Extremal sizes of subspace partitions*

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Abstract

A *subspace partition* Π of $V = V(n, q)$ is a collection of subspaces of V such that each 1-dimensional subspace of V is in exactly one subspace of Π. The *size* of Π is the number of its subspaces. Let $\sigma_q(n, t)$ denote the *minimum size* of a subspace partition of V in which the largest subspace has dimension t, and let $\rho_q(n, t)$ denote the *maximum size* of a subspace partition of V in which the smallest subspace has dimension t. In this paper, we determine the values of $\sigma_q(n, t)$ and $\rho_q(n, t)$ for all positive integers n and t. Furthermore, we prove that if $n \geq 2t$, then the minimum size of a maximal partial t-spread in $V(n + t - 1, q)$ is $\sigma_q(n, t)$.

Keywords. Subspace partition; Vector space partitions; Partial t-spreads.

1 Introduction

Let $V = V(n, q)$ denote a vector space of dimension n over a finite field with q elements. A *subspace partition* Π of V is a collection of subspaces of V such that each 1-dimensional subspace of V is in exactly one subspace of Π. A subspace partition Π is also called a *vector space partition* (or simply a *partition*) of V. There is a rich literature about vector space partitions, see e.g. [1, 3, 5, 15, 24] and the references therein.

The *size* of Π is the number of its subspaces. Let $\sigma_q(n, t)$ denote the *minimum size* of a subspace partition of V in which the largest subspace has dimension t, and let $\rho_q(n, t)$ denote the *maximum size* of a subspace partition of V in which the smallest subspace has dimension t. The purpose of this study is to find these numbers. Since $\sigma_q(n, n) = \rho_q(n, n) = 1$, and $\sigma_q(n, 1) = \rho_q(n, 1) = (q^n - 1)/(q - 1)$, we will focus on the case $1 < t < n$. Moreover, if t divides n, then $\sigma_q(n, t) = \rho_q(n, t)$ is the size of a t-spread in V, i.e., a subspace partition of V in which all the subspaces have dimension t.

We will prove the following theorem:

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Theorem 1 *Let $n, k, t,$ and r be integers such that $0 \leq r < t$, $k \geq 2$, and $n = kt + r$. Then*

$$\rho_q(n,t) = q^{t+r} \sum_{i=0}^{k-2} q^{it} + 1 \ ,$$

and if furthermore $1 \leq r < t$, then

$$\sigma_q(n,t) = q^{t+r} \sum_{i=0}^{k-2} q^{it} + q^{\lceil \frac{t+r}{2} \rceil} + 1 \ .$$

This theorem improves a result of Beutelspacher [2] who in 1980 proved that

$$\sigma_q(n,t) \geq q^{\lceil \frac{n}{2} \rceil} + 1.$$

We must also remark that the last two authors of this paper recently found the value of $\sigma_q(2t+1,t)$, see [22]. They used some equations for subspace partitions derived by the first two authors in [18]. Furthermore, our derivation of the value of $\sigma_q(n,t)$ uses arguments quite similar to those used in [22].

After some preliminary results in Section 2, we will prove our theorem in Section 3 and Section 4. Finally, in Section 5, we combine our result on $\sigma_q(n,t)$ with a construction of P. Govaerts [14] to show that the minimum size of a maximal partial t-spread in $V(n+t-1,q)$ is $\sigma_q(n,t)$ for any integer $n \geq 2t$.

2 Preliminary Results

Let Π be a subspace partition of $V = V(n,q)$, $n \geq 2$, with m_i subspaces of dimension i, $1 \leq i \leq n-1$. Let H be any hyperplane, i.e., any $(n-1)$-dimensional subspace of V, and let $b_i \leq m_i$ be the number of subspaces of Π that are contained in H. We say that $(m_{n-1}, \ldots, m_1)$ is the *type* of Π and $b = (b_{n-1}, \ldots, b_1)$ is the *type of the hyperplane H* (with respect to Π). Let s_b denote the number of hyperplanes in V of type b and define the set

$$B = \{b : \ s_b > 0\}.$$

For $1 \leq i \leq n$, let

$$\theta_i = \frac{q^i - 1}{q-1}$$

denote the number of 1-dimensional subspaces in an i-space; then

$$h_q(n,i) = \max\{0, \theta_{n-i}\}$$

denotes the number of hyperplanes containing a given i-dimensional subspace. The following two lemmas were derived in [18].

