PARTITIONS OF $V(n,q)$ INTO 2-AND s-DIMENSIONAL SUBSPACES

G. SEELINGER, P. SISSOKHO, L. SPENCE, AND C. VANDEN EYNDEN

Abstract. Let $V = V(n,q)$ denote a vector space of dimension n over the field with q elements. A set $\mathcal{P}$ of subspaces of V is a (vector space) *partition* of V if every nonzero element of V is contained in exactly one subspace in $\mathcal{P}$. Suppose that $\mathcal{P}$ is a partition of V with x_i subspaces of dimension d_i for $1 \le i \le k$. Then we call $d_1^{x_1} \ldots d_k^{x_k}$ the *type* of the partition $\mathcal{P}$. Which possible types correspond to actual partitions is in general an open question. We prove that for any odd integer $s \ge 3$ and for any integer $n \ge 2s$, the existence of partitions of $V(n,q)$ across a suitable range of types $s^x 2^y$ guarantees the existence of partitions of $V(n+js,q)$ of essentially all the types $s^x 2^y$ for any integer $j \ge 1$. We then apply this result to construct new classes of partitions of V.

1. Introduction and supporting results

Let $V = V(n,q)$ denote a vector space of dimension n over the field with q elements. A set $\mathcal{P}$ of subspaces of V is a (vector space) *partition* of V if every nonzero element of V is contained in exactly one subspace in $\mathcal{P}$. Let $x_1, \ldots, x_k$ and $d_1, \ldots, d_k$ be positive integers such that the d_i's are distinct. Suppose that $\mathcal{P}$ is a partition of V with x_i subspaces of dimension d_i for $1 \le i \le k$. Then we call $d_1^{x_1} \ldots d_k^{x_k}$ the *type* of the partition $\mathcal{P}$. If $d_1^{x_1} \ldots d_k^{x_k}$ is the type of some partition of V, then $x_1, \ldots, x_k$ must satisfy the Diophantine equation

$$\sum_{i=1}^{k} (q^{d_i} - 1)x_i = q^n - 1.$$

A second necessary condition comes from dimension considerations. If U and W are subspaces of V with $U \cap W = \{0\}$, then $U \oplus W$ is a subspace of dimension $\dim(U) + \dim(W)$. Therefore,

$$\text{if } x_i \ge 2, \text{ then } d_i \le n/2, \quad \text{and} \quad \text{if } i \ne j, \text{ then } d_i + d_j \le n.$$

Other necessary conditions are given by Blinco et al. [4], Bu [5], Heden [12], and Heden and Lehmann [14].

The problem of determining which possible types $d_1^{x_1} \ldots d_k^{x_k}$ correspond to actual partitions of $V(n,q)$ is in general an open question. A few special cases are known. For future reference in this paper, we state three of these results as lemmas. In 1956, André proved the following lemma, which he used to construct translation planes.

Lemma 1.1 (André [1]). *Let n and d be positive integers such that d divides n. Then $V(n,q)$ can be partitioned into $\frac{q^n-1}{q^d-1}$ subspaces of dimension d.*

Later, Beutelspacher and Bu independently proved the following result.

Date: December 5, 2011.

Key words and phrases. Vector space partition; Subspace partition; Partition type.

Lemma 1.2 (Beutelspacher [3] and Bu [5]). *Let n, d be integers such that $1 \le d < n/2$. Then $V(n,q)$ can be partitioned into one subspace of dimension $n-d$ and q^{n-d} subspaces of dimension d.*

Recently, Blinco et al. proved the following lemma, which can be viewed as a generalization of André's result (Lemma 1.1).

Lemma 1.3 (Blinco et al. [4]). *Let r and t be positive integers with $rt = n$, and let x and y be nonnegative integers such that*

$$x(q^r - 1) + y(q^t - 1) = q^n - 1.$$

Then there exists a partition of $V(n,q)$ into x subspaces of dimension r and y subspaces of dimension t.

All partition types of $V(n,q)$ are known in the following cases: for $q = 2$, $k = 2$, $d_2 = 3$, and $d_1 = 2$ (El-Zanati et al. [7]); for $q = 2$ and $n \le 7$ (El-Zanati et al. [8]); for $q = 2$, $n = 8$, and $d_i \ge 2$ for all $1 \le i \le k$ (El-Zanati et al. [9]); for $q = 2$, $k = 3$, $n \ge 9$, $d_3 = n - 3$, $d_2 = 3$, and $d_1 = 2$ (Heden [12]); and finally for $k \ge q + 1$ and $d_{q+1} = d_{q+2} = \ldots = d_k$ (Heden [13]).

In this paper, we generalize the method used by El-Zanati et al. [7] who constructed all the partitions of $V(n,2)$ of types $3^x 2^y$. In particular, we consider partitions of $V(n,q)$ of types $s^x 2^y$, where q is a prime power and $s \ge 3$ is an odd integer (see Remark 1.5 for the case s even). In essence, we prove that the partitions of almost all possible types $s^x 2^y$ can be recursively constructed from a suitable number of base partitions. To be more precise, we first introduce some notation. Let q be a fixed prime power, and let n and s be positive integers such that $s \ge 3$ is odd. Let c be the remainder in the division of n by s, and let $\theta = \theta(c)$ be the least residue of c modulo 2. Finally, define the integers

$$m_c = \frac{q^{s\theta+c} - 1}{q^2 - 1} \text{ and } M_n = \frac{q^n - q^{s\theta+c}}{q^s - 1}. \tag{1}$$

The main result in this paper is the following theorem.

Theorem 1.4. *Let q be a fixed prime power, and let n and s be positive integers such that $n \ge 2s$ and $s \ge 3$ is odd. Let c be the remainder in the division of n by s, and let m_c and M_n be as defined in (1). Finally, let a be an integer such that*

$$0 \le a \le \frac{q^{n-s}(q^s - q) - (q^2 - 1)m_c}{(q^s - 1)(q + 1)}.$$

Suppose there exists a partition of $V(n,q)$ with type $s^{x_i} 2^{y_i}$ for all

$$x_i = M_n - (q+1)i,\ y_i = m_c + \frac{q^s - 1}{q - 1}\, i,\ \text{and } a \le i \le \left\lfloor \frac{M_n}{q+1} \right\rfloor - \frac{q^{s-1} - 1}{q + 1}.$$

Then there exists a partition of $V(n+s,q)$ with type $s^{x_i} 2^{y_i}$ for all

$$x_i = M_{n+s} - (q+1)i,\ y_i = m_c + \frac{q^s - 1}{q - 1}\, i,\ \text{and } a \le i \le \left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor.$$

Remark 1.5.
(*a*) *If s is even, then we can use Lemmas 1.1 and 1.3 to prove Theorem 1.4 by induction.*

(b) *If $x > M_n$, then there is no vector space partition of $V(n,q)$ of type $s^x 2^y$. This follows from a Drake-Freeman* [6] *upper bound on the maximum number of s dimensional subspaces in any partition of $V(n,q)$. Consequently, for any partition of $V(n,q)$ of type $s^x 2^y$, we have $x = x_i$ and $y = y_i$, where*

$$x_i = M_n - (q+1)i,\ y_i = m_n + \frac{q^s-1}{q-1}\, i,\ \text{for some } i \text{ with } 0 \le i \le \left\lfloor \frac{M_n}{q+1} \right\rfloor.$$

The rest of the paper is organized as follows. In Section 2, we prove Theorem 1.4, and in Section 3 we apply it to construct partitions of $V(n,2)$ of almost all possible types $5^x 2^y$ for $n \ge 14$.

