Partitions of finite vector spaces over $GF(2)$ into subspaces of dimensions 2 and s

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Abstract

A vector space partition of a finite vector space V over the field of q elements is a collection of subspaces whose union is all of V and whose pairwise intersections are trivial. While a number of necessary conditions have been proved for certain types of vector space partitions to exist, the problem of the existence of partitions meeting these conditions is still open. In this note, we consider vector space partitions of a finite vector space over the field $GF(2)$ into subspaces of dimension 2 and dimension $s \neq 2$.

While certain cases have been done previously ($s = 1$, $s = 3$, and s even), in our main theorem we build upon these general results to give a constructive proof for the existence of vector space partitions over $GF(2)$ into subspaces of dimensions s and 2 of almost all types. In doing so, we introduce techniques that identify subsets of our vector space which can be viewed as the union of subspaces having trivial pairwise intersection in more than one way. These subsets are used to transform a given partition into another partition of a different type. This technique will also be useful in constructing further partitions of finite vector spaces.

1 Introduction

Let q be a prime power, $F = GF(q)$ be the field of q elements, and $V = V(n, q)$ be the n-dimensional F-vector space F^n, and let $S \subseteq V$ be a subset containing the zero vector. We say a collection of nonzero subspaces $\mathcal{P}$ of V is a *subspace partition* of S if the union of all the subspaces of $\mathcal{P}$ equals S and any two distinct subspaces of $\mathcal{P}$ have trivial intersection. For the sake of simplicity, in this paper we will use the term partition when we are refering to subspace partitions. We say $\mathcal{P}$ is a partition of *type* $s_1^{x_1} s_2^{x_2} \cdots s_k^{x_k}$ if $\mathcal{P}$ consists of x_i subspaces of dimension s_i for all $1 \leq i \leq k$, where the s_i are positive and distinct.

Note that, in order for a partition of $S \subseteq V(n, q)$ of type $s_1^{x_1} s_2^{x_2} \cdots s_k^{x_k}$ to exist, there are two obvious necessary conditions. First is the diophantine equation:

$$\sum_{i=1}^{k} x_i(q^{s_i} - 1) = |S| - 1. \tag{1}$$

The second comes from dimension considerations and can be stated as follows:

$$\text{If } i \neq j \text{ and } x_i \geq x_j > 0, \text{ then } s_i + s_j \leq n. \text{ Furthermore, if } x_i \geq 2, \text{ then } 2s_i \leq n. \tag{2}$$

In general, we say the expression $s_1^{x_1} s_2^{x_2} \cdots s_k^{x_k}$ is a *partition type* of $V(n, q)$ if the s_i are distinct, the x_i are nonnegative, and conditions (1) and (2) are satisfied for $S = V(n, q)$.

We note that there are partition types of $V(n, q)$ for which no partition exists. For example, $2^{10}1^1$ is a partition type of $V(5, 2)$, yet by [7, Lemma 2.1] no partition of $V(5, 2)$ of this type exists. A number of papers investigate the partition types for which a corresponding partition exists. (See, for example, [10], [11], [12], [13], as well as [3], [7], [8], [9], and [16].) Note also that partial spreads, considered, for example, in [5] and [15], become vector space partitions if we include the remaining 1-dimensional subspaces in our partitions.

This paper is an extension of [7] and [16]. In particular, we give the main theorem of the present paper.

Theorem 1 (main theorem) *Let $s \geq 3$ and $n \geq 2s$ be integers, c be the least residue of n modulo s, $\epsilon = \begin{cases} 1 & \text{if } c \text{ is even} \\ 0 & \text{if } c \text{ is odd,} \end{cases}$ and $h = \dfrac{2^{s+c} - \epsilon(2^s - 1) - 1}{3}$. If x and y are nonnegative integers such that $x(2^s - 1) + 3y = 2^n - 1$ and $y \geq h$, then there exists a partition of $V(n, 2)$ of type $s^x 2^y$.*

In the conclusion of Theorem 1, it is easy to see that the minimum possible value of x is 0 and the maximum value of x is

$$m = \frac{2^n - 1 - 3h}{2^s - 1} = \frac{2^n - 2^{s+c}}{2^s - 1} + \epsilon.$$

The number

$$\frac{2^n - 2^{s+c}}{2^s - 1} + 1$$

was conjectured in a 1972 paper [15] by Hong and Patel to be the maximum number of s-dimensional subspaces in $V(n, 2)$ having trivial pairwise intersections. Although the Hong-Patel conjecture was recently disproved for $n \geq 8$, $s = 3$, and $c = 2$ (see [6]), for no n is there a presently known partition of $V(n, 2)$ of type $s^x 2^y$ in which x exceeds m. Thus the conclusion of Theorem 1 accounts for all the partition types of $V(n, 2)$ into subspaces of dimensions 2 and s except those that would contradict the Hong-Patel Conjecture. Furthermore, Corollary 22 states that when $c = 0, 1$, or 2, our main theorem is actually an "if and only if" statement.

Note that the main theorem does not include the $s = 1$ case. We discuss this case in the next section and see that something similar is true.

The rest of the paper is organized as follows. In Section 2, we first treat the case when s is even, showing that a more general version of our main theorem for arbitrary q follows as a corollary from Lemmas 2 and 3. Therefore, we can reduce to the case when s is odd. Furthermore, since [7, Theorem 4.4] implies our main theorem for $s = 3$, we only consider the case when $s \geq 5$. Section 3 gives some general constructions of partitions for arbitrary q. In particular, in Section 3 we identify particular subsets of $V(n, q)$ that can be partitioned in multiple ways. This allows us to tranform a given vector space partition of $V(n, q)$ into another vector space partition of $V(n, q)$ of a different type. In Section 4, we restrict our attention to the case when $q = 2$ and use the constructions in Section 3 to form some initial partitions of $V(n, 2)$. Finally, in Section 5, we use the partitions in Section 4 to prove the initial cases for our main theorem (see Theorems 18, 19, and 20) and then combine these cases with [16, Theorem 1.4] to give an inductive proof of our main theorem.

While some of the techniques used in this paper are introduced in [16, Section 3] in the case when $s = 5$, here we consider arbitrary odd values of s that are at most $\lfloor \frac{n}{2} \rfloor$. (Note that when $s > \lfloor \frac{n}{2} \rfloor$, there can be at most one s-dimensional subspace in a partition of $V(n, 2)$ by condition (2). Furthermore, in order to have a partition of type $s^1 2^y$, we must have $n \equiv s \pmod 2$.)

2 Some known partitions of type $s^x 2^y$ for arbitrary q

Let q be a power of a prime and $F = GF(q)$. Let V be an n-dimensional vector space over F. In this section, we bring together some previous results and sometimes rephrase them to be consistent with the statement of our main theorem. Much of what we do in here follows from two known results that we restate here for the reader's convenience.

Lemma 2 (André [1]) *Let $m > 0$ be a divisor of n. Then there exists a partition of $V(n, q)$ of type m^k, where $k = \dfrac{q^n - 1}{q^m - 1}$.*

Lemma 3 (Beutelspacher [2] and Bu [4]) *Let* $1 < d < \frac{1}{2}n$*. Then there exists a partition of* $V(n,q)$ *of type* $(n-d)^1 d^{q^{n-d}}$*.*

By recursively applying the above lemma, we get the following as a direct consequence.

Lemma 4 *Let* $d_1 < d_2 < n$ *be positive integers such that* $n \equiv d_2 \pmod{d_1}$*. Then there exists a partition of* $V(n,q)$ *of type* $d_2^1 d_1^g$ *for* $g = \dfrac{q^n - q^{d_2}}{q^{d_1}-1}$*.*

We start with the $s = 1$ case. If n is even, by Lemma 2, there exists a partition of V of type $1^0 2^u$ where $u = \dfrac{q^n-1}{q^2-1}$. Since every two-dimensional subspace can also be viewed as the union of $q+1$ one-dimensional subspaces with pairwise trivial intersection, we can transform any partition of type $1^x 2^y$ into a partition of type $1^{x+q+1}2^{y-1}$ in this way. As a result, when n is even there exists a partition of type $1^x 2^y$ for every nonnegative integer solution to the diophantine equation $x(q-1)+y(q^2-1) = q^n-1$.

If $n \geq 3$ is odd, we note that by successively applying Lemma 3 when $d = 2$ and $n \geq 5$, we have a partition of type $3^1 2^z$, where $z = \dfrac{q^n - q^3}{q^2-1}$. Since there exists a partition of any three-dimensional subspace of type $2^1 1^{q^2}$, we get a partition of V of type $1^{q^2} 2^{z+1}$. So, by reconfiguring the two-dimensional subspaces into $q+1$ one-dimensional subspaces, one two-dimensional subspace at a time, we get a partition of type $1^x 2^y$ for all nonnegative solutions to the diophantine equation $x(q-1)+y(q^2-1) = q^n-1$, where $x \geq q^2$. Note that it follows from [12, Theorem 1] that $x \geq 2q$ when n is odd. Hence for $q = 2$ and $q = 3$, the above solutions account for all partition types of the form $1^x 2^y$ for which a partition of V exists. We summarize these results below.

Theorem 5 *Let* $q = 2$ *or* $q = 3$, $n \geq 2$*, and let* b *be the least residue of* n *modulo* 2*. Then there exists a partition of* $V(n,q)$ *of type* $1^x 2^y$ *if and only if* x *and* y *are nonnegative solutions to the diophantine equation* $x(q-1)+y(q^2-1) = q^n-1$ *such that* $0 \leq y \leq \dfrac{q^n-1}{q^2-1} - \left(\dfrac{q^2}{q+1}\right) b$*.*

Next, we consider the case when $s \geq 4$ is even. Note that if n is odd, there does not exist a partition of V of type $s^x 2^y$ since, if there were such a partition, then $2^n - 1 \equiv (-1)^n - 1 \equiv 1 \pmod 3$, while $x(2^s-1)+3y \equiv x((-1)^s-1) \equiv 0 \pmod 3$, contradicting (1). Hence we can assume that $n = \dim(V)$ is even.

In this case, we can use Lemmas 2 and 3 to obtain the following theorem.