Lemma 1 *Let Π be a subspace partition of $V = V(n,q)$ of type $(m_{n-1},\dots,m_1)$ and let $b = (b_{n-1},\dots,b_1)$ be the type of the hyperplane H with respect to Π. Let s_b denote the number of hyperplanes in V with type b. Assume furthermore that Π contains a subspace of dimension d and a subspace of dimension d', with $1 \le d, d' \le n-2$. Then*

(i) $\sum\limits_{b\in B} s_b = \frac{q^n-1}{q-1} = h_q(n,0)$,
(ii) $\sum\limits_{b\in B} b_d s_b = m_d h_q(n,d)$,
(iii) $\sum\limits_{b\in B} \binom{b_d}{2} s_b = \binom{m_d}{2} h_q(n,2d)$,
(iv) $\sum\limits_{b\in B} b_d b_{d'} s_b = m_d m_{d'} h_q(n,d+d')$.

Lemma 2 *Let Π be a subspace partition of $V = V(n,q)$ and let $(b_{n-1},\dots,b_1)$ be the type of the hyperplane H with respect to Π. Then the number of subspaces in Π is*

$$|\Pi| = 1 + \sum_{i=1}^{n-1} b_i q^i.$$

We will also use the following lemma due to Herzog and Schönheim [19] and independently Beutelspacher [1] and Bu [5].

Lemma 3 *Let n and d be integers such that $1 \le d \le n/2$. Then $V = V(n,q)$ admits a partition with one subspace of dimension $n-d$ and q^{n-d} subspaces of dimension d.*

For $n = kt + r$, $0 \le r < t$, and $k \ge 2$, let

$$\ell = q^r \sum_{i=0}^{k-2} q^{it}. \tag{1}$$

The following proposition is an immediate consequence of Lemma 3.

Proposition 1 *Let $n, k, t,$ and r be integers such that $0 \le r < t$, $k \ge 2$, and $n = kt + r$. Then $V = V(n,q)$ admits a partition Π_m of size*

$$|\Pi_m| = \ell \cdot q^t + 1,$$

consisting of ℓq^t subspaces of dimension t and one subspace of dimension $t + r$. If furthermore, $1 \le r < t$, then V admits a partition Π_M of size

$$|\Pi_M| = \ell \cdot q^t + q^{\lceil \frac{t+r}{2} \rceil} + 1\ ,$$

consisting of ℓq^t subspaces of dimension t, $q^{\lceil (t+r)/2 \rceil}$ subspaces of dimension $\lfloor (t+r)/2 \rfloor$ and one subspace of dimension $\lceil (t+r)/2 \rceil$.

We close this section by giving three relations that will be frequently used. They follow easily from the definitions of ℓ and the function θ_i; the third is an immediate consequence of the first two:

$$\theta_{n-t} - \theta_r = \ell\theta_t, \tag{2}$$

$$\theta_{a+b} - \theta_b = q^b\theta_a\ , \tag{3}$$

$$\theta_n - \ell q^t \theta_t = \theta_{t+r}\ . \tag{4}$$

3 The minimum size

In this section we will find $\sigma_q(n,t)$, as indicated in Theorem 1. We will need the following lemma, which may be of independent interest.

Lemma 4 *Let $n,k,t,$ and r be integers such that $k \geq 2, 1 \leq r < t,$ and $n = kt + r$. Let Π be a subspace partition of $V = V(n,q)$ with no subspace of dimension higher than t. Assume furthermore that Π contains a subspace of dimension t and a subspace of dimension d, with $0 \leq d < t$. Then*

$$|\Pi| \geq q^{t+r} \sum_{i=0}^{k-2} q^{it} + q^d + 1.$$

Proof. Let Π be a subspace partition of V containing subspaces of dimension t and d with $t > d$. Since there exist subspaces of dimensions t and d in Π, we have $m_t > 0$ and $m_d > 0$. So it follows from Lemma 1(iv) that

$$\sum_{b \in B} b_t b_d s_b = m_t m_d \theta_{n-t-d} \neq 0. \tag{5}$$

Additionally,

$$\sum_{b \in B} b_t b_d s_b = \sum_{\substack{b \in B \\ 0 \leq b_t \leq \ell - 1}} b_t b_d s_b + \sum_{\substack{b \in B \\ b_t \geq \ell}} b_t b_d s_b.$$