Convention. If S is a subspace of some vector space $V(n,q)$, we sometimes use the term "subspace" to also refer to the set $S \setminus \{0\}$.

2. Proof of the main theorem (Theorem 1.4)

The proof of Theorem 1.4 uses several lemmas. Three of them (Lemmas 1.1–1.3) are known results that were already introduced in Section 1.

If U and W are subspaces such that $U \cap W = \{0\}$, we define

$$U \boxplus W = \{0\} \cup [(U+W)\backslash(U \cup W)].$$

We say that $L \boxplus W$ has a partition of type $s^x 2^y$ if there exist x subspaces of dimension s and y subspaces of dimension 2 such that each nonzero vector in $L \boxplus W$ is in exactly one of these subspaces. We introduce two more known lemmas.

Lemma 2.1 (El-Zanati et al. [7]). *Let U and W be subspaces of $V(n,q)$ with $U \cap W = \{0\}$ such that the dimension of U is no more than that of W. Then $U \boxplus W$ can be partitioned into $|W\backslash\{0\}|$ subspaces with the same dimensions as U.*

Lemma 2.2 (El-Zanati et al. [7]). *Let U and W be subspaces of a vector space with $U \cap W = \{0\}$, and let $U_1, \ldots, U_k$ be a partition of U into subspaces. Then each nonzero vector of $U \boxplus W$ is in exactly one of the sets in $\{U_1 \boxplus W, \ldots, U_k \boxplus W\}$.*

In order to prove Theorem 1.4, we prove the next three lemmas.

Lemma 2.3. *Let s and n be integers with $s \ge 3$ odd, and let U and W be subspaces of $V(n,q)$ of dimension s with $U \cap W = \{0\}$. Then for all $0 \le i \le \frac{q^s-q}{q+1}$, $U \boxplus W$ can be partitioned into $(q^s-1)-(q+1)i$ subspaces of dimension s and $\frac{q^s-1}{q-1}\, i$ subspaces of dimension 2.*

Proof. By Lemma 1.3, $V(2s,q)$ has a partition of type $s^x 2^y$ for any nonnegative integers x and y such that

$$x(q^s-1) + (q^2-1)y = q^{2s}-1.$$

Note that $x_i = (q^s+1)-(q+1)i$ and $y_i = \frac{q^s-1}{q-1}\, i$, $0 \le i \le \frac{q^s+1}{q+1}$, are all the nonnegative integer solutions of the above equation. So for each i, $0 \le i \le \frac{q^s+1}{q+1}$, there exists a partition P_i of $V(2s,q)$ with type $s^{x_i} 2^{y_i}$. Since $x_i \ge q+1 \ge 3$, we can select two s-dimensional subspaces U and W from P_i for any i such that $0 \le i \le \frac{q^s+1}{q+1} - 1$. Then $U \cap W = \{0\}$. Hence by Lemma 2.2, $U \boxplus W$ admits a partition into $x_i - 2 = (q^s-1)-(q+1)i$ subspaces of dimension s and $y_i = \frac{q^s-1}{q-1}\, i$ subspaces of dimension 2 for all $0 \le i \le \frac{q^s+1}{q+1} - 1 = \frac{q^s-q}{q+1}$. $\square$

Lemma 2.4. *Let n and s be positive integers with $s \geq 3$ odd. Let L and W be subspaces of $V(n+s,q)$ of dimensions n and s, respectively, such that $L \cap W = \{0\}$. If L has a partition of type $r^w s^x 2^y$ with $r \geq s$, then $L \boxplus W$ admits partitions of all types $s^{x_j} 2^{y_j}$, where*

$$x_j = (q^n - 1) - (q+1)\,j,\ y_j = \left(\frac{q^s - 1}{q-1}\right) j,\ \text{and } (q-1)y \leq j \leq (q-1)y + \left(\frac{q^s - q}{q+1}\right) x.$$

Proof. We first consider the cases when $(w,x,y) = (1,0,0)$, $(w,x,y) = (0,0,1)$, and $(w,x,y) = (0,1,0)$. If $(w,x,y) = (1,0,0)$, by Lemma 2.1 $L \boxplus W$ can be partitioned into $q^r - 1$ subspaces of dimension s. Similarly, if $(w,x,y) = (0,0,1)$, then $L \boxplus W$ can be partitioned into $q^s - 1$ subspaces of dimension 2. If $(w,x,y) = (0,1,0)$, then by Lemma 2.3, we can partition $L \boxplus W$ into $(q^s - 1) - (q+1)i$ subspaces of dimension s and $\left(\frac{q^s-1}{q-1}\right) i$ subspaces of dimension 2, for all $0 \leq i \leq \frac{q^s - q}{q+1}$.

Now for general (w,x,y), we use Lemma 2.2 and the above special cases to partition $L \boxplus W$ into $x_i = w(q^r - 1) - (q+1)i$ subspaces of dimension s and $y_i = y(q^s - 1) + \left(\frac{q^s-1}{q-1}\right) i$ subspaces of dimension 2, for all $0 \leq i \leq x\left(\frac{q^s-q}{q+1}\right)$. Since $L \boxplus W$ has $(q^n - 1)(q^s - 1)$ elements, for each $0 \leq i \leq x\left(\frac{q^s-q}{q+1}\right)$, we have

$$x_i = (q^n - 1) - \left(\frac{q^2 - 1}{q^s - 1}\right) y_i = q^n - 1 - (q^2 - 1)y - (q+1)i.$$

So our result follows by letting $j = (q-1)y + i$. □

Lemma 2.5. *Let q be a fixed prime power, and let n and s be integers such that $n \geq 2s$ and $s \geq 3$ is odd. Let k and c be integers defined by $n = ks + c$ and $0 \leq c < s$. Let $\theta = \theta(c)$ be the least residue of c modulo 2, and let δ_n be the least residue of n modulo 2. Let $m_c = (q^{s\theta + c} - 1)/(q^2 - 1)$ and $M_n = (q^n - q^{s\theta+c})/(q^s - 1)$. Then*

(i) *$M_n, m_c \geq 0$ are integers and $(q^s - 1)M_n + (q^2 - 1)m_c = q^n - 1$.*

(ii) *$M_{n+s} = M_n + q^n$.*

(iii) $\frac{q^s-q}{q+1} M_n + (q-1)\left(\left\lfloor \frac{M_n}{q+1} \right\rfloor + m_c\right) = \left\lfloor \frac{q^n}{q+1} \right\rfloor$.