Theorem 6 *Let* $n > s \geq 4$ *be even integers such that* $2s \leq n$*, and let* c *be the least residue of* n *modulo* s*. Then for every integer* i *such that* $0 \leq i \leq \dfrac{q^n - q^{s+c}}{q^s-1}+1$ *there exists a partition of* $V(n,q)$ *of type* $s^{x_i} 2^{y_i}$*, where*

$$x_i = \frac{q^n - q^{s+c}}{q^s-1} + 1 - i \quad \text{and} \quad y_i = \frac{q^{s+c}-q^s}{q^2-1} + \left(\frac{q^s-1}{q^2-1}\right) i.$$

Proof. Let $n = sj + c$, where $j \geq 2$ and c is even. When $i = 0$, we use induction on j to prove this case of the theorem. So let $j = 2$. Then by Lemma 3. there exists a partition of V of type $(s+c)^1 s^{q^{s+c}}$. Let W be the subspace in this partition of dimension $s+c$. Now by Lemmas 2 and 3, there is a partition of W of type $s^1 2^y$ where $y = \dfrac{q^{s+c}-q^s}{q^2-1}$. Therefore, V has a partition of type $s^x 2^y$ where $x = q^{s+c}+1 = \dfrac{q^n - q^{s+c}}{q^s-1}+1$.

Next, assume the theorem is true for $i = 0$ and for a given $j \geq 2$. If V is a vector space of dimension $n = s(j+1)+c$, by Lemma 3 there exists a partition of V of type $(sj+c)^1 s^{q^{sj+c}}$. Let W be the subspace in this partition of dimension $sj+c$. By our induction hypothesis, there exists a partition of W of type $s^{x'}2^r$, where $x' = \dfrac{q^{sj+c}-q^{s+c}}{q^s-1}+1$ and $r = \dfrac{q^{s+c}-q^s}{q^2-1}$. Therefore, we have a partition of V of type $s^x 2^r$, where

$$x = x' + q^{sj+c} = \frac{q^{sj+c}-q^{s+c}}{q^s-1} + 1 + q^{sj+c} = \frac{q^{sj+c}-q^{s+c}+q^{s(j+1)+c}-q^{sj+c}}{q^s-1} + 1 = \frac{q^n-q^{s+c}}{q^s-1} + 1.$$

Hence the theorem is established for $i = 0$.

Since $s \geq 4$ is even, each s-dimensional subspace can be reconfigured into $\dfrac{q^s-1}{q^2-1}$ two-dimensional subspaces by Lemma 2. By starting with the $i = 0$ case and making this conversion one s-dimensional subspace at a time, we get the remaining partitions. ∎

3 Some general results

As above, let q be a power of a prime, $F = GF(q)$, and V be an n-dimensional vector space over F. Let $t = \lfloor \frac{n}{2} \rfloor$, where $n \geq 5$. Let $V_1, V_2 \subseteq V$ be two t-dimensional subspaces of V with trivial intersection, so that either $V = V_1 \oplus V_2$ or $V = V_1 \oplus V_2 \oplus R$ when n is even or odd, respectively, and $R \subseteq V$ is a one-dimensional subspace such that $R \cap (V_1 \oplus V_2) = \{0\}$. Then we can think of each V_i as a copy of a degree t field extension K of F. Because K is a finite field extension of F, we know that K is Galois over F with a cyclic Galois group $G = \langle \rho \rangle$ of order t, where $\rho(x) = x^q$ for all $x \in K$. For any field L, we will denote the set of nonzero elements of L by $L^\times$.

3.1 n even

We start by considering the case when n is even, so that $n = 2t$. In this case, we can view V as a $2t$-dimensional F-vector space or as a 2-dimensional K-vector space. Using this latter characterization, we write $V = K \oplus K$. One way to realize an F-vector space partition of type t^{q^t+1} is to consider the one-dimensional K-subspaces of V that we identify with the projective line over K, which is usually denoted by $\mathbb{P}^1_K$. The elements of $\mathbb{P}^1_K$ are of the form Kv for some nonzero $v \in V$. In general, for any $(a,b) \in K \oplus K = V$, we let $K(a,b) = \{(ka,kb) : k \in K\} \subseteq V$. So for every $\alpha \in K$, let $K_\alpha = K(1,\alpha)$, and let $K_\infty = K(0,1)$. We call this partition the *projective line partition.* If $W \subseteq K$ is an F-subspace, we write U_α to denote the subspace $\{(u,\alpha u) : u \in U\}$.

We now consider some constructions that allow us to group subspaces in such a way that we can reconfigure each group independently to get F-vector space partitions of V of different types. First, we use the F-algebra Galois automorphism ρ.

For any subspace $U \subseteq K$ and any $\alpha, x \in K$ with $x \neq 0$, let us define

$$U_\alpha(x) = \{(xu, \rho(x)u + \alpha xu) : u \in U\} = \{(xu, x^q u + \alpha xu) : u \in U\}.$$

Note that $U_\alpha(x) \subseteq K_\alpha(x) = K(x, x^q + \alpha x) = K(1, x^{q-1} + \alpha)$.

Next, we include the following well-known lemma.

Lemma 7 *Let $x, y \in K^\times$ be such that $x^{q-1} = y^{q-1}$. Then there exists $a \in F$ such that $y = ax$.*

Proof. Assume $x, y \in K^\times$ are such that $x^{q-1} = y^{q-1}$. Then $(yx^{-1})^{q-1} = 1$; hence yx^{-1} is a zero of $z^{q-1}-1$. But the zeros of $z^{q-1}-1$ are exactly the nonzero elements in F. Therefore, $yx^{-1} = a$ for some $a \in F^\times$, and we get $y = ax$, as claimed. ∎

It follows that

$$K_\alpha(x) = K_\alpha(x') \Leftrightarrow x^{q-1} = (x')^{q-1}. \tag{3}$$

Furthermore, it follows from definitions that

$$U_\alpha(x) = \{x(u, (x^{q-1} + \alpha)u) : u \in U\} = x(U_{x^{q-1}+\alpha}). \tag{4}$$

Similarly, for any F-subspace $W \subseteq K$ and for any $y \in K^\times$, we can define the subset

$$W^\alpha(y) = \{(wy, (w^q + \alpha w)y) : w \in W\}.$$

Theorem 8 *Let $\alpha \in K$ and $W \subseteq K$ an F-subspace of dimension s. Then the following hold:*

1. *For any $y \in K^\times$, the set $W^\alpha(y)$ is an F-subspace of V of dimension s.*
2. *For any $y, y' \in K^\times$, we have*

$$W^\alpha(y) \cap W^\alpha(y') \neq \{0\} \Leftrightarrow W^\alpha(y) = W^\alpha(y') \Leftrightarrow Fy = Fy'.$$

3. *For any F-subspace $U \subseteq K$ of dimension r, we have*

$$\bigcup_{0 \neq x \in W} U_\alpha(x) = \bigcup_{0 \neq y \in U} W^\alpha(y) = \{(xy, (x^q + \alpha x)y) : x \in W, y \in U\}.$$

Furthermore, the above set can be partitioned into either $\dfrac{q^s-1}{q-1}$ subspaces of dimension r or $\dfrac{q^r-1}{q-1}$ subspaces of dimension s.

Proof. (1) Let $\alpha, y \in K$ with $y \neq 0$. To show the $W^\alpha(y)$ are subspaces of dimension s, we define the function $\phi_{\alpha,y} : K \to V$ by $\phi_{\alpha,y}(x) = (xy, (x^q + \alpha x)y)$ for all $x \in K$. We claim that $\phi_{\alpha,y}$ is an injective F-linear transformation.

Indeed, for any $x, x' \in K$ and any $b \in F$, we have

$$\phi_{\alpha,y}(x) + b\phi_{\alpha,y}(x') = (xy, (x^q + \alpha x)y) + b(x'y, ((x')^q + \alpha x')y) = \big((x + bx')y, ((x^q + \alpha x) + b((x')^q + \alpha x'))y\big)$$

$$= \big((x + bx')y, ((x + bx')^q + \alpha(x + bx'))y\big) = \phi_{\alpha,y}(x + bx'),$$

since $x \mapsto x^q$ is an F-linear transformation. Furthermore, $\phi_{\alpha,y}(x) = 0 \Rightarrow xy = 0 \Rightarrow x = 0$ since $y \in K^\times$, and hence $\phi_{\alpha,y}$ is injective.

Finally, for any F-subspace $W \subseteq K$, we have $W^\alpha(y) = \phi_{\alpha,y}(W)$, so that $W^\alpha(y)$ is an F-subspace of V of dimension s, as claimed.

(2) Let $\alpha, y, y' \in K$ with $y \neq 0 \neq y'$. If $Fy = Fy'$, then there exists $b \in F$ such that $y' = by$. So for any $w \in W$ we have $(wy', (w^q + \alpha w)y') = (bwy, b(w^q + \alpha w)y) = (bwy, ((bw)^q + \alpha bw)y)$. Hence $W^\alpha(y) = W^\alpha(y')$.

Conversely, assume $0 \neq v \in W^\alpha(y) \cap W^\alpha(y')$. Then there exist $w, w' \in W$ such that $(wy, (w^q + \alpha w)y) = v = (w'y', ((w')^q + \alpha w')y')$. Hence $wy = w'y'$, and since $w \neq 0 \neq w'$, we have

$$(w^{q-1} + \alpha)(wy) = ((w')^{q-1} + \alpha)(w'y') \Rightarrow w^{q-1} + \alpha = (w')^{q-1} + \alpha \Rightarrow w^{q-1} = (w')^{q-1}.$$

Therefore $Fw = Fw'$ by Lemma 7. Thus there exists $b \in F$ such that $w' = bw$, and therefore $bwy' = w'y' = wy \Rightarrow by' = y \Rightarrow Fy' = Fy$. It follows from the preceding paragraph that $W^\alpha(y) = W^\alpha(y')$.

