 + A_1^6$, and $C_{i+3} = C_i + C_{i+1}$ for $i > 0$. The sequence C_i is periodic modulo 7, and

(12)
$$(C_1, \dots, C_7) = (x_3 + W^1, x_1 + A_0^4, x_2 + A_1^6, x_2 + A_5^4, x_3 + A_2^3, A_3^3, x_1 + A_4^6).$$

For example,

$$C_1 + C_2 = (x_3 + W^1) + (x_1 + A_0^4) = x_2 + W^1 + U^4 + W^4 \overset{\text{by (11)}}{=} x_2 + U^4 + W^2 = x_2 + A_5^4.$$

Thus if we fix any particular coordinate of the sequences $C_1, \dots, C_7$, we obtain the nonzero elements of a subspace of dimension 3. Hence the sets listed in (12) (with 0 included) can be partitioned into 7 subspaces of dimension 3.

In the same way, let $D_1 = x_2 + W^1$, $D_2 = x_3 + A_1^1$, $D_3 = A_4^3$, $E_1 = x_1 + W^1$, $E_2 = x_1 + A_1^2$, $E_3 = x_2 + A_2^3$, $F_1 = W^1$, $F_2 = x_3 + A_0^4$, and $F_3 = x_1 + A_2^1$. Furthermore, let the sequences D_i, E_i, and F_i satisfy the same recursion that v_i does in (9). Then these sequences are periodic modulo 7, and

(13)
$$(D_1, \dots, D_7) = (x_2 + W^1, x_3 + A_1^1, A_4^3, x_1 + A_3^1, x_3 + A_6^7, x_1 + A_5^7, x_2 + A_0^3),$$

(14)
$$(E_1, \dots, E_7) = (x_1 + W^1, x_1 + A_1^2, x_2 + A_2^3, A_5^2, x_3 + A_4^5, x_2 + A_6^5, x_3 + A_3^3),$$

(15)
$$(F_1, \dots, F_7) = (W^1, x_3 + A_0^4, x_1 + A_2^1, x_3 + A_5^4, x_2 + A_4^2, x_2 + A_3^2, x_1 + A_6^1).$$

Note that equations (12) to (15) use exactly 28 of the 32 sets mentioned above, as shown in Table 1. These sets form 28 subspaces of dimension 3 (when we include 0 in each of them).

	W	A_0	A_1	A_2	A_3	A_4	A_5	A_6
$0\,+$	(15)				(12)	(13)	(14)	
$x_1\,+$	(14)	(12)	(14)	(15)	(13)	(12)	(13)	(15)
$x_2\,+$	(13)	(13)	(12)	(14)	(15)	(15)	(12)	(14)
$x_3\,+$	(12)	(15)	(13)	(12)	(14)	(14)	(15)	(13)

TABLE 1. The Equation Containing a Given Set

The 4 sets not used, A_0, A_1, A_2, and A_6, form subspaces of dimension 3 when 0 is included. Thus we have a partition $\mathcal{P}$ of $V_8(2)$ of type $[(1,5),(32,3)]$ that consists of $X \oplus U$, A_0, A_1, A_2, A_6, and the 28 subspaces produced by equations (12) through (15).

From each list of sets (12) through (15), we can also obtain two sets of seven 2-dimensional subspaces and one of the 3-dimensional subspaces $A_i \cup \{0\}$ in a similar manner. The nonzero elements in these sets of 2-dimensional subspaces are described by the equations below. Some of the sequences are permuted to get these equations, but note that W^i corresponds to the same set W^* no matter what i is, likewise the set of elements of A_i^j depends only on i.

$$(x_1 + A_0^4) + (x_2 + A_1^6) = (x_3 + A_2^3) \quad \text{and}$$
$$(x_3 + W^1) + (x_2 + A_5^1) = (x_1 + A_4^1) \tag{16}$$

$$(x_2 + A_0^3) + (x_3 + A_1^1) = (x_1 + A_5^7) \quad \text{and}$$
$$(x_2 + W^5) + (x_1 + A_3^1) = (x_3 + A_6^1) \tag{17}$$

$$(x_1 + A_1^2) + (x_2 + A_2^3) = (x_3 + A_4^5) \quad \text{and}$$
$$(x_1 + W^5) + (x_3 + A_3^1) = (x_2 + A_6^1) \tag{18}$$

$$(x_3 + A_0^4) + (x_1 + A_2^1) = (x_2 + A_4^2) \quad \text{and}$$
$$(x_3 + A_5^1) + (x_1 + A_6^6) = (x_2 + A_3^5) \tag{19}$$

These sets of equations omit the sets A_3, A_4, A_5, and W^* from equations (12) through (15), respectively. By using any of these sets of two equations instead of the corresponding one of equations (12) through (15), we replace 7 subspaces of dimension 3 with 1 subspace of dimension 3 and 14 subspaces of dimension 2. Using k of the equation sets gives a partition of type $[(1,5),(32-6k,3),(14k,2)]$ for $0 \le k \le 4$.

We can also form any 3 of the unused subspaces $A_i \cup \{0\}$ into a set of 7 subspaces of dimension 2. For example, each of the following equations allows us to replace 3 subspaces of dimension 3 with 7 subspaces of dimension 2.

$$A_0^1 + A_1^3 = A_2^7 \tag{20}$$

$$A_3^1 + A_4^3 = A_5^7 \tag{21}$$

We can always do this once using (20), and if $k = 4$ in a partition of type $[(1,5),(32-6k,3),(14k,2)]$, we can use (21) to do it again.

Hence we can obtain a partition of $V_8(2)$ of type

$$[(1,5),(32-3j,3),(7j,2)]$$

for any $j = 0,\ldots,10$ by combining equations (12)–(21) appropriately. Of course, condition (2) implies that a partition of $V_8(2)$ can have at most 1 subspace of dimension 5. Thus we can obtain all partitions of $V_8(2)$ into subspaces of dimensions 5, 3, and 2 for which there is exactly 1 subspace of dimension 5.

We can use Lemma 1.2 to partition a subspace of dimension 5 into 1 subspace of dimension 3 and 8 subspaces of dimension 2. Doing this with partitions of $V_8(2)$ of type

$$[(1,5),(32-3j,3),(7j,2)] \quad \text{for } 0 \le j \le 10$$

gives partitions of type

$$[(33-3j,3),(7j+8,2)] \quad \text{for } 0 \le j \le 10.$$

This provides partitions of $V_8(2)$ of type $[(x,3),(y,2)]$ for all solutions of the equation $7x + 3y = 255$ except $x = 36, y = 1$ and $x = 0, y = 85$. A partition with $y = 1$ is impossible by Lemma 2.1, while a partition into all subspaces of dimension 2 exists by Lemma 1.1.

This completes the proof of Theorem 6.1, and so proves the main theorem (Theorem 4.4) when $n = 8$.

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