(iv) $\left\lfloor \frac{M_n}{q+1} \right\rfloor + \left\lfloor \frac{q^n}{q+1} \right\rfloor = \left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor - \delta_n$.

Proof. Parts (i) and (ii) follow easily from the definitions of c, θ, m_c, and M_n. We now prove (iii) and (iv). By (i), we have $(q^s - 1)M_n + (q^2 - 1)m_c = q^n - 1$, which yields

$$\frac{(q^s - q)M_n}{q+1} + \frac{(q-1)M_n}{q+1} + (q-1)m_c = \frac{q^n - 1}{q+1}. \tag{2}$$

Since $n = ks + c \geq 2s$ and $\theta = c \pmod 2$, we have

$$M_n = \frac{q^n - q^{s\theta+c}}{q^s - 1} = \frac{q^{s\theta+c}(q^{(k-\theta)s} - 1)}{q^s - 1} = \sum_{i=1}^{k-\theta} q^{(k-i)s+c}.$$

Since s is odd and c is fixed, M_n is a sum of powers of q in which any two consecutive terms have different parity. Since $q \equiv -1 \pmod{q+1}$, the terms alternate being congruent to 1 and -1 modulo $q+1$. Thus if n (and so $k - \theta$) is even, $M_n \equiv 0 \pmod{q+1}$, while if n is odd, then $M_n \equiv q^{(k-1)s+c} \equiv q^{n-s} \equiv 1 \pmod{q+1}$. Hence

$$\left\lfloor \frac{M_n}{q+1} \right\rfloor = \frac{M_n - \delta_n}{q+1}, \tag{3}$$

where δ_n is 1 if n is odd and 0 if n is even. Similarly, we have

$$\left\lfloor \frac{q^n}{q+1} \right\rfloor = \frac{q^n - 1 - \delta_n(q-1)}{q+1}. \tag{4}$$

Hence

$$\begin{aligned} & \frac{q^s - q}{q+1} M_n + (q-1)\left(\left\lfloor \frac{M_n}{q+1} \right\rfloor + m_c\right) \\ = \ & \frac{q^s - q}{q+1} M_n + (q-1)\left(\frac{M_n - \delta_n}{q+1} + m_c\right) \\ = \ & \frac{q^n - 1}{q+1} - \frac{\delta_n(q-1)}{q+1} = \left\lfloor \frac{q^n}{q+1} \right\rfloor, \end{aligned}$$

where the first equality follows from (3), the second from (2), and the last from (4). This proves (iii).

Finally, we prove (iv). We have

$$\begin{aligned} \left\lfloor \frac{M_n}{q+1} \right\rfloor + \left\lfloor \frac{q^n}{q+1} \right\rfloor &= \frac{M_n - \delta_n}{q+1} + \frac{q^n - 1 - \delta_n(q-1)}{q+1} \\ &= \frac{M_{n+s} - (\delta_{n+s} + \delta_n) - q\delta_n}{q+1} = \left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor - \delta_n, \end{aligned}$$

where the first equality follows from (3) and (4), the second from (ii) and the fact that $\delta_{n+s} + \delta_n = 1$ (because s is odd), and the last from (3).

□

Proof of Theorem 1.4. Let $i \geq 0$ be an integer and set

$$\begin{aligned} & b = \frac{q^{s-1} - 1}{q+1}, \quad B_i = (q-1)m_c + (q^s - 1)i, \\ & R_i = \frac{q^s - q}{q+1} M_n + B_i - (q^s - q)i, \quad \text{and } I = \left\lfloor \frac{M_n}{q+1} \right\rfloor - b, \end{aligned} \tag{5}$$

where n, q, m_c, and s are as defined in the statement of the theorem.

Let L and W be subspaces of $V(n+s, q)$ of dimensions $n \geq 2s$ and s, respectively, such that $L \cap W = \{0\}$. For convenience, we let

$$\begin{gathered} P_j \text{ denote a partition of } L \boxplus W \text{ of type } s^{u_j} 2^{v_j} \text{ with} \\ u_j = q^n - 1 - (q+1)j \text{ and } v_j = \frac{q^s - 1}{q-1} j, \text{ for some integer } j \geq 0. \end{gathered} \tag{6}$$

By Lemma 1.2, L has a partition of type $(n-s)^1 s^{q^{n-s}}$. By Lemma 2.4, this partition yields

$$\text{partitions } P_j \text{ for } 0 \leq j \leq \frac{q^{n-s}(q^s - q)}{q+1}. \tag{7}$$

By the hypothesis of the theorem on the upper bound for a, we have

$$B_a = (q-1)m_c + (q^s - 1)a \leq \frac{q^{n-s}(q^s - q)}{q+1}. \tag{8}$$

Now (7) and (8) yield

$$\text{partitions } P_j \text{ for } 0 \le j \le B_a \le \frac{q^{n-s}(q^s-q)}{q+1}. \tag{9}$$

It again follows from the hypothesis of the theorem that

$$\begin{gathered} L \text{ has partitions } Q_i \text{ of types } s^{x_i}2^{y_i}, \text{ where} \\ x_i = M_n - (q+1)i \text{ and } y_i = m_c + \frac{q^s-1}{q-1}\, i, \\ \text{for all } i \text{ with } a \le i \le \lfloor M_n/(q+1) \rfloor - b. \end{gathered} \tag{10}$$

By Lemma 2.4, each partition Q_i of L of type $s^{x_i}2^{y_i}$ yields

$$\begin{gathered} \text{partitions } P_j \text{ for } B_i \le j \le B_i + \frac{q^s-q}{q+1}\,(M_n-(q+1)i) = R_i, \\ \text{where } a \le i \le \lfloor M_n/(q+1) \rfloor - b. \end{gathered} \tag{11}$$

Using the definitions of B_i and I in (5), we see that $R_{i-1} \ge B_i$ holds for all $a < i \le I$. Hence, the intervals (for the index j) defined in (9) and (11) overlap. By setting $i = I$ in (11), we then obtain

$$\text{partitions } P_j \text{ for } 0 \le j \le R_I = \left\lfloor \frac{q^n}{q+1} \right\rfloor - (q-1)b, \tag{12}$$

where the value of R_I is computed using (5) and Lemma 2.5(iii) as follows:

$$\begin{aligned} R_I &= B_I + \frac{q^s-q}{q+1} M_n - (q^s-q)I \\ &= \frac{q^s-q}{q+1} M_n + (q-1)\left(\left\lfloor \frac{M_n}{q+1} \right\rfloor + m_c - b\right) = \left\lfloor \frac{q^n}{q+1} \right\rfloor - (q-1)b. \end{aligned}$$

Since $L \boxplus W = \{0\} \cup [(L+W) \backslash (L \cup W)]$ and $\dim(W) = s$, it follows from (6), (10), and (12) that

$$\begin{gathered} V(n+s,q) = L+W \text{ has partitions of types } s^{x_{ij}}2^{y_{ij}}, \text{ where} \\ x_{ij} = (M_n+1) + q^n - 1 - (q+1)(i+j) \text{ and } y_{ij} = m_c + \frac{q^s-1}{q-1}\,(i+j), \\ \text{for all } i,j \text{ with } a \le i \le \lfloor M_n/(q+1) \rfloor - b \text{ and } 0 \le j \le \left\lfloor \frac{q^n}{q+1} \right\rfloor - (q-1)b. \end{gathered} \tag{13}$$