(3) First, the equality of the above unions follows immediately from definitions. Furthermore, because there are $q^r - 1$ nonzero elements in U, there are $\frac{q^r-1}{q-1}$ distinct s-dimensional subspaces in the set $\{W^\alpha(y) : 0 \neq y \in U\}$ by (1) and (2). Furthermore, if we let $U_\alpha(W) = \bigcup_{0\neq y\in U} W^\alpha(y)$, it follows from (1) and a counting argument that $U_\alpha(W)$ is the union of $\frac{q^s-1}{q-1}$ subspaces of the form $U_\alpha(w)$ for $0 \neq w \in W$. As each grouping of these subspaces have pairwise trivial intersections, we get the claimed partitions. ∎

For convenience, for any subspaces $U, W \subseteq K$ and any $\alpha \in K$, we define the notation

$$U_\alpha(W) = \bigcup_{0\neq w\in W} U_\alpha(w) = \bigcup_{0\neq u\in U} W^\alpha(u) = W^\alpha(U). \tag{5}$$

In general, we will write $U_\alpha(W)$ to emphasize the above set as a union of the subspaces $U_\alpha(w)$, and we write $W^\alpha(U)$ to emphasize the above set as a union of the subspaces $W^\alpha(u)$.

Hence it follows from Theorem 8 that if we can find a collection of subspaces of a partition of V whose unions are of the form $U_\alpha(W)$ for subspaces $U, W \subseteq K$, then we can transform this collection into the appropriate number of subspaces of dimensions $\dim(U)$ and $\dim(W)$. This will be our general strategy in Section 5.

3.2 n odd

Now assume $V = K \oplus K \oplus R$, where R is an F-vector space of dimension r, and let $\lambda : K \to R$ be an F-linear transformation. While we are primarily interested in the case when $R = Fv$ for some nonzero $v \in V$, the initial construction is more general, and so we include the more general formulation here.

Let V_0 be the subspace $K \oplus K \oplus \{0\} \subseteq V$, and let us identify R with the subspace $\{0\} \oplus \{0\} \oplus R$. Let $\alpha \in K$ and $U, W \subseteq K$ be F-subspaces. Let $U_\alpha(W) \subseteq V_0$ and $W^\alpha(U) \subseteq V_0$ be the unions of subspaces defined in equation (5). Also, for any nonempty subset $S \subseteq V$ and any vector $y \in V$, define $S + y = \{u + y : u \in S\}$ and the sets

$$U_{\alpha,\lambda}(x) = \{(xy, (x^q + \alpha x)y, \lambda(x)) : y \in U\} = U_\alpha(x) + \lambda(x) \tag{6}$$

for each $x \in K^\times$,

$$W^{\alpha,\lambda}(y) = \{(wy, (w^q + \alpha w)y, \lambda(w)) : w \in W\} \tag{7}$$

for each $y \in K^\times$, and

$$U_{\alpha,\lambda}(W) = \bigcup_{0\neq w\in W} U_{\alpha,\lambda}(w) = \bigcup_{0\neq u\in U} W^{\alpha,\lambda}(u) = W^{\alpha,\lambda}(U).$$

Note that $U_{\alpha,\lambda}(x)$ is a subspace of V if and only if $x \in \ker(\lambda)$. For $W^{\alpha,\lambda}(y)$ we have the following lemma.

Lemma 9 *Let $\alpha, y \in K$ with $y \neq 0$, and let $W \subseteq K$ be an F-subspace. Then $W^{\alpha,\lambda}(y)$ is an F-subspace of V of dimension* $\dim(W)$.

Proof. Define a function $\psi_{\alpha,y} : W \to V$ by $\psi_{\alpha,y}(x) = (xy, (x^q + \alpha x)y, \lambda(x))$. Then for any $x, x' \in W$ and any $a \in F$ we have

$$\begin{aligned}
\psi_{\alpha,y}(x + ax') &= ((x + ax')y, ((x + ax')^q + \alpha(x + ax'))y, \lambda(x + ax')) \\
&= (xy + ax'y, (x^q + \alpha x)y + a((x')^q + \alpha x')y, \lambda(x) + a\lambda(x')) \\
&= \psi_{\alpha,y}(x) + a\psi_{\alpha,y}(x').
\end{aligned}$$

Hence $\psi_{\alpha,y}$ is F-linear. It is straightforward to show that $\psi_{\alpha,y}$ is injective since $y \in K^\times$. Finally, the image of $\psi_{\alpha,y}$ is $W^{\alpha,\lambda}(y)$, completing the proof. ∎

We can now prove a result that parallels the forward implication of Theorem 8(2).

Proposition 10 *Let $\alpha \in K$ and let $W \subseteq K$ be a subspace of dimension s. Then for any linear transformation $\lambda : K \to R$ and any $y, y' \in K^\times$ we have*

$$W^{\alpha,\lambda}(y) \cap W^{\alpha,\lambda}(y') \neq \{0\} \Leftrightarrow W^{\alpha,\lambda}(y) = W^{\alpha,\lambda}(y') \Rightarrow Fy = Fy'.$$

Proof. Let $\lambda : K \to R$ be an F-linear transformation, and let $y, y' \in K^\times$ be nonzero.

If $0 \neq v \in W^{\alpha,\lambda}(y) \cap W^{\alpha,\lambda}(y')$, then there exist nonzero $w, w' \in W$ such that $(wy, (w^{q-1} + \alpha)wy, \lambda(w)) = (w'y', ((w')^{q-1} + \alpha)w'y', \lambda(w'))$. Hence $wy = w'y' \neq 0 \Rightarrow w^{q-1} + \alpha = (w')^{q-1} + \alpha \Rightarrow w^{q-1} = (w')^{q-1}$. Thus by Lemma 7 there exists $b \in K$ such that $w' = bw$; so $wy = w'y' = bwy' \Rightarrow y = by'$. Hence $Fy = Fy'$, as claimed.

If $w \notin \ker(\lambda)$, we get $\lambda(w) = \lambda(w') = \lambda(bw) = b\lambda(w) \neq 0 \Rightarrow b = 1 \Rightarrow w = w'$. Since $wy = w'y'$, we have $y = y'$, and hence $W^{\alpha,\lambda}(y) = W^{\alpha,\lambda}(y')$. Finally, if $w \in \ker(\lambda)$, then $w' \in \ker(\lambda)$. So the equality $W^{\alpha,\lambda}(y) = W^{\alpha,\lambda}(y')$ follows from Theorem 8. ∎

4 The $q = 2$ case

When $q = 2$ and $V = K \oplus K$, we have for any $x, \alpha \in K$ with $x \neq 0$ that $K_\alpha(x) = K_{x+\alpha}$. Therefore, if $K = W \oplus W'$ for some subspaces W and W', then Proposition 11 gives us $K_\alpha(W) \cap K_\beta(W) = \{0\}$ for distinct elements $\alpha, \beta \in W'$. Then we can apply Theorem 8 to reconfigure collections of subspaces of the form $K_\alpha(W)$, where $\alpha \in K$ and W is a proper subspace of K of dimension s. In Proposition 13, we see that something similar will hold for the case when the dimension of V is odd. So let us fix a proper subspace W of K of dimension $s < t$.

4.1 n even

Let $n = 2t$. Then $K_\alpha(W)$ has a partition of type t^{2^s-1} consisting of subspaces of the form $K_{\alpha+w}$ for $w \in W$ which are contained in the projective line partition of V. Furthermore, we get the following proposition.

Proposition 11 *Let $W' \subseteq K$ be a subspace such that $K = W \oplus W'$. Then for any $\alpha \neq \beta \in W'$ we have $K_\alpha(W) \cap K_\beta(W) = \{0\}$.*

Proof. Let $w \in K_\alpha(W) \cap K_\beta(W)$. Then there exist $x, y \in W$ such that $w \in K_\alpha(x) \cap K_\beta(y)$. If $w \neq 0$, then $x^{-1}(x^2+\alpha x) = y^{-1}(y^2+\beta y) \Rightarrow x+\alpha = y+\beta$. But this gives $x-y = \beta-\alpha \in W \cap W' = \{0\} \Rightarrow \beta = \alpha$, which is a contradiction. Therefore, $w = 0$, and so $K_\alpha(W) \cap K_\beta(W) = \{0\}$. ∎

It follows from equation (3) that $K_\alpha(w) = K_\alpha(w') \Leftrightarrow w = w'$. Therefore, the subspaces $K_\alpha(w)$ for $w \in W$ partition the set $K_\alpha(W)$.

Theorem 12 *Let $V = K \oplus K$ and $t = \dim(K)$. Then for any $s < t$ and any $0 \leq j \leq 2^{t-s}$, there exists a partition $\mathcal{P}_j$ of V of type $t^{a(j)}s^{b(j)}$, where $a(j) = 2^t + 1 - (2^s - 1)j$ and $b(j) = (2^t - 1)j$. Furthermore, this partition contains the subspaces K_0, K_∞, and K_β for all $\beta \in W'$, where W' is a subspace of K of F-dimension $t - s$.*

Proof. Let W and W' be F-subspaces of K such that $K = W \oplus W'$ and $\dim(W) = s$, so that $\dim(W') = t - s$. When $j = 0$, we just use the projective line partition.

Next, let $1 \leq j \leq 2^{t-s}$, and let $S \subseteq W'$ be a subset of exactly j elements. Then for each $\beta \in S$ we can reconfigure the $2^s - 1$ subspaces in $K_\beta(W)$ into the $2^t - 1$ subspaces $W^\beta(y)$ of dimension s for all $0 \neq y \in K$. Since by Proposition 11 the $K_\beta(W)$ are distinct for distinct β in S, we get the appropriate partition. It is also straightforward to check that $K_0 \cap K_\beta(W) = \{0\} = K_\infty \cap K_\beta(W)$ for all $\beta \in W'$; hence K_0 and K_∞ are both in the resulting partition.

Furthermore, since $K_\beta(w) = \{(wy, (w + \beta)wy) : y \in K\}$, where $0 \neq w \in W$, we see that for any $\gamma \in W'$ we have $K_\gamma \cap K_\beta(W) \neq \{0\} \Rightarrow (wy, (w + \beta)wy) = (xy', \gamma xy')$ for some $0 \neq w \in W$, $x, y, y' \in K \setminus \{0\}$, and $\beta \in W'$. As $wy = xy' \neq 0$, we have $w + \beta = \gamma \Rightarrow \beta - \gamma = w \neq 0$. But then $w \in W \cap W' = \{0\}$, which is a contradiction. So $K_\gamma \cap K_\beta(W) = \{0\}$ for all $\beta, \gamma \in W'$. ∎

4.2 n odd

Let $n = 2t + c$ for some $0 < c < t$ and let R be a c-dimensional F-vector space. Let $V = K \oplus K \oplus R$. It follows from Proposition 10 that for any $\alpha \in K$, any F-linear transformation $\lambda : K \to R$, and any F-subspace W of K we have $W^{\alpha,\lambda}(y) \cap W^{\alpha,\lambda}(y') \neq \{0\} \Leftrightarrow y = y'$. In addition, we have the following proposition.