If

$$\sum_{b \in B,\, b_t \geq \ell} b_t b_d s_b \neq 0,$$

then there exists $b \in B$ such that $b_t \geq \ell$, $b_d \geq 1$, and $s_b \geq 1$. In this case, Lemma 2 yields

$$|\Pi| = \sum_{i=1}^{n-1} b_i q^i + 1 \geq b_t q^t + b_d q^d + 1 \geq \ell\, q^t + q^d + 1,$$

and the lemma follows. So we may assume that $\sum_{b \in B,\, b_t \geq \ell} b_t b_d s_b = 0$. This assumption, combined with (5) and Lemma 1(iv), yields

$$\begin{aligned}
(\ell - 1) m_d \theta_{n-d} &= \sum_{b \in B} (\ell - 1) \cdot b_d s_b \\
&= \sum_{\substack{b \in B \\ 0 \leq b_t \leq \ell - 1}} (\ell - 1) \cdot b_d s_b + \sum_{\substack{b \in B \\ b_t \geq \ell}} (\ell - 1) \cdot b_d s_b \\
&\geq \sum_{\substack{b \in B \\ 0 \leq b_t \leq \ell - 1}} b_t \cdot b_d s_b + 0 \\
&= \sum_{\substack{b \in B \\ 0 \leq b_t \leq \ell - 1}} b_t \cdot b_d s_b + \sum_{\substack{b \in B \\ b_t \geq \ell}} b_t \cdot b_d s_b
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{b\in B} b_t b_d s_b \\
&= m_t m_d \theta_{n-t-d} \qquad (6)
\end{aligned}
$$

Since $m_d > 0$, dividing both sides of (6) by m_d yields

$$m_t \le \frac{(\ell-1)\,\theta_{n-d}}{\theta_{n-t-d}}.$$

We now show that this implies that

$$m_t \le (\ell-1)q^t + q^d \;. \qquad (7)$$

From (3) we obtain that $\theta_{n-d} = \theta_t + q^t\theta_{n-d-t}$, and hence it remains to prove that

$$\frac{(\ell-1)\theta_t}{\theta_{n-d-t}} \le q^d \;.$$

This fact follows from Equations (2), (3) and (4):

$$q^d\theta_{n-d-t} - \ell\theta_t + \theta_t = \theta_{n-t} - \theta_d - \theta_{n-t} + \theta_r + \theta_t = \theta_t + \theta_r - \theta_d \;,$$

as $\theta_t > \theta_d$.

Note that Π is the disjoint union of $\mathcal{A} = \{W \in \Pi :\ \dim(W) = t\}$ and $\mathcal{B} = \{W \in \Pi : \dim(W) \le t-1\}$. By counting the 1-dimensional subspaces not taken up by $\mathcal{A}$, we can bound the size of $\mathcal{B}$ by

$$|\mathcal{B}| \ge \frac{\theta_n - |\mathcal{A}|\cdot\theta_t}{\theta_{t-1}} \;.$$

Since $|\mathcal{A}| = m_t$, we obtain from (7) that

$$|\Pi| = |\mathcal{A}| + |\mathcal{B}| \ge m_t + \frac{\theta_n - m_t\cdot\theta_t}{\theta_{t-1}} \ge \frac{\theta_n - (\ell\, q^t - q^t + q^d)(\theta_t - \theta_{t-1})}{\theta_{t-1}} \;. \qquad (8)$$

By using Equation (4), the above inequality can be further simplified

$$|\Pi| \ge \ell q^t + q^d + \frac{\theta_{t+r} + q^t(\theta_t - \theta_{t-1}) - q^d\theta_t}{\theta_{t-1}} > \ell q^t + q^d + \frac{q^t(\theta_t - \theta_{t-1}) - q^d\theta_t}{\theta_{t-1}} \;.$$

As furthermore,

$$q^t(\theta_t - \theta_{t-1}) = q^{2t-1} > q^d\theta_t \;,$$

we finally obtain

$$|\Pi| \ge \ell\, q^t + q^d + 1.$$

This conclusion the proof of the lemma. □

We now prove that under the assumptions of Theorem 1, $\sigma_q(n,t) = \ell\, q^t + q^{\lceil \frac{t+r}{2} \rceil} + 1$.