By Lemma 2.5(ii), we have $M_{n+s} = q^n + M_n$. So by letting $h = i+j$, we deduce from (13) that

$$\begin{gathered} V(n+s,q) \text{ has partitions of types } s^{x_h}2^{y_h}, \text{ where} \\ x_h = M_{n+s} - (q+1)h \text{ and } y_h = m_c + \frac{q^s-1}{q-1}h, \ \text{ and for all } h \text{ with} \\ a \le h \le \left\lfloor \frac{M_n}{q+1} \right\rfloor - b + \left\lfloor \frac{q^n}{q+1} \right\rfloor - (q-1)b = \left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor - qb - \delta_n, \end{gathered} \tag{14}$$

where the last equality again comes from Lemma 2.5(iv).

For convenience, we let

$$\Pi_h \text{ denote a partition of } V(n+s,q) \text{ of type } s^{x_h}2^{y_h} \text{ with}$$
$$x_h = M_{n+s} - (q+1)h \text{ and } y_h = m_c + \frac{q^s-1}{q-1}h, \text{ for some integer } h \geq 0.$$

Then to finish the proof of the theorem, it remains to show the existence of

$$\text{partitions } \Pi_h \text{ for } \left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor - qb - \delta_n \leq h \leq \left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor. \tag{15}$$

We consider the cases n odd and n even.

Case 1: n is odd. By Lemma 1.2, there is a partition F of $V(n+s,q)$ into subspaces of dimensions $2s$ and $(n+s)-2s = n-s$. (The number of subspaces of each dimension is at least one and depends on whether $2s < (n+s)/2$ or not.) Since both $2s$ and $n-s$ are even, this partition F can be changed (by Lemma 1.1) into a partition F' of $V(n+s,q)$ of type $(2s)^1 2^y$, with $y = (q^{n+s} - q^{2s})/(q^2-1)$.

We now apply Lemma 1.3 to the $2s$-dimensional subspace in the partition F' to obtain

$$\text{partitions of } V(n+s,q) \text{ of types } s^{e_i}2^{f_i}, \text{ with} \tag{16}$$
$$e_i = q^s + 1 - (q+1)i,\ f_i = \frac{q^{n+s}-q^{2s}}{q^2-1} + \frac{q^s-1}{q-1}\,i, \text{ for } 0 \leq i \leq \frac{q^s+1}{q+1}.$$

Since $n+s$ is even, it follows from (3) that

$$\left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor = \frac{M_{n+s}}{q+1}. \tag{17}$$

Using Lemma 2.5(i) and (17), we can describe the partitions in (16) as

$$\text{partitions } \Pi_h \text{ for } H = \left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor - \frac{q^s+1}{q+1} \leq h \leq \left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor. \tag{18}$$

Finally, by (18) and the definition of b in (5), we have

$$\text{partitions } \Pi_h \text{ for } H = \left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor - qb - 1 \leq h \leq \left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor,$$

as required in (15), because $\delta_n = 1$ for n odd.

Case 2: n is even. Since s odd and $n \geq 2s$ is even, there exist integers k and c such that $n = 2s + (k-2)s + c$, $k \geq 2$, $0 \leq c < s$, and $n - 2s$ is even. Similarly to the argument used at the start of Case 1, we can use Lemma 1.2 (based on whether $2s < n - 2s$ or not) and Lemma 1.1 (since both $2s$ and $n-2s$ are even) to infer the existence of a partition F_1 of the n-dimensional subspace L of type $(2s)^1 2^y$, with $y = (q^n - q^{2s})/(q^2-1)$. Moreover, since n is even, Lemma 1.1 yields a partition F_1' of L of type $2^{y'}$, with $y' = (q^n-1)/(q^2-1)$. By applying Lemma 2.4 with the partition F_1' of L and the partition of W of type s^1, we obtain a partition F_2 of $L \boxplus W$ of type 2^z, with $z = (q^n-1)(q^s-1)/(q^2-1)$. By combining the other partition F_1 of L, the partition F_2 of $L \boxplus W$, and the partition of W of type s^1, we obtain a partition F of $V(n+s,q) = L \oplus W$ of type

$$(2s)^1 s^1 2^{y+z}, \text{ with } y + z = \frac{q^{n+s} - q^{2s} - q^s + 1}{q^2-1}.$$

We now apply Lemma 1.3 to the $2s$-dimensional subspace in the partition F above to obtain

$$\text{partitions of } V(n+s,q) \text{ of types } s^{\alpha_i}2^{\beta_i}, \text{ with} \tag{19}$$

$$\alpha_i = q^s + 2 - (q+1)i,\ \beta_i = \frac{q^s-1}{q-1}i + \frac{q^{n+s}-q^{2s}-q^s+1}{q^2-1}, \quad \text{for } 0 \le i \le \frac{q^s+1}{q+1}.$$

Since $n+s$ is odd, it follows from (3) that

$$\left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor = \frac{M_{n+s}-1}{q+1}. \tag{20}$$

Using Lemma 2.5(i) and (20), we can describe the partitions in (19) as

$$\text{partitions } \Pi_h \text{ for } H = \left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor - \frac{q^s+1}{q+1} \le h \le \left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor. \tag{21}$$

Finally, by (21) and the definition of b in (5), we have

$$\text{partitions } \Pi_h \text{ for } H < \left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor - qb \le h \le \left\lfloor \frac{M_{n+s}}{q+1} \right\rfloor,$$

as required in (15), because $\delta_n = 0$ for n even. □

3. Application: Partitions of $V(n,2)$ of type 5^x2^y

In this section, we consider the case when $q = 2$ to construct partitions of $V(n,2)$ of type 5^x2^y for all $n \ge 14$. Since this construction is based on a recursive application of our main theorem (Theorem 1.4), we only need to consider the base cases $n \in \{10, 12, 14, 16, 18\}$. Let $F = \mathbb{F}_2$ and V be an F-vector space of dimension $n = 2t$ for $5 \le t \le 9$. Let K be a field extension of degree t over F and $V = K \oplus K$. Then V can be considered a $2t$-dimensional F-vector space or a 2-dimensional K-vector space. One way to realize an F-vector space partition of V of type t^{2^t+1} is to consider the 1-dimensional K-subspaces of V, which we can identify with the projective line over K, $\mathbb{P}^1_K$. The elements of $\mathbb{P}^1_K$ are of the form Kv for some nonzero $v \in V$. We call this partition the *projective line partition*. For any $(a,b) \in K \oplus K$, let $K(a,b) = \{(ka,kb) : k \in K\}$. Let $K_\infty = K(0,1)$ and for $\alpha \in K$, let $K_\alpha = K(1,\alpha)$. If $W \subseteq K$ is an F-subspace, we use W_α to denote the subspace $\{(w,\alpha w) : w \in W\}$.