Proposition 13 *Let $W \subseteq K$ be an F-subspace, and let $\alpha, \beta \in K$ such that $\alpha - \beta \notin \ker(\lambda) \cap W$. Then for any $y, y' \in K^\times$ we have $W^{\alpha,\lambda}(y) \cap W^{\beta,\lambda}(y') = \{0\}$.*

Proof. Assume $0 \neq z \in W^{\alpha,\lambda}(y) \cap W^{\beta,\lambda}(y')$. Then there exists nonzero $x, x' \in W$ such that

$$(xy, (x^2 + \alpha x)y, \lambda(x)) = z = (x'y', ((x')^2 + \beta x')y', \lambda(x')).$$

Hence $xy = x'y'$, and so we get

$$(x + \alpha)xy = (x^2 + \alpha x)y = ((x')^2 + \beta x')y' = (x' + \beta)x'y'.$$

Therefore $x + \alpha = x' + \beta \Rightarrow x' = x + \alpha - \beta$, and $\alpha - \beta \in W$. Now, by looking at the last component, we also have $\lambda(x) = \lambda(x') = \lambda(x) + \lambda(\alpha - \beta) \Rightarrow \alpha - \beta \in \ker(\lambda)$. ∎

In what follows, we will primarily be interested in the case when $R = Fv$ for some nonzero vector $v \in V$, so that $c = \dim_F(R) = 1$. So if $W \subseteq K$ is a subspace of dimension $s \geq 2$ not contained in the kernel of the linear functional $\lambda : K \to F$, we can construct a direct sum decomposition of K of the form

$$K = (W \cap \ker(\lambda)) \oplus Fw \oplus W',$$

where $W = (W \cap \ker(\lambda)) \oplus Fw$ and $\ker(\lambda) = (W \cap \ker(\lambda)) \oplus W'$. As such, for each $\alpha \in W' \oplus Fw$, Lemma 9 shows that the set $W^{\alpha,\lambda}(K)$ is the union of $2^t - 1$ subspaces of dimension $s = \dim(W)$, and Proposition 13 shows that for any $\alpha \neq \beta \in W' \oplus Fw$ we have $W^{\alpha,\lambda}(K) \cap W^{\beta,\lambda}(K) = \{0\}$.

Lemma 14 *Let $V = K \oplus K \oplus Fv$ for some nonzero $v \in V$, let V_0 be the subspace $K \oplus K \oplus \{0\}$, let $\lambda : K \to Fv$ be a linear transformation, and let $\alpha \in K$. For any subspace $U \subseteq K$ we have the following.*

1. *If $U \subseteq \ker(\lambda)$, then $K_{\alpha,\lambda}(U) = K_\alpha(U) \subseteq V_0$.*

2. *If $U \not\subseteq \ker(\lambda)$, then for any $x \in U \setminus \ker(\lambda)$*

$$K_{\alpha,\lambda}(U) = K_\alpha(U \cap \ker(\lambda)) \cup (K_{\alpha+x}(U \cap \ker(\lambda)) + v) \cup (K_{\alpha+x} + v)\,.$$

Proof. (1) Assume $0 \neq u \in U \subseteq \ker(\lambda)$. Then $K_{\alpha,\lambda}(u) = K_\alpha(u) + \lambda(u) = K_\alpha(u)$, and hence the equality follows.

(2) If $U \not\subseteq \ker(\lambda)$, then $\dim(U \cap \ker(\lambda)) = \dim(U) - 1$ since $\dim(\ker(\lambda)) = t - 1$. So for any $x \in U \setminus \ker(\lambda)$ we have a corresponding direct sum decomposition $U = Fx \oplus (U \cap \ker(\lambda))$. Since $q = 2$, we get $U = (U \cap \ker(\lambda)) \cup ((U \cap \ker(\lambda)) + x)$. So for any $0 \neq u \in U \cap \ker(\lambda)$ we see that $K_{\alpha,\lambda}(u) = K_\alpha(u)$ and $K_{\alpha,\lambda}(u + x) = K_\alpha(u + x) + v = K_{\alpha+x}(u) + v$. Finally, we have $K_{\alpha,\lambda}(x) = K_\alpha(x) + v = K_{x+\alpha} + v$. Therefore,

$$K_{\alpha,\lambda}(U) = K_\alpha(U \cap \ker(\lambda)) \cup (K_{\alpha+x}(U \cap \ker(\lambda)) + v) \cup (K_{\alpha+x} + v)\,.$$

∎

Proposition 15 *Let $q = 2$ and V be a vector space such that $\dim(V) = 2t + 1$. Then for any $s \leq t$ there exists a partition of type $(t+1)^1 t^x s^y$, where $x = 2^{t-s+1}$ and $y = 2^{2t-s+1} - 2^{t-s+1}$.*

Proof. As above, let K be a field extension of degree t over F, and let $V = K \oplus K \oplus Fv$ for some nonzero $v \in V$. Let $\lambda : K \to F$ be a nontrivial linear transformation and $U \subseteq K$ be a subspace of dimension s such that $U \not\subseteq \ker(\lambda)$. Then $\dim(U \cap \ker(\lambda)) = s - 1$, and we can choose a subspace $W \subseteq K$ of dimension $t - s + 1$ such that $K = W \oplus (U \cap \ker(\lambda))$, $\dim(W \cap \ker(\lambda)) = t - s$, and $\dim(W \cap U) = 1$.

Then, for each $\alpha \in W$, the set $K_{\alpha,\lambda}(U)$ has a partition consisting of $2^t - 1$ subspaces of dimension s by Lemma 9. So $\bigcup_{\alpha \in W} K_{\alpha,\lambda}(U)$ has a partition consisting of $2^{t-s+1}(2^t - 1) = 2^{2t-s+1} - 2^{t-s+1}$ subspaces of dimension s by Proposition 13.

Observe that $K_\infty \oplus Fv$ is a subspace of dimension $t+1$ not included in the above sets. Furthermore, for any $w \in K$ we have $w = w_0 + w_1$ for some $w_0 \in U \cap \ker(\lambda)$ and $w_1 \in W$. Choose $u \in W \cap U$ so that $\lambda(u) = 1$. Then $K_w + v = K_{w_1}(w_0) + v = K_{w_1+\lambda(w)u,\lambda}(w_0 + \lambda(w)u) \subseteq K_{w_1+\lambda(w)u,\lambda}(U)$. Hence

$$K_w + v \subseteq \bigcup_{\alpha \in W} K_{\alpha,\lambda}(U)$$

for all $w \in K$.

Finally, for each $\beta \in W$ we have $K_\beta \cap K_{\alpha,\lambda}(U) = \{0\}$ for all $\alpha \in W$. Hence $\{K_\beta : \beta \in W\}$ is a set of 2^{t-s+1} subspaces of dimension t that completes the desired partition. ∎

5 Partitions of type $s^x 2^y$ for s odd

Let $s \geq 3$ be odd. In this section, we use the partitions given in Theorem 12 and Proposition 15 to construct the partitions of type $s^x 2^y$ needed to prove the initial cases for our main theorem (see Theorems 18, 19, and 20). Once we establish these cases, we can combine them with [16, Theorem 1.4] to give an inductive proof of our main theorem.

First, we will assume $\dim(V) = 2t$. If $s = t$, then for any solution x and y of the diophantine equation $(2^s - 1)x + 3y = 2^{2s} - 1$ there exists a partition of V of type $s^x 2^y$ by [3, Lemma 2.1].

Next, assume $s < t < 2s$. Decompose $K = W \oplus W'$, where $\dim(W) = s$. Then

$$\{K_\infty\} \cup \{K_\alpha : \alpha \in W'\} \cup \left(\bigcup_{\alpha \in W'} \{K_\beta : K_\beta \subseteq K_\alpha(W)\} \right), \tag{8}$$

is a t^{2^t+1}-partition of V, which is our initial projective line partition. Also, by Theorem 8, each set $K_\alpha(W)$ can be partitioned into $2^t - 1$ subspaces of dimension s. Therefore, we can also view the above decomposition as a partition of type $t^{2^{t-s}+1}s^{2^{t-s}(2^t-1)}$, where for each $\alpha \in W'$ we have

$$K_\alpha(W) = W^\alpha(K).$$

Since we are interested in partitions of type $s^x 2^y$, we first restate [16, Lemmas 3.5 and 3.6] which we will use to reconfigure the subspaces K_α.

Lemma 16 [16, Lemma 3.5] *Let A_1, A_2, A_3 be subspaces of $V = K \oplus K$ of dimension t such that $A_i \cap A_j = \{0\}$ for $i \neq j$. Then $V = A_1 \oplus A_2$. Let $\pi_i : V \to A_i$ for $i = 1, 2$ be the corresponding projection. Then:*

1. *For any $0 \neq x \in A_3$ the set $B_x = \{0, x, \pi_1(x), \pi_2(x)\}$ is a two-dimensional subspace of V contained in $A_1 \cup A_2 \cup A_3$. Hence there exists a partition of $A_1 \cup A_2 \cup A_3$ of type 2^{2^t-1}.*

2. *For any $s < t$, $A_1 \cup A_2 \cup A_3$ has a partition of type $s^3 2^{2^t-2^s}$.*

Let $K = U \oplus U'$ be a direct sum decomposition and choose $0 \neq c \in U'$. If we consider the direct sum decomposition $U' = U'' \oplus Fc$ for some subspace U'', then we get the following lemma.

Lemma 17 [16, Lemma 3.6] *For any $\alpha \in U''$ we have*

$$K_\alpha(U \oplus Fc) = K_\alpha(U) \cup K_{\alpha+c}(U) \cup K_{\alpha+c}.$$

Now we can prove the first of a series of three theorems.

Theorem 18 *Let $V = K \oplus K$, where $t = \dim(K)$, so that $\dim(V)$ is even. Let s be an odd integer such that $s < t < 2s$. Then for each $0 \leq i \leq \frac{1}{3}\left(2^{2t-s}+1\right)$ there exists a partition $\mathcal{P}_i$ of V of type $s^{x_i}2^{y_i}$, where*

$$x_i = (2^{2t-s}+1) - 3i \quad \text{and} \quad y_i = \frac{1}{3}(2^{2t-s} - 2^s) + (2^s - 1)i.$$

Proof. We break this proof up into two cases. First we consider when t is even (so $n \equiv 0 \pmod 4$), then we consider the case when t is odd (so $n \equiv 2 \pmod 4$).