Next, we give some constructions that will let us reconfigure some of these subspaces to get F-vector space partitions of V of different types.

Given an F-subspace $U \subseteq K$, $\alpha \in K$, and $x \in K^\times = K \setminus \{0\}$, we define

$$U_\alpha(x) = \{(xy, (x+\alpha)xy) : y \in U\} \subseteq K_{x+\alpha} = K(1, x+\alpha).$$

Therefore, we see that

$$U_\alpha(x) = U_\alpha(x') \Leftrightarrow x = x'. \tag{22}$$

Similarly, for any F-subspace $W \subseteq K$ and for any $\alpha \in K$, $y \in K^\times$, we define

$$W^\alpha(y) = \{(wy, (w+\alpha)wy) : w \in W\}.$$

Lemma 3.1. *Let $\alpha \in K$ and $U, W \subseteq K$ be F-subspaces of dimensions r and s, respectively. Then for any $y \in K^\times$, the sets $U_\alpha(y)$ and $W^\alpha(y)$ are subspaces of V of dimensions r and s, respectively. Furthermore,*

$$\bigcup_{x \in W\setminus\{0\}} U_\alpha(x) = \bigcup_{y \in U\setminus\{0\}} W^\alpha(y) = \{(xy, (x+\alpha)xy) : x \in W, y \in U\}.$$

Proof. Let $\alpha, y \in K$ with $y \neq 0$. Since the function $\nu_{\alpha,y} : K \to V$ defined by $\nu_{\alpha,y}(x) = x(y, (y+\alpha)y) = (yx, (y+\alpha)yx)$ is an injective linear transformation, $U_\alpha(y)$ is a subspace of dimension r.

To show that $W^\alpha(y)$ is a subspace of dimension s, we define the function $\phi_{\alpha,y} : K \to V$ by $\phi_{\alpha,y}(x) = (xy, (x+\alpha)xy)$ for all $x \in K$. We claim that $\phi_{\alpha,y}$ is an injective F-linear transformation. Indeed, for any $x, x' \in K$, we have

$$\begin{aligned} \phi_{\alpha,y}(x) + \phi_{\alpha,y}(x') &= (xy, (x+\alpha)xy) + (x'y, (x'+\alpha)x'y) \\ &= \big((x+x')y, (x^2 + (x')^2 + \alpha(x+x'))y\big) \\ &= \big((x+x')y, ((x+x')^2 + \alpha(x+x'))y\big) \\ &= \phi_{\alpha,y}(x+x'), \end{aligned}$$

since $q = 2$. Furthermore, $\phi_{\alpha,y}(x) = 0 \Rightarrow xy = 0 \Rightarrow x = 0$ since $y \in K^\times$, hence $\phi_{\alpha,y}$ is injective.

Finally, for any F-subspace $W \subseteq K$ of dimension s, we have $W^\alpha(y) = \phi_{\alpha,y}(W)$. Hence $W^\alpha(y)$ is an F-subspace of V of dimension s.

The last equality in the statement of the lemma is immediate. □

For any $\alpha \in K$ and subspaces $U, W \subseteq K$ define

$$U_\alpha(W) = \bigcup_{w \in W\setminus\{0\}} U_\alpha(w) = \bigcup_{u \in U\setminus\{0\}} W^\alpha(u) = W^\alpha(U). \tag{23}$$

In general, we use $U_\alpha(W)$ to emphasize the above set as a union of the subspaces $U_\alpha(w)$ and we write $W^\alpha(U)$ to emphasize the above set as the union of the subspaces $W^\alpha(u)$. Furthermore, we observe that $K_\infty \cap K_\alpha(W) = \{0\}$ since in $K_\alpha(w)$ the first component is never zero except when we consider the zero element. Furthermore, $K_0 \cap K_\alpha(W) = \{0\}$ if and only if $\alpha \notin W$.

Next, we want to show that Equation (22) has an analogy for the $W^\alpha(y)$.

Theorem 3.2. *Let $\alpha \in K$ and let $W \subseteq K$ be a subspace of dimension s. Then for any $y, y' \in K^\times$,*

$$W^\alpha(y) \cap W^\alpha(y') \neq \{0\} \Leftrightarrow W^\alpha(y) = W^\alpha(y') \Leftrightarrow y = y'.$$

Therefore, for any subspace $U \subseteq K$ of dimension r, the set $U_\alpha(W)$ consisting of $2^s - 1$ subspaces of dimension r can also be considered a set of $2^r - 1$ subspaces of dimension s having pairwise trivial intersection.

Proof. Let $\alpha \in K$, let $W \subseteq K$ be a subspace of dimension s, and let $y, y' \in K^\times$. Assume $v \in W^\alpha(y) \cap W^\alpha(y')$ and $v \neq 0$. Then there exist nonzero $w, w' \in W$ such that $(wy, (w+\alpha)wy) = v = (w'y', (w'+\alpha)w'y')$. Hence $wy = w'y'$, so $(w+\alpha)wy = (w'+\alpha)w'y' \Rightarrow w+\alpha = w'+\alpha \Rightarrow w = w'$. Since $w = w' \neq 0$, we have $wy = w'y' \Rightarrow y = y'$. Hence $W^\alpha(y) = W^\alpha(y')$.

Since there are $2^r - 1$ elements in $U^\times = U \setminus \{0\}$, our proven condition tells us there are $2^r - 1$ distinct s-dimensional subspaces in the set $\{W^\alpha(y) : y \in U^\times\}$ with pairwise trivial intersection. Since $U_\alpha(W) = \bigcup_{y \in U^\times} W^\alpha(y)$, it follows from Lemma 3.1 and a counting argument that $U_\alpha(W)$ is the union of $2^s - 1$ subspaces with pairwise trivial intersection of the form $U_\alpha(w)$ for $w \in W \setminus \{0\}$. □

Proposition 3.3. *Let $K = W \oplus W'$. Then for any $\alpha,\ \beta \in W'$ with $\alpha \neq \beta$, we have $K_\alpha(W) \cap K_\beta(W) = \{0\}$.*

Proof. Let $w \in K_\alpha(W) \cap K_\beta(W)$. Then there exist $x, y \in W$ such that $w \in K_\alpha(x) \cap K_\beta(y)$. If $w \neq 0$, then we have $x + \alpha = y + \beta$. But this gives $x - y = \beta - \alpha \in W \cap W' = \{0\} \Rightarrow \beta = \alpha$, which is a contradiction. Therefore, $w = 0$ and so $K_\alpha(W) \cap K_\beta(W) = \{0\}$. □

It follows from Equation (22) that

$$K_\alpha(w) = K_\alpha(w') \Leftrightarrow w + \alpha = w' + \alpha \Leftrightarrow w = w'.$$

Therefore, the subspaces in $\{K_\alpha(w) :\ w \in W\}$ partition the set $K_\alpha(W)$.

Theorem 3.4. *Let V and K be as above. Then for any $s < t$, there exists a partition $\mathcal{Q}$ of type $t^a s^b$, where $a = 2^{t-s} + 1$ and $b = 2^{2t-s} - 2^{t-s}$. Furthermore, the t-dimensional subspaces in this partition are K_∞ and K_β for all $\beta \in W'$, where W' is a subspace of K of F-dimension $t - s$.*

Proof. Let W and W' be F-subspaces of K such that $K = W \oplus W'$ and $\dim(W) = s$, so $\dim(W') = t - s$. Then for each $\beta \in W'$ we can use Theorem 3.2 to reconfigure the $2^s - 1$ subspaces in $K_\beta(W)$ into the $2^t - 1$ subspaces $W^\beta(y)$ of dimension s for all $y \in K \setminus \{0\}$. Since by Proposition 3.3 pairwise intersections of the $K_\beta(W)$ are trivial for distinct β in W', we get the appropriate partition. It is also straightforward to check that $K_0 \cap K_\beta(W) = \{0\} = K_\infty \cap K_\beta(W)$ for all $\beta \in W'$, and hence K_0 and K_∞ are both in the resulting partition.

Furthermore, since $K_\beta(w) = \{(wy, (w + \beta)wy) : y \in K\}$, where $w \in W \setminus \{0\}$, we see that for any $\gamma \in W'$ we have

$$z \in (K_\gamma \cap K_\beta(W)) \setminus \{0\} \Rightarrow (wy, (w + \beta)wy) = (xy', \gamma xy')$$

for some $w \in W \setminus \{0\}$, $x, y, y' \in K \setminus \{0\}$ and $\beta \in W'$. As $wy = xy' \neq 0$, we have $w + \beta = \gamma \Rightarrow \beta - \gamma = w \neq 0$. However, then $w \in W \cap W' = \{0\}$, which is a contradiction. So $K_\gamma \cap K_\beta(W) = \{0\}$ for all $\beta, \gamma \in W'$. □

Now we are able to construct partitions of $V(n, 2)$ of type $5^x 2^y$ for $n = 2t$ with $5 \leq t \leq 9$. The case $t = 5$ is trivial since for any solution (x, y) of the Diophantine equation $31x + 3y = 1023$, there exists a partition of $V(10, 2)$ of type $5^x 2^y$ by Lemma 1.3. For the cases $5 < t \leq 9$, we consider the decomposition $K = X \oplus X'$, where X and X' are subspaces with $\dim(X) = 5$ and $\dim(X') = t - 5$.

Then

$$\{K_\infty\} \cup \{K_\alpha \,|\, \alpha \in X'\} \cup \{K_\alpha(X) \,|\, \alpha \in X'\}$$

is a partition of V of type t^{2^t+1}, namely our initial projective line partition. Also, by Theorem 3.2, each $K_\alpha(X)$ can be reconfigured to $2^t - 1$ subspaces of dimension 5. Therefore, we get the partition of type $t^a 5^b$, with $a = 2^{t-5} + 1$ and $b = 2^{t-5}(2^t - 1)$. This partition is given in Theorem 3.4 with $W = X$, where $K_\alpha(X) = X^\alpha(K)$ for each $\alpha \in X'$.

The next lemma allows us to change 3 subspaces of dimension t into subspaces of dimensions 2 and 5.

Lemma 3.5. *Let A_1, A_2, A_3 be subspaces of V of dimension t such that $A_i \cap A_j = \{0\}$ for all $1 \leq i < j \leq 3$. For $i = 1, 2$, let $\pi_i : V = A_1 \oplus A_2 \to A_i$ be the corresponding projection. Then:*

(1) *for any $x \in A_3 \setminus \{0\}$ the set $B_x = \{0, x, \pi_1(x), \pi_2(x)\}$ is a subspace of dimension 2 contained in $A_1 \cup A_2 \cup A_3$.*

(2) $A_1 \cup A_2 \cup A_3$ *is the union of* 3 *subspaces of dimension* 5 *and* $2^t - 2^5$ *subspaces of dimension* 2 *whose pairwise intersections are trivial.*

Proof. (1) Since $V = A_1 \oplus A_2$, for any $v \in V$ we have $v = \pi_1(v) + \pi_2(v)$. Therefore, since $q = 2$, it follows that B_x is a subspace of dimension 2 contained in $A_1 \cup A_2 \cup A_3$.
(2) Let $C \subseteq A_3$ be a subspace of dimension 5. Then we take C, $\pi_1(C)$, $\pi_2(C)$ to be the 3 subspaces of dimension 5 and B_x for $x \in A_3 \setminus C$ to be the $2^t - 2^5$ subspaces of dimension 2. □

Next, we let $z \in X' \setminus \{0\}$ and consider a direct sum decomposition $X' = X'' \oplus Fz$ for some (possibly trivial) subspace X''.

Lemma 3.6. *For any $\alpha \in X''$, we have*

$$K_\alpha(X \oplus Fz) = K_\alpha(X) \cup K_{\alpha+z}(X) \cup K_{\alpha+z}.$$

Proof. It follows from the definition of $K_\alpha(X \oplus Fz)$ that

$$K_\alpha(X \oplus Fz) = K_\alpha(X) \cup K_\alpha(X + z) \cup K_\alpha(z).$$

Furthermore, for any $x \in X \setminus \{0\}$, we have

$$K_\alpha(x + z) = (x + z)K_{x+z+\alpha} = x^{-1}(x + z)K_{z+\alpha}(x) = K_{z+\alpha}(x).$$

Similarly, $K_\alpha(z) = zK_{\alpha+z} = K_{\alpha+z}$. □

Now we can prove the main theorem of this section, which we will combine with Theorem 2.1 to get the existence of a range of partitions of type 5^x2^y when $n \geq 14$

Theorem 3.7. *Let* $\dim(V) = n \in \{12, 14, 16, 18\}$. *Then for each* $0 \leq i \leq \frac{1}{3}\left(2^{n-5} + 1\right)$, *there exists a partition $\mathcal{P}_i$ of V of type $5^{x_i}2^{y_i}$, where*

$$x_i = (2^{n-5} + 1) - 3i \quad \textit{and} \quad y_i = \frac{1}{3}(2^{n-5} - 32) + 31i.$$

Proof. Let K be a field of degree $t = \frac{n}{2}$ over F, and identify V with $K \oplus K$. We again consider the decomposition $K = X \oplus X'$, where X and X' are subspaces such that $\dim(X) = 5$ and $\dim(X') = t-5$. We break this proof up into the case when t is even and the case when t is odd.

Case 1: Assume that t is equal to 6 or 8. To prove that $\mathcal{P}_0$ exists, we start with the partition $\mathcal{Q}$ given in Theorem 3.4 consisting of $2^{n-5} - 2^{t-5}$ subspaces of dimension 5 and $2^{t-5} + 1$ subspaces of dimension t, where $W = X$ and for every $\alpha \in X'$, it follows from Theorem 3.2 that the set $K_\alpha(X)$ consists of $2^t - 1$ subspaces of dimension 5. In addition, we have $2^{t-5} + 1$ subspaces of dimension t, which are the subspaces $\{K_\alpha : \alpha \in X'\} \cup \{K_\infty\}$.