Case 1: Assume t is even. To prove $\mathcal{P}_0$ exists, we start with the projective line partition as described in (8). By Theorem 8, for every $\alpha \in W'$ we can partition $K_\alpha(W)$ into $2^t - 1$ subspaces of dimension s. Since there are 2^{t-s} of these sets, we have partitioned their union into $2^{2t-s} - 2^{t-s}$ subspaces of dimension s. In addition, we have $2^{t-s}+1$ subspaces of dimension t, which are the subspaces $\{K_\alpha : \alpha \in W'\} \cup \{K_\infty\}$.

Since t is even, $t-s$ is odd, so $2^{t-s}+1 \equiv 0 \pmod 3$. As a result, we can divide up these t-dimensional subspaces into $\frac{1}{3}(2^{t-s}+1)$ triples. Using Lemma 16, the union each of these triples can be partitioned into three subspaces of dimension s and $2^t - 2^s$ subspaces of dimension 2. Hence, from the union of all of these triples of t-dimensional subspaces we get a partition conisisting of $2^{t-s}+1$ subspaces of dimension s and $\frac{1}{3}(2^{t-s}+1)(2^t - 2^s) = \frac{1}{3}(2^{2t-s} - 2^s)$ two-dimensional subspaces. We now combine all of the above to get a partition of V. Since there is a total of $(2^{2t-s} - 2^{t-s}) + (2^{t-s}+1) = 2^{2t-s}+1$ subspaces of dimension s in this partition, we have a partition of type $s^{x_0}2^{y_0}$.

Next, to get the remaining $\mathcal{P}_i$, it is sufficient to create an appropriate number of triples of s-dimensional subspaces that can be reconfigured into $2^s - 1$ two-dimensional subspaces.

Since t is even, there exists a partition of K consisting of $d = \frac{1}{3}(2^t - 1)$ subspaces of dimension 2. Let $U_1, \ldots, U_d$ be the subspaces of this partition. Then for any $\alpha \in W'$, the $2^t - 1$ subspaces of dimension s in $K_\alpha(W)$ can be grouped into sets of three $\{W^\alpha(u) : 0 \neq u \in U_j\}$ for $1 \leq j \leq d$. But

$$W^\alpha(U_j) = \bigcup_{0 \neq u \in U_j} W^\alpha(u) = (U_j)_\alpha(W),$$

and so by Theorem 8 can partitioned into $2^s - 1$ subspaces of dimension 2. Hence we need only consider the partitions of the K_α for $\alpha \in W'$ and K_∞. But in $\mathcal{P}_0$ these were grouped in triples. Since t is even, we can take any three of these subspaces and partition each individual subspace into $\frac{1}{3}(2^t - 1)$ subspaces of dimension 2. This reconfigures 3 subspaces of dimension s and $2^t - 2^s$ of dimension 2 into $2^t - 1$ subspaces of dimension 2. Thus the theorem follows when t is even.

Case 2: Now assume t is odd. To prove $\mathcal{P}_0$ exists, we again start with the partition given in (8). As above, by Theorem 8 $K_\alpha(W)$ can be partitioned into $2^t - 1$ subspaces of dimension s for each $\alpha \in W'$. Hence the union of all of these sets can be partitioned into $2^{2t-s} - 2^{t-s}$ subspaces of dimension s. In addition, we have $2^{t-s} + 1$ subspaces of dimension t, which are the subspaces $\{K_\alpha : \alpha \in W'\} \cup \{K_\infty\}$.

Since both s and t are odd, by Lemma 4 we can partition each t-dimensional subspace above into one subspace of dimension s and $\frac{1}{3}(2^t - 2^s)$ subspaces of dimension 2. Therefore, the union $K_\infty \cup \bigcup_{\alpha\in W'} K_\alpha$ can be partitioned into $2^{t-s} + 1$ subspaces of dimension s and $\frac{1}{3}(2^t - 2^s)(2^{t-s} + 1) = \frac{1}{3}(2^{2t-s} - 2^s)$ subspaces of dimension 2. Hence by combining these partitions of subsets we again have a partition of V of type $s^{x_0} 2^{y_0}$.

Next, to get the remaining $\mathcal{P}_i$, it is sufficient to create an appropriate number of triples of s-dimensional subspaces that can be reconfigured into $2^s - 1$ two-dimensional subspaces.

Choose $0 \neq c \in W'$ and $W'' \subseteq W'$ such that $W' = W'' \oplus Fc$. Then we can rearrange our initial projective line partition of V as

$$\{K_\infty\} \cup \left\{K_\alpha : \alpha \in W''\right\} \cup \left(\bigcup_{\alpha\in W''} \{K_\beta : K_\beta \subseteq K_\alpha(W \oplus Fc)\} \right).$$

Since $\dim(W'') = t - s - 1$ is odd, the number of subspaces in the set $\{K_\infty\} \cup \{K_\alpha : \alpha \in W''\}$ is divisible by 3, and so we can group these subspaces into triples and use Lemma 16 to change the partition of each triple from a partition of three s-dimensional subspaces and $2^t - 2^s$ two-dimensional subspaces to a partition of $2^t - 1$ two-dimensional subspaces.

Since t is odd, we can create a partition of K of type $s^1 2^b$ where $b = \frac{1}{3}(2^t - 2^s)$. Let W be the subspace of dimension s in this partition, and let $U_1, \ldots, U_b$ be the two-dimensional subspaces in this partition. As s is odd, there exists a partition of W of type $2^{\frac{1}{3}(2^s-5)} 1^4$. Let $U_{b+1}, \ldots, U_g$ be the subspaces of dimension 2 in this latter partition, where $g = \frac{1}{3}(2^t - 5)$.

Next, we look at two ways to partition the set $K_\alpha(W \oplus Fc)$ into subspaces of dimensions 2 and s for each $\alpha \in W''$. Our first method will give us nine subspaces of dimension s or more, while our second method will give us nine subspaces of dimension s or fewer.

To describe our first method, for each $\alpha \in W''$ we apply Lemma 17 to the set $K_\alpha(W \oplus Fc) = K_\alpha(W) \cup K_{\alpha+c}(W) \cup K_{\alpha+c}$. As with K, we can partition $K_{\alpha+c}$ into one s-dimensional subspace and b two-dimensional subspaces. Next, by Theorem 8, each $(U_j)_\alpha(W)$ and $(U_j)_{\alpha+c}(W)$ can be partitioned into three s-dimensional subspaces or $2^s - 1$ two-dimensional subspaces. In this way, we can reconfigure $K_\alpha(W \oplus Fc)$ three s-dimensional subspaces at a time so that there are only nine s-dimensional subspaces in this set and the rest are two-dimensional subspaces. Hence we have reorganized the set $K_\alpha(W \oplus Fc)$ from a partition of $2^{t+1} - 1$ subspaces of dimension s and $\frac{1}{3}(2^t - 2^s)$ subspaces of dimension 2 into a partition of nine subspaces of dimension s and $\frac{1}{3}(2^t - 2^s + (2^{t+1} - 10)(2^s - 1)) = \frac{1}{3}(2^{t+s+1} - 2^t - 11(2^s) + 10)$ subspaces of dimension 2 by changing partitions of $(U_j)_\alpha(W)$ of type s^3 to partitions of type 2^{2^s-1} one set at a time.

To describe our second method, we note we can also partition the set $K_\alpha(W \oplus Fc)$ into $2^{s+1} - 1$ subspaces of the form K_β. As $s + 1$ is even, $2^{s+1} - 1 \equiv 0 \pmod 3$, hence we can group these K_β into triples and use Lemma 16 to partition each triple into three subspaces of dimension s and $2^t - 2^s$ subspaces of dimension 2. In this way, we can partition $K_\alpha(W \oplus Fc)$ into $2^{s+1} - 1$ subspaces of dimension s and $\frac{1}{3}(2^{s+1} - 1)(2^t - 2^s) = \frac{1}{3}(2^{t+s+1} - 2^t - 2^{2s+1} + 2^s)$ subspaces of dimension 2. Since $s \geq 3$, the number of s-dimensional subspaces is $2^{s+1} - 1 > 9$, so we have partitioned $K_\alpha(W \oplus Fc)$ into $2^{s+1} - 1$

subspaces of dimension s and $\frac{1}{3}(2^{s+1}-1)(2^t-2^s)$ of dimension 2. Now we can use Lemma 16 to change the partition each triple of subspaces of the form K_β from a partition of type $s^3 2^{2^t-2^s}$ to a partition of type 2^{2^t-1}.

By converting $K_\alpha(W \oplus Fc)$ for each $\alpha \in W''$ in this way, one at a time, we get all the remaining partition types, and hence the theorem is proven. ∎

Next, we consider the case when $\dim(V) = 2t+1$ is odd. If $s = 3$, then our main theorem follows from [7, Theorem 4.4]. So assume $s \geq 5$. Let $V = K \oplus K \oplus Fc$, where $\dim_F(K) = t$. Write $K = W \oplus W'$, where $\dim(W) = s$, and let $\lambda : K \to F$ be a linear functional such that $W \not\subseteq \ker(\lambda)$ and $W' \subseteq \ker(\lambda)$. So $W = (\ker(\lambda) \cap W) \oplus Fw$ for some $0 \neq w \in W$. Then by Lemma 9 and Proposition 13, for each $\alpha \in W' \oplus Fw$, we can partition $K_{\alpha,\lambda}(W) = W^{\alpha,\lambda}(K)$ into $2^t - 1$ subspaces of dimension s. We use this to prove the following theorem.

Theorem 19 *Let* $\dim(V) = n \equiv 1 \pmod 4$*, where* $n = 2t+1$ *with* $5 \leq s \leq t < 2s$*. Then for each* $0 \leq i \leq \frac{1}{3}(2^{2t-s+1}-1)$ *there exists a partition* $\mathcal{Q}_i$ *of* V *of type* $s^{x_i}2^{y_i}$*, where*

$$x_i = 2^{2t-s+1} - 3i \quad \text{and} \quad y_i = \frac{1}{3}(2^{2t-s+1}-1) + (2^s-1)i.$$

Proof. Let V, K, W, W', λ, w, and c be as above. As in Theorem 18, we first prove $\mathcal{Q}_0$ exists. For each $\alpha \in W' \oplus Fw$ we can partition the set $W^{\alpha,\lambda}(K)$ into $2^t - 1$ subspaces of dimension s. Hence we can partition the union of the $W^{\alpha,\lambda}(K)$ over all $\alpha \in W' \oplus Fw$ into $2^{t-s+1}(2^t-1)$ subspaces of dimension s. Furthermore, for any $\beta \in K = (W \cap \ker(\lambda)) \oplus W' \oplus Fw$, we can find $\alpha \in W' \oplus Fw$ and $\gamma \in W \cap \ker(\lambda)$ such that $\beta - w = \alpha + \gamma$. By equation (4), we have $K_\beta + c = K_\alpha(w+\gamma) + c$. Therefore, by equation (6), we have $K_\alpha(w+\gamma) + c = K_{\alpha,\lambda}(w+\gamma) \subseteq K_{\alpha,\lambda}(W) = W^{\alpha,\lambda}(K) \subseteq \bigcup_{\sigma \in W' \oplus Fw} W^{\sigma,\lambda}(K)$. Thus the complement of the above union can be partitioned into the 2^{t-s+1} subspaces $\{K_\alpha : \alpha \in W' \oplus Fw\}$ of dimension t and the subspace $K_\infty \oplus Fc$ of dimension $t+1$. Since t is even and s is odd, $2^{t-s+1} \equiv 1 \pmod 3$; so we can group $2^{t-s+1}-1$ of these t-dimensional subspaces into triples and apply Lemma 16 to each of these triples to get $2^{t-s+1}-1$ more subspaces of dimension s and $\frac{1}{3}(2^{t-s+1}-1)(2^t-2^s) = \frac{1}{3}(2^{2t-s+1}-2^{t+1}-2^t+2^s)$ subspaces of dimension 2. Since t is even, the one remaining subspace of dimension t can be partitioned into $\frac{1}{3}(2^t-1)$ subspaces of dimension 2. Finally, since $\dim(K_\infty \oplus Fc) = t+1$ is odd, we can partition this into one subspace of dimension s and $\frac{1}{3}(2^{t+1}-2^s)$ subspaces of dimension 2.

So, in total, the number of subspaces of dimension s in our partition of V is

$$2^{t-s+1}(2^t-1) + (2^{t-s+1}-1) + 1 = 2^{2t-s+1} = x_0,$$

and the number of two-dimensional subspaces is

$$\frac{1}{3}(2^{2t-s+1}-2^{t+1}-2^t+2^s) + \frac{1}{3}(2^t-1) + \frac{1}{3}(2^{t+1}-2^s) = \frac{1}{3}(2^{2t-s+1}-1) = y_0.$$

Hence we have $\mathcal{Q}_0$.

To find the other $\mathcal{Q}_i$, we show that we can partition unions of triples of s-dimensional subspaces into sets of 2^s-1 two-dimensional subspaces one triple at a time. We leave the partition of $K_\infty \oplus Fc$ alone, as well as the one K_α not used in the preceding triples. For any triple of the K_α mentioned above, we use Lemma 16 to partition the union of three s-dimensional subspaces and the 2^t-2^s two-dimensional subspaces into 2^t-1 two-dimensional subspaces.

Next, let $Y_1, \ldots, Y_b$ be a partition of K into two-dimensional subspaces, where $b = \frac{1}{3}(2^t-1)$. Now we look at partitioning the set $W^{\alpha,\lambda}(Y_j)$ into 2^s-1 two-dimensional subspaces. Let $W_0, Z_1, \ldots, Z_d$ be a partition of W, where $W_0 \subseteq \ker(\lambda)$ is of dimension $s-2$, Z_i is of dimension 2, and $d = \frac{1}{3}(2^s-2^{s-2})$.

(Such a partition exists since we are assuming $s \geq 5$.) Now each $(Z_i)^{\alpha,\lambda}(Y_j) \subseteq W^{\alpha,\lambda}(Y_j)$ is a two-dimensional subspace. Furthermore, $(Y_j)_{\alpha,\lambda}(x)$ is a two-dimensional subspace for every $0 \neq x \in W_0$ by Lemma 14. Hence we can partition $W^{\alpha,\lambda}(Y_j)$ into either three s-dimensional subspaces or $2^s - 1$ two-dimensional subspaces. This completes the proof. ∎

Theorem 20 *Let* $\dim(V) = n \equiv 3 \pmod 4$, *and let* $n = 2t + 1$ *with* $5 \leq s \leq t < 2s$. *Then for each* $0 \leq i \leq \frac{1}{3}(2^{2t-s+1} - 1)$ *there exists a partition* $\mathcal{Q}_i$ *of* V *of type* $s^{x_i}2^{y_i}$, *where*

$$x_i = 2^{2t-s+1} - 3i \quad \text{and} \quad y_i = \frac{1}{3}(2^{2t-s+1} - 1) + (2^s - 1)i.$$

Proof. Let V, K, W, W', λ, w, and c be as in the paragraph preceding Theorem 19. As before, we first prove $\mathcal{Q}_0$ exists. Consider the partition of type $(t+1)^1 t^g s^h$, where $g = 2^{t-s+1}$ and $h = 2^{2t-s+1} - 2^{t-s+1}$, given by

$$\{K_\infty \oplus Fc\} \cup \{K_\alpha : \alpha \in W' \oplus Fw\} \cup \left(\bigcup_{\alpha \in W' \oplus Fw} \left\{ W^{\alpha,\lambda}(k) : k \in K^\times \right\} \right).$$

For each $\alpha \in W' \oplus Fw$ the set $W^{\alpha,\lambda}(K)$ consists of $2^t - 1$ subspaces of dimension s; hence this gives us $2^{t-s+1}(2^t - 1) = 2^{2t-s+1} - 2^{t-s+1}$ subspaces of dimension s. Furthermore, for any $\beta \in K = (W \cap \ker(\lambda)) \oplus W' \oplus Fw$, we can find $\alpha \in W' \oplus Fw$ and $\gamma \in W \cap \ker(\lambda)$ such that $\beta - w = \alpha + \gamma$. By equation (4), we have $K_\beta + c = K_\alpha(w + \gamma) + c$. Therefore, by equation (6), we have $K_\alpha(w+\gamma) + c = K_{\alpha,\lambda}(w+\gamma) \subseteq K_{\alpha,\lambda}(W) = W^{\alpha,\lambda}(K) \subseteq \bigcup_{\sigma \in W' \oplus Fw} W^{\sigma,\lambda}(K)$. Thus we are left with the 2^{t-s+1} subspaces $\{K_\alpha : \alpha \in W' \oplus Fw\}$ of dimension t and the subspace $K_\infty \oplus Fc$ of dimension $t+1$. Since $n \equiv 3 \pmod 4$, necessarily t is odd. So we can partition each K_α into one subspace of dimension s and $\frac{1}{3}(2^t - 2^s)$ subspaces of dimension 2. Also, $\dim(K_\infty \oplus Fc) = t + 1$ is even; so we can partition $K_\infty \oplus Fc$ into $\frac{1}{3}(2^{t+1} - 1)$ subspaces of dimension 2.

Therefore, in our partition of V the total number of subspaces of dimension s is

$$(2^{2t-s+1} - 2^{t-s+1}) + 2^{t-s+1} = 2^{2t-s+1} = x_0,$$

and the total number of subspaces of dimension 2 is

$$2^{t-s+1}\left(\frac{1}{3}(2^t - 2^s)\right) + \frac{1}{3}(2^{t+1} - 1) = \frac{1}{3}(2^{2t-s+1} - 1) = y_0.$$

Hence we have $\mathcal{Q}_0$.

To find the other $\mathcal{Q}_i$, we show that we can partition the union of triples of s-dimensional subspaces into 2-dimensional subspaces one triple at a time. We leave the partition of $K_\infty \oplus Fc$ unchanged.

Let $v \in W' \subseteq \ker(\lambda)$, and create a direct sum decomposition $W' = W'' \oplus Fv$. For each $\alpha \in W'' \oplus Fw$, we consider the set $K_{\alpha,\lambda}(W) \cup K_{\alpha+v,\lambda}(W) \cup K_{\alpha+v}$. We will use two methods to partition this set into subspaces of dimensions 2 and s. We will apply the first method to get at least 9 subspaces of dimension s and we will apply the second method to get 9 or fewer subspaces of dimension s.

Now we describe our first method. Since t is odd, there exists a partition of K of type $s^1 2^b$, where $b = \frac{1}{3}(2^t - 2^s)$. Let us choose this partition so that W is the subspace of dimension s in this partition. Let $U_1, \ldots, U_b$ be the two-dimensional subspaces in this partition. In this way, we can also partition $K_{\alpha+v}$ into one subspace of dimension s and b subspaces of dimension 2.

Furthermore, as s is odd, by Lemma 4, there exists a partition of W of type $3^1 2^{\frac{1}{3}(2^s-8)}$, to which we can apply Lemma 3 to get a partition of type $2^{\frac{1}{3}(2^s-5)}1^4$. Let $U_{b+1}, \ldots, U_g$ be the subspaces of dimension 2 in this partition, where $g = \frac{1}{3}(2^t - 5)$.