Since t is equal to 6 or 8, $2^{t-5} + 1$ is equal to 3 or 9. As a result, we can group the remaining subspaces of dimension t into $\frac{1}{3}(2^{t-5} + 1)$ triples. Using Lemma 3.5, each of these triples gives 3 subspaces of dimension 5 and $2^t - 32$ subspaces of dimension 2. Hence, from these $\frac{1}{3}(2^{t-5}+1)$ triples of subspaces of dimension t, we get $2^{t-5} + 1$ subspaces of dimension 5 and $\frac{1}{3}(2^{t-5} + 1)(2^t - 32) = \frac{1}{3}(2^{n-5} - 32)$ subspaces of dimension 2. Since there is a total of $(2^{n-5} - 2^{t-5}) + (2^{t-5} + 1) = 2^{n-5} + 1$ subspaces of dimension 5, we have a partition of type $5^{x_0}2^{y_0}$.

Next, to get the remaining partitions $\mathcal{P}_i$, it is sufficient to create an appropriate number M of triples of subspaces of dimension 5 that can be reconfigured into $31 \cdot M$ subspaces of dimension 2.

As t is even, there exists a partition of K consisting of $c = \frac{1}{3}(2^t - 1)$ subspaces of dimension 2. Let $U_1, \ldots, U_c$ be the subspaces of this partition. Then for any $\alpha \in X'$, it follows from Theorem 3.2 that the 2^t-1 subspaces of dimension 5 in $K_\alpha(X)$ can be grouped into sets of three $\{X^\alpha(u) : u \in U_j \setminus \{0\}\}$ for $1 \le j \le c$. Since

$$X^\alpha(U_j) = \bigcup_{u \in U_j \setminus \{0\}} X^\alpha(u) = (U_j)_\alpha(X),$$

these sets of three can be reconfigured to 31 subspaces of dimension 2. Thus, we need only consider the partitions of the subspaces K_α, for $\alpha \in X'$, and K_∞. But in $\mathcal{P}_0$ these were grouped in triples. Since t is even, we can take any three of these subspaces and partition each individual one into $\frac{1}{3}(2^t-1)$ subspaces of dimension 2. Hence a group of 3 subspaces of dimension 5 and 2^t-32 subspaces of dimension 2 can be reconfigured into 2^t-1 subspaces of dimension 2. So the theorem follows when t is even.

Case 2: Assume that t is equal to 7 or 9. To prove that $\mathcal{P}_0$ exists, we again start with the partition $\mathcal{Q}$ in Theorem 3.4 (with $W = X$). This partition gives us $2^{n-5}-2^{t-5}$ subspaces of dimension 5 and $2^{t-5}+1$ subspaces of dimension t. Again, for every $\alpha \in X'$, it follows from Theorem 3.2 that the set $K_\alpha(X)$ consists of 2^t-1 subspaces of dimension 5 in our partition and the $2^{t-5}+1$ subspaces of dimension t are the subspaces $\{K_\alpha : \alpha \in X'\} \cup \{K_\infty\}$.

Since t is odd, we can partition each of the subspaces of dimension t above into one subspace of dimension 5 and $\frac{1}{3}(2^t-32)$ subspaces of dimension 2. Therefore, we have an additional $2^{t-5}+1$ subspaces of dimension 5 and $\frac{1}{3}(2^t-32)(2^{t-5}+1) = \frac{1}{3}(2^{2t-5}-32)$ subspaces of dimension 2. Hence, here again, we have a partition of V of type $5^{x_0}2^{y_0}$.

As in Case 1, we get the remaining partitions $\mathcal{P}_i$ by taking an appropriate number N of triples of subspaces of dimension 5 and reconfiguring them into $31 \cdot N$ subspaces of dimension 2.

Choose $z \in X' \setminus \{0\}$ and $X'' \subseteq X'$ such that $X' = X'' \oplus Fz$. Then we can write V as the union of subspaces

$$V = K_\infty \cup \left(\bigcup_{\alpha \in X''} K_\alpha \right) \cup \left(\bigcup_{\alpha \in X''} K_\alpha(X \oplus Fz) \right).$$

Since $\dim(X'') = t-6 > 0$ is odd, the set $\{K_\infty\} \cup \{K_\alpha : \alpha \in X''\}$ has order divisible by 3. Hence, we can group these into triples and use Lemma 3.5 to reconfigure 3 subspaces of dimension 5 and 2^t-32 subspaces of dimension 2 into a set of 2^t-1 subspaces of dimension 2.

Since t is equal to 7 or 9, we use Lemma 1.1 and Lemma 1.2 to construct a partition of K of type $5^1 2^b$, where $b = \frac{1}{3}(2^t-32)$. Let X be the subspace of dimension 5 and $Y_1, \ldots, Y_b$ be the subspaces of dimension 2 this partition. We can recursively apply Lemma 1.2 to construct a partition of X of type $2^9 1^4$. Let $Y_{b+1}, \ldots, Y_g$ be the subspaces of dimension 2 in this partition, where $g = \frac{1}{3}(2^t-5)$.

Next, for each $\alpha \in X''$, we use two methods to group the elements of the sets $K_\alpha(X \oplus Fz)$ into pairwise trivially intersecting sets of subspaces of dimensions 2 and 5. For each $\alpha \in X''$, our first method will give us nine or more subspaces of dimension 5, while our second method will give us nine or fewer subspaces of dimension 5.

In our first method, we apply Lemma 3.6 to the set $K_\alpha(X \oplus Fz) = K_\alpha(X) \cup K_{z+\alpha}(X) \cup K_{z+\alpha}$. As with K, we can partition $K_{z+\alpha}$ into 1 subspace of dimension 5 and b subspaces of dimension 2. Next, by Theorem 3.2, the elements of each of the sets $(Y_j)_\alpha(X')$ and $(Y_j)_{z+\alpha}(X')$ can be grouped as 3 subspaces of dimension 5 or 31 subspaces of dimension 2. In this way, the elements of $K_\alpha(X \oplus Fz)$ can be grouped, using 3 subspaces of dimension 5 at a time, so that there are at least 9 subspaces of dimension 5 in this set and the rest are subspaces of dimension 2. Hence we have reorganized the set $K_\alpha(X \oplus Fz)$ from a grouping of $2^{t+1}-1$ subspaces of dimension 5 and $\frac{1}{3}(2^t-32)$ subspaces of dimension 2 into nine subspaces of dimension 5 and $\frac{1}{3}\left(2^t - 32 + 31(2^{t+1} - 10)\right) = (21 \cdot 2^t - 114)$ subspaces of dimension 2 with pairwise trivial intersections.

In our second method, we regard the set $K_\alpha(X \oplus Fz)$ as the union of 63 subspaces of the form K_β. Then we can group these subspaces K_β into triples and use Lemma 3.5 to group each triple into 3 subspaces of dimension 5 and $2^t - 32$ subspaces of dimension 2. In this way, we can group the elements of $K_\alpha(X \oplus Fz)$ into 63 subspaces of dimension 5 and $21(2^t - 32)$ subspaces of dimension 2 all of which have pairwise trivial intersections. Since the number of subspaces of dimension 5 here is greater than 9, this gives us another way to group the elements of $K_\alpha(X \oplus Fz)$ into 63 subspaces of dimension 5 and $21(2^t - 32)$ of dimension 2. Now we can use Lemma 3.5 to convert each triple of subspaces of the form K_β from 3 subspaces of dimension 5 and $2^t - 32$ subspaces of dimension 2 into $2^t - 1$ subspaces of dimension 2 with pairwise trivial intersections.

By using the two methods above to convert $K_\alpha(X \oplus Fz)$ for each $\alpha \in X''$, one at a time, we get all the remaining partition types. This concludes the proof. $\square$

We conclude this section with the following corollary, which follows from Theorem 1.4 and Theorem 3.7. We first recall the definitions of m_c and M_n (see (1)) when $s = 5$ and $q = 2$. Let c be the remainder in the division of n by 5, and let $\theta = \theta(c)$ be the least residue of c modulo 2. In this case,

$$M_n = \frac{2^n - 2^{5\theta+c}}{31} \text{ and } m_c = \frac{2^{5\theta+c} - 1}{3}. \tag{24}$$

Corollary 3.8. *Let $n \geq 14$ and c be the remainder in the division of n by 5. Let m_c and M_n be as defined in (24). Then for any integer i with $m_c \leq i \leq \lfloor M_n/3 \rfloor$, there exists a partition $\mathcal{P}_i$ of $V(n,2)$ with type $5^{x_i}2^{y_i}$, where*

$$x_i = M_n - 3i \text{ and } y_i = m_c + 31i.$$

Proof. For any integer $k \geq 2$ and $c = 0$, the corollary follows from Lemma 1.3.

Let c be a fixed integer such that $0 < c < 5$, and let k_c be the smallest integer such that $k_c \geq 2$ and $k_c + c$ is even. Then for any integer $k \geq k_c$, we can prove the corollary for $n = 5k + c$ by using induction on k where the base cases $n = 5k_c + c$ are given by Theorem 3.7, and the inductive step is given by Theorem 1.4.

For instance, let us consider the case $n = 12$. Then $c = 2$ and $k_2 = 2$. Now it follows from Theorem 3.7 that for each integer i with $0 \leq i \leq 43$, there exists a partition $\mathcal{P}_i$ of $V(12,2)$ of type $5^{x_i}2^{y_i}$, where $x_i = 129 - 3i$ and $y_i = 32 + 31i$.

Let M_n and m_c be as defined in (24). Then we can easily compute $M_{12} = 132$ and $m_{12} = 1$. If we set a to $m_2 = 1$ in Theorem 1.4, then for any integer i with $1 \leq i \leq \lfloor M_{12}/3 \rfloor = 44$, there exists a partition $\mathcal{P}_i$ of $V(12,2)$ with type $5^{x_i}2^{y_i}$, where

$$x_i = 132 - 3i = M_{12} - 3i \text{ and } y_i = 1 + 31i = m_2 + 31i.$$

This establishes the base case (i.e., $k = k_2 = 2$) of the induction argument when $n = 5k + 2$.

Now assume that for any integer $k \geq 2$ and for any integer i with $1 \leq i \leq \lfloor M_n/3 \rfloor$, there exists a partition $\mathcal{P}_i$ of $V(n,2)$ with type $5^{x_i}2^{y_i}$, where $x_i = M_n - 3i$ and $y_i = m_2 + 31i$. Then by Theorem 1.4, for any integer i with $1 \leq i \leq \lfloor M_{n+5}/3 \rfloor$, there exists a partition $\mathcal{P}_i$ of $V(n+5,2)$ with type $5^{x_i}2^{y_i}$, where $x_i = M_{n+5} - 3i$ and $y_i = m_{n+5} + 31i$.

Since $n+5 = 5(k+1)+2$, this proves the inductive step. Hence, the corollary holds for all integers $n = 5k + 2$ and $k \geq k_2 = 2$.

For $c = 4, 1$, and 3, we have $k_c = 2, 3$, and 3, respectively. The corresponding base cases $n = 14, 16$, and 18 are given by Theorem 3.7. Then an argument similar to the one used for $n = 5k + 2$ above yields the appropriate partition $\mathcal{P}_i$ of $V(n, 2)$ for $n = 5k + c$ with $k \geq k_c$ when $m_c \leq i \leq \lfloor M_n/3 \rfloor$. $\square$

Acknowledgements: The authors thank one the referees for his/her detailed comments that led to an improved version of the paper.

References

[1] J. André, Über nicht-Desarguessche Ebenen mit transitiver Translationsgruppe, *Math. Z.* (1954), 156–186.
[2] A. Beutelspacher, Partial spreads in finite projective spaces and partial designs, *Math. Z.* (1975), 211–229.
[3] A. Beutelspacher, Partitions of finite vector spaces: an application of the Frobenius number in geometry, *Arch. Math.* (1978), 202–208.
[4] A. Blinco, S. El-Zanati, G. Seelinger, P. Sissokho, L. Spence, and C. Vanden Eynden, On Vector space partitions and uniformly resolvable designs, *Des. Codes and Crypt.* (2008), 69–77.
[5] T. Bu, Partitions of a vector space, *Discrete Math.* (1980), 79–83.
[6] D. Drake and J. Freeman, Partial t-spreads and group constructible (s, r, μ)-nets, *J. of Geometry* (1979), 211–216.
[7] S. El-Zanati, G. Seelinger, P. Sissokho, L. Spence, and C. Vanden Eynden Partitions of finite vector spaces into subspaces, *J. Combin. Des.* (2008), 329–341.
[8] S. El-Zanati, G. Seelinger, P. Sissokho, L. Spence, and C. Vanden Eynden On partitions of finite vector spaces of low dimension over $GF(2)$, *Discrete Math.* (2009), 4727–4735.
[9] S. El-Zanati, O. Heden, G. Seelinger, P. Sissokho, L. Spence, and C. Vanden Eynden, Partitions of the 8-dimensional vector subspace over GF(2), *J. Combin. Des.* (2010), 462–474.
[10] O. Heden, Partitions of finite abelian groups, *Europ. J. Combin.* (1986), 11–25.
[11] O. Heden, On partitions of finite vector spaces of small dimensions, *Arch. Math.* (1984), 507–509.
[12] O. Heden, On the length of the tail of a vector space partition, *Discrete Math.*, (2009), 6169–6180.
[13] O. Heden, Necessary and sufficient conditions for the existence of a class of partitions of a finite vector space, *Des. Codes and Crypt.* (2009), 69–73.
[14] O. Heden and J. Lehmann, Some necessary conditions for vector space partitions, *Discrete Math.* (2012), 351–361.

Mathematics Department, Illinois State University, Campus Box 4520, Stevenson Hall 313, Normal, IL 61790-4520

E-mail address: {gfseeli|psissok|spence|cve}@ilstu.edu