For $\alpha \in W' \oplus Fw$ consider the sets $W^{\alpha,\lambda}(K) = K_{\alpha,\lambda}(W)$ and $W^{\alpha+v,\lambda}(K) = K_{\alpha+v,\lambda}(W)$. As stated above, we can partition each of these sets into $2^t - 1$ subspaces of dimension s. Let $1 \leq k \leq g$ and consider the three s-dimensional subspaces in $W^{\alpha,\lambda}(U_k)$. This set can be partitioned into $2^s - 1$ two-dimensional subspaces as follows. Let $W_0, Z_1, \ldots, Z_d$ be a partition of W, where $W_0 \subseteq \ker(\lambda)$ is of dimension $s-2$, Z_i is of dimension 2, and $d = \frac{1}{3}(2^s - 2^{s-2})$. (Such a partition exists by Lemma 3 since we are assuming $s \geq 5$.) Each $(Z_i)^{\alpha,\lambda}(u) \subseteq W^{\alpha,\lambda}(u)$ is a two-dimensional subspace for each $0 \neq u \in U_k$. Furthermore, $(U_k)_{\alpha,\lambda}(x)$ is a two-dimensional subspace for every $0 \neq x \in W_0$ by Lemma 14. Hence we can regard $W^{\alpha,\lambda}(U_k)$ as either three s-dimensional subspaces or $(2^s - 2^{s-2}) + (2^{s-2} - 1) = 2^s - 1$ two-dimensional subspaces. As a result, we can partition the set $W^{\alpha,\lambda}(K)$ into $(2^t - 1) - 3j$ subpaces of dimension s and $(2^s - 1)j$ subspaces of dimension 2 for any $0 \leq j \leq g$. The same procedure can be carried out on $W^{\alpha+v,\lambda}(K)$ as well. Hence we can partition $K_{\alpha,\lambda}(W) \cup K_{\alpha+v,\lambda}(W) \cup K_{\alpha+v}$ into $(2^{t+1} - 1) - 3j$ subspaces of dimension s and $b + (2^s - 1)j$ subspaces of dimension 2 for $0 \leq j \leq 2g$. (Note that when $j = 2g$, we get 9 subspaces of dimension s.)

In our second method, for each $\alpha \in W'' \oplus Fw$, we partition the set $K_{\alpha,\lambda}(W) \cup K_{\alpha+v,\lambda}(W) \cup K_{\alpha+v}$ into subspaces of dimensions s and 2 in ways that include the cases when the number of s-dimensional subspaces is fewer than 9.

As above, we first use the two-dimensional subspaces $(Z_i)^{\alpha,\lambda}(u)$ and $(Z_i)^{\alpha+v,\lambda}(u)$ for all $0 \neq u \in K$ and $1 \leq i \leq d$ to get $\frac{1}{3}(2^{t+s+1} - 2^{t+s-1} - 2^{s+1} + 2^{s-1})$ subspaces of dimension 2. Once we exclude the union of these subspaces, we are left with

$$K_{\alpha,\lambda}(W_0) \cup K_{\alpha+v,\lambda}(W_0) \cup K_{\alpha+v} = K_\alpha(W_0) \cup K_{\alpha+v}(W_0) \cup K_{\alpha+v},$$

since $W_0 \subseteq \ker(\lambda)$. Now, by Lemma 17, we have $K_\alpha(W_0) \cup K_{\alpha+v}(W_0) \cup K_{\alpha+v} = K_\alpha(W_0 \oplus Fv)$, which we partition into $2^{s-1} - 1$ subspaces of dimension t in $K \oplus K \subseteq V$ of the form K_β for $0 \neq \beta \in W_0 \oplus Fv$. Since s is odd, we have $2^{s-1} - 1 \equiv 0 \pmod{3}$, hence we can group the K_β into triples and use Lemma 16 to partition the union of each triple into either 3 subspaces of dimension s and $2^t - 2^s$ subspaces of dimension 2 or into $2^t - 1$ subspaces of dimension 2.

Using this second method, for each of the triples of t-dimensional subspaces above, we can partition their union into two-dimensional subspaces. As a result, we can partition $K_{\alpha,\lambda}(W) \cup K_{\alpha+v,\lambda}(W) \cup K_{\alpha+v}$ into $2^{s-1} - 1 - 3j$ subspaces of dimension s and $\frac{1}{3}(2^{t+s+1} - 2^{2s-1} - 2^t - 2^{s-1}) + (2^s - 1)j$ subspaces of dimension 2 for $0 \leq j \leq \frac{1}{3}(2^{s-1} - 1)$. Note that when $j = 0$ we have the number of s-dimensional subspaces is $2^{s-1} - 1 \geq 15 > 9$, since $s \geq 5$, hence this accounts for the cases when there are 9 or fewer subspaces of dimension s.

Hence we have proven, for each $\alpha \in W'' \oplus Fw$, the set $K_{\alpha,\lambda}(W) \cup K_{\alpha+v,\lambda}(W) \cup K_{\alpha+v}$ can be partitioned into $(2^{t+1} - 1) - 3j$ subspaces of dimension s and $b + (2^s - 1)j$ subspaces of dimension 2 for $0 \leq j \leq \frac{1}{3}(2^{t+1} - 1)$. Since $|W'' \oplus Fw| = 2^{t-s}$, the union

$$\bigcup_{\alpha \in W'' \oplus Fw} (K_{\alpha,\lambda}(W) \cup K_{\alpha+v,\lambda}(W) \cup K_{\alpha+v})$$

can be partitioned into $2^{t-s}(2^{t+1} - 1) - 3j$ subspaces of dimension s and $2^{t-s}b + (2^s - 1)j$ subspaces of dimension 2 for all $0 \leq j \leq \frac{1}{3}2^{t-s}(2^{t+1} - 1) = \frac{1}{3}(2^{2t-s+1} - 2^{t-s})$.

Note that in the above partitions we have not used the subspaces contained in the K_α for $\alpha \in W'' \oplus Fw$, each of which contains the subspace W_α of dimension s. In all, there are 2^{t-s} such subspaces of dimension s. Since t is odd, $2^{t-s} \equiv 1 \pmod{3}$; so we can group $2^{t-s} - 1$ of these subspaces into triples, and then use Lemma 16 to partition the union of each triple into $2^s - 1$ two-dimensional subspaces. In this way, we have a total of $2^{t-s}(2^{t+1} - 1) + 2^{t-s} - 3j$ subspaces of dimension s and $2^{t-s}b + \frac{1}{3}2^{t-s}(2^t - 2^s) + (2^s - 1)j$ subspaces of dimension 2 in our partition for $0 \leq j \leq \frac{1}{3}(2^{2t-s+1} - 1)$. Therefore we have constructed all the $\mathcal{Q}_j$, and the proof is complete. ∎

To get our main result, we combine Theorems 18–20 with [16, Theorem 1.4], which we restate here for $q = 2$.

Theorem 21 (Theorem 1.4 of [16]) *Let n and s be integers such that $n \geq 2s$ and s is odd. Let c be the least residue of n modulo s, and let θ be the least residue of c modulo 2. Define $m_c = \frac{1}{3}(2^{s\theta+c} - 1)$ and $M_n = \dfrac{2^n - 2^{s\theta+c}}{2^s - 1}$ for all positive integers n. Let a be an integer such that*

$$0 \leq a \leq \frac{2^n - 2^{n-s+1} - 2^{s\theta+c} + 1}{3(2^s - 1)}.$$

Suppose there exists a partition of $V(n, 2)$ with type $s^{x_i}2^{y_i}$ for all

$$x_i = M_n - 3i \quad \text{and} \quad y_i = m_c + (2^s - 1)i$$

such that $a \leq i \leq \lfloor \frac{1}{3}(M_n - (2^{s-1} + 1)) \rfloor$. Then for every integer $a \leq i \leq \lfloor \frac{1}{3} M_{n+s} \rfloor$ there exists a partition of $V(n + s, 2)$ of type $s^{x_i}2^{y_i}$ where

$$x_i = M_{n+s} - 3i \quad \text{and} \quad y_i = m_c + (2^s - 1)i.$$

5.1 Proof of the main theorem

We are now ready to prove our main theorem.

Proof of main theorem: As in the statement of the main theorem, let $s \geq 3$ and $n \geq 2s$ be integers, c be the least residue of n modulo s, $\epsilon = \begin{cases} 1 & \text{if } c \text{ is even} \\ 0 & \text{if } c \text{ is odd,} \end{cases}$ and $h = \dfrac{2^{s+c} - \epsilon(2^s - 1) - 1}{3}$.

We first note that if s is even and n is odd, then there are no solutions to the diophantine equation

$$x(2^s - 1) + 3y = 2^n - 1, \tag{9}$$

so the theorem is vacuously true in this case. Also, for $s = 3$, [7, Theorem 4.4] implies our main theorem.

If s is even and n is even, then c is also even; hence $\epsilon = 1$. So the main theorem is just a restatement of Theorem 6 when $q = 2$ since the nonnegative solutions of $x(2^s-1)+3y = 2^n-1$ with $y \geq h = \frac{1}{3}(2^{s+c}-2^s)$ have the form

$$x = \frac{2^n - 2^{s+c}}{2^s - 1} + 1 - i = \frac{2^n - 1 - 3h}{2^s - 1} - i \quad \text{and} \quad y = \frac{1}{3}(2^{s+c} - 2^s + (2^s - 1)i) = h + \frac{1}{3}(2^s - 1)i$$

for $1 \leq i \leq \left(\dfrac{2^n - 2^{s+c}}{2^s - 1} + 1 \right)$.

Now assume $s \geq 5$ is odd. Write $n = sr + c$ for some integer $r \geq 2$ and $0 \leq c < s$. We will consider two cases according to whether c is even or odd, proceeding by induction on r in each case.

Case 1: Assume c is even, which gives us $\epsilon = 1$ and $h = \frac{1}{3}(2^{s+c} - 2^s)$. Define $m_c = \frac{1}{3}(2^c - 1)$ and $M_n = \dfrac{2^n - 2^c}{2^s - 1}$. (Note that m_c and M_n have the same values as in Theorem 21.) Then $h = 2^s m_c$, and so $x = M_n - 3m_c$ and $y = h$ is a nonnegative integer solution to equation (9). Furthermore, it is straightforward to show all other nonnegative integer solutions of equation (9) must have the form

$$x = M_n - 3(m_c + i) \quad \text{and} \quad y = 2^s m_c + (2^s - 1)i$$

for some integer $0 \le i \le \frac{1}{3}M_n - m_c$. Therefore, it is sufficient to prove that for every integer $0 \le i \le \frac{1}{3}M_n - m_c$, there exists a partition of V of type $s^{x_i}2^{y_i}$, where

$$x_i = M_n - 3(m_c + i) \quad \text{and} \quad y_i = 2^s m_c + (2^s - 1)i.$$

If $r = 2$, then $n = 2s + c$ is even. Let $t = n/2$. In this case, our theorem claims that for each integer $0 \le i \le \frac{1}{3}\left(\frac{2^n - 2^{s+c}}{2^s - 1} + 1\right)$ there exists a partition of V of type $s^{x_i}2^{y_i}$, where

$$x_i = \frac{2^{2s+c} - 2^{s+c}}{2^s - 1} + 1 - 3i = 2^{s+c} + 1 - 3i = 2^{2t-s} + 1 - 3i$$

and

$$y_i = \frac{1}{3}(2^{s+c} - 2^s) + (2^s - 1)i = \frac{1}{3}(2^{2t-s} - 2^s) + (2^s - 1)i.$$

Since $\frac{1}{3}\left(\frac{2^n - 2^{s+c}}{2^s - 1} + 1\right) = \frac{1}{3}\left(2^{s+c} + 1\right) = \frac{1}{3}\left(2^{2t-s} + 1\right)$, this is exactly the result of Theorem 18.

Now assume case 1 is true for $n = s(r-1) + c$, where $r > 2$. Let $a = m_c$. Then, in order to use Theorem 21, we need to show that $a = m_c \le \frac{2^n - 2^{n-s+1} - 3m_c}{3(2^s - 1)}$. So the above inequality is true if and only if

$$2^s(3m_c) = 2^s(2^c - 1) \le 2^n - 2^{n-s+1} = 2^{n-s+1}(2^{s-1} - 1).$$

Since $c \le s - 1$ and $n \ge 2s \Rightarrow n - s + 1 > s$, the above inequality holds. Therefore, the hypotheses for Theorem 21 are satisified, and we conclude that for all $0 \le i \le \frac{1}{3}M_{n+s} - m_c$ there exists a partition of $V(n+s, 2)$ of type $s^{x_i}2^{y_i}$, where

$$x_i = M_{n+s} - 3(m_c + i) \quad \text{and} \quad y_i = 2^s m_c + (2^s - 1)i.$$

This completes the proof of case 1.

Case 2: Assume c is odd, which gives us $\epsilon = 0$ and $h = \frac{1}{3}(2^{s+c} - 1)$. Define $m_c = \frac{1}{3}(2^{s+c} - 1) = h$ and $M_n = \frac{2^n - 2^{s+c}}{2^s - 1}$. (Again, m_c and M_n have the same values as in Theorem 21.) Now we see that $x = M_n$ and $y = h$ is a nonnegative integer solution to equation (9). Furthermore, it is straightforward to show all other nonnegative integer solutions of equation (9) must have the form

$$x = M_n - 3i \quad \text{and} \quad y = m_c + (2^s - 1)i$$

for some integer $0 \le i \le \frac{1}{3}M_n$. Therefore, it is sufficient to prove that for every integer $0 \le i \le \frac{1}{3}M_n$ there exists a partition of V of type $s^{x_i}2^{y_i}$ where

$$x_i = M_n - 3i \quad \text{and} \quad y_i = m_c + (2^s - 1)i.$$

If $r = 2$, then $n = 2s + c$ is odd. Let $t = (n-1)/2$. In this case our theorem claims that for each integer $0 \le i \le \frac{1}{3}M_n = \frac{1}{3}2^{s+c}$ there exists a partition of type $s^{x_i}2^{y_i}$, where

$$x_i = \frac{2^n - 2^{s+c}}{2^s - 1} - 3i = \frac{2^{2s+c} - 2^{s+c}}{2^s - 1} - 3i = 2^{s+c} - 3i = 2^{n-s} - 3i = 2^{2t-s+1} - 3i$$

and

$$y_i = \frac{2^{s+c} - 1}{3} + (2^s - 1)i = \frac{1}{3}(2^{n-s} - 1) + (2^s - 1)i = \frac{1}{3}(2^{2t-s+1} - 1) + (2^s - 1)i.$$

This is exactly what was proved in Theorems 19 and 20. (Note that since $2^{s+c} \equiv 1 \pmod 3$ and i is restricted to integer values, the bounds for i match what is given in both Theorems 19 and 20.)

Now assume $r > 2$ and case 2 is true for $n = (r-1)s + c$. Then for each $0 \leq i \leq \frac{1}{3}M_n$ there exists a partition of $V(n,2)$ of type $s^{x_i}2^{y_i}$, where

$$x_i = M_n - 3i \quad \text{and} \quad y_i = m_c + (2^s - 1)i.$$

Hence, by Theorem 21 we know that for each $0 \leq i \leq \frac{1}{3}M_{n+s}$ there exists a partition of $V(n+s,2)$ of type $s^{x_i}2^{y_i}$, where

$$x_i = M_{n+s} - 3i \quad \text{and} \quad y_i = m_c + (2^s - 1)i.$$

This completes the proof. ∎

5.2 A necessary and sufficient condition when $n \equiv 0, 1, 2 \pmod{s}$

Let $s \geq 3$ be an odd integer, n be an integer such that $2s \leq n$, c be the least residue of n modulo s, $\epsilon = \begin{cases} 1 & \text{if } c \text{ is even} \\ 0 & \text{if } c \text{ is odd,} \end{cases}$ and $h = \dfrac{2^{s+c} - \epsilon(2^s - 1) - 1}{3}$. Our main theorem shows that if

$$x_i = \frac{2^n - 2^{s+c}}{2^s - 1} - 3i + \epsilon = \frac{2^n - 1 - 3h}{2^s - 1} - 3i \quad \text{and} \quad y_i = h + (2^s - 1)i,$$

there exists a partition of $V(n,2)$ of type $s^{x_i}2^{y_i}$ for all integers $0 \leq i \leq \dfrac{1}{3}\left(\dfrac{2^n - 2^{s+c}}{2^s - 1} + \epsilon\right)$.

By using [12, Theorem 1], we can show the converse of our main theorem in some cases. Suppose that there exists a partition of $V(n,2)$ type s^x2^y for $y < y_0 = h$. Then x and y have the forms

$$x = x_0 + 3i \qquad \text{and} \qquad y = y_0 - (2^s - 1)i$$

for some positive integer i. By [12, Theorem 1], we have

$$y \geq \begin{cases} 2^s & \text{if } 2^{s-2} \text{ divides } y \\ 2^{s-1} + 1 & \text{otherwise.} \end{cases}$$

Hence $y_0 - (2^s - 1)i = y \geq 2^{s-1} + 1$. So

$$2^{s+c} - 1 - (2^s - 1)\epsilon \geq 3(2^s - 1)i + 3(2^{s-1} + 1).$$

Thus if

$$2^{s+c} - 1 - (2^s - 1)\epsilon < 3(2^s - 1)i + 3(2^{s-1} + 1) \tag{10}$$

for every positive integer i, then there can be no partition of $V(n,2)$ of type s^x2^y with $y < y_0$. Note that if inequality (10) holds for $i = 1$, then it holds for every positive integer i. So we get the following corollary.

Corollary 22 *Let $s \geq 3$ be an odd integer, $k \geq 2$ an integer, $c = 0, 1,$ or 2, and $\epsilon = \begin{cases} 1 & \text{if } c \text{ is even} \\ 0 & \text{if } c \text{ is odd.} \end{cases}$ Then a partition of $V(ks+c,2)$ of type s^x2^y exists if and only if*

$$x(2^s - 1) + 3y = 2^{ks+c} - 1 \quad \textit{and} \quad y \geq \frac{1}{3}(2^{s+c} - \epsilon(2^s - 1) - 1).$$

Proof. Given our main theorem and the preceeding discussion, it is sufficient to show that inequality (10) holds for $i = 1$ and $c = 0, 1,$ and 2. Since

$$2^{s+c} - 1 - (2^s - 1)\epsilon \leq 2^{s+c} - 1 = 2^{c+1}2^{s-1} - 1 < 9 \cdot 2^{s-1}$$

for $c + 1 \leq 3$, the corollary follows. ∎

References

[1] J. André, Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe, *Math Z.* 60 (1954), 156-186.

[2] A. Beutelspacher, Partitions of finite vector spaces: an application of the Frobenius number in geometry, *Arch. Math.* **31** (1978), 202–208.

[3] A.D. Blinco, S.I. El-Zanati, G.F. Seelinger, P.A. Sissokho, L.E. Spence, C. Vanden Eynden, On vector space partitions and uniformly resolvable designs, *Des. Codes Cryptogr.,* **15**(6) (2008), 69–77.

[4] T. Bu, Partitions of a vector space, *Discrete Math.,* **31** (1980), 79–83.

[5] D. Drake and J. Freeman, Partial t-spreads and group constructible (s, r, μ)-nets. *J. of Geometry* **13/2** (1979), 211–216.

[6] S. El-Zanati, H. Jordon, G. Seelinger, P. Sissokho, L. Spence, The maximum size of a partial 3-spread in a finite vector space over $GF(2)$, *Designs, Codes and Crypt.* **54**(2) (2010), 101–107.

[7] S. El-Zanati, G. Seelinger, P. Sissokho, L. Spence, C. Vanden Eynden, Partitions of finite vector spaces into subspaces, *J. Comb. Designs*, **16**(4) (2008), 329–341.

[8] S. El-Zanati, G. Seelinger, P. Sissokho, L. Spence, and C. Vanden Eynden *On partitions of finite vector spaces of low dimension over* $GF(2)$. Discrete Math 309 (2009), 4727-4735.

[9] S. El-Zanati, O. Heden, G. Seelinger, P. Sissokho, L. Spence, and C. Vanden Eynden, Partitions of the 8-dimensional vector subspace over GF(2), *J. Combin. Des.* 18 (2010), 462-474.

[10] O. Heden, On partitions of finite vector spaces of small dimensions, *Arch. Math.* **43** (1984), 507–509.

[11] O. Heden, Partitions of finite abelian groups, *Europ. J. Combin.* **7** (1986), 11–25.

[12] O. Heden, On the length of the tail of a vector space partition, *Discrete Mathematics*, 309 (2009), 6169–6180.

[13] O. Heden, Necessary and sufficient conditions for the existence of a class of partitions of a finite vector space, *Designs, Codes and Crypt.* **53** (2009), 69–73.

[14] O. Heden and J. Lehmann, Some necessary conditions for vector space partitions, submitted.

[15] S. Hong and A. Patel, A general class of maximal codes for computer applications. *IEEE Trans. Comput.*, **C-21** (1972), 1322–1331.

[16] G. Seelinger, P. Sissokho, L. Spence, C. Vanden Eynden, Partitions of $V(n, q)$ into 2- and s-dimensional subspaces, submitted.