Proof. Let Π be a subspace partition of $V = V(n,q)$ in which the largest subspace has dimension t. Let $\beta = \lceil (t+r)/2 \rceil$. If there is a subspace of dimension d in Π with $\beta \le d < t$, then by Lemma 4

$$|\Pi| \ge \ell\, q^t + q^d + 1 \ge \ell\, q^t + q^\beta + 1. \tag{9}$$

It remains to consider the case where every subspace in Π has either dimension t or a dimension less than or equal to $\beta - 1$.

If there exists a hyperplane H of type b with $b_t \ge \ell + 1$, then by Lemma 2

$$|\Pi| = \sum_{i=1}^{n-1} b_i q^i + 1 \ge (\ell+1)q^t + 1 \ge \ell\, q^t + q^\beta + 1, \tag{10}$$

where the last inequality holds since $\beta \le t$.

So now assume that if $s_b \ne 0$ then $b_t \le \ell$. Then Lemma 1(ii) yields

$$m_t \theta_{n-t} = \sum_{b \in B} b_t s_b \le \ell \cdot \sum_{b \in B} s_b = \ell \cdot \theta_n\ . \tag{11}$$

From (2), we derive $\ell\theta_t < \theta_{n-t}$. By combining this inequality with (3), we obtain

$$\ell\theta_n = \ell(q^t\theta_{n-t} + \theta_t) = \ell q^t \theta_{n-t} + \ell\theta_t < \ell q^t\theta_{n-t} + \theta_{n-t}\ .$$

Consequently, (11) yields

$$m_t < \ell q^t + 1\ . \tag{12}$$

Note that Π is the disjoint union of $\mathcal{A} = \{W \in \Pi :\ \dim(W) = t\}$ and $\mathcal{B} = \{W \in \Pi : \dim(W) \le \beta - 1\}$. By Equation (12) and since m_t is an integer, we may assume that $m_t \le \ell\, q^t$. So by using Equation (3), we obtain that

$$|\Pi| = |\mathcal{A}| + |\mathcal{B}|\ \ge\ m_t + \frac{\theta_n - m_t \cdot \theta_t}{\theta_{\beta-1}} = \frac{\theta_n - m_t(\theta_t - \theta_{\beta-1})}{\theta_{\beta-1}} \ge \frac{\theta_n - \ell q^t(\theta_t - \theta_{\beta-1})}{\theta_{\beta-1}},$$

and hence from (4), and the fact that $\theta_{\beta-1}(q^\beta + 1) \le \theta_{t+r}$, we conclude that

$$|\Pi| \ge \ell q^t + \frac{\theta_n - \ell q^t \theta_t}{\theta_{\beta-1}} = \ell q^t + \frac{\theta_{t+r}}{\theta_{\beta-1}} \ge \ell q^t + q^\beta + 1\ . \tag{13}$$

Summarizing the distinct cases we have considered, we thus obtain

$$\sigma_q(n,t) \ge \ell\, q^t + q^\beta + 1. \tag{14}$$

Finally, by using Proposition 1 we may conclude that

$$\sigma_q(n,t) = \ell\, q^t + q^\beta + 1.$$

□

4 The maximum size

In this section we now prove that under the assumptions of Theorem 1, $\rho_q(n,t) = \ell\, q^t + 1$.

Proof. Let Π be a subspace partition of $V = V(n,q)$ in which the smallest subspace has dimension t. Suppose $|\Pi| > \ell\, q^t + 1$. The type of Π is $(m_{n-1}, \ldots, m_t, 0, \ldots, 0)$. Let H be any hyperplane of V, and let $(b_{n-1}, \ldots, b_t, 0, \ldots, 0)$ be the type of H with respect to Π. Then by Lemma 2, we have

$$|\Pi| = 1 + \sum_{i=t}^{n-1} b_i q^i = 1 + q^t \sum_{i=t}^{n-1} b_i q^{i-t}.$$

Thus, $|\Pi| \equiv 1 \pmod{q^t}$, and by our above assumption on $|\Pi|$, we have $|\Pi| \geq \ell\, q^t + q^t + 1$. As the dimension of each member of Π is at least t, we may use relation (4) and the fact that $(q^t + 1)\theta_t = \theta_{2t}$ to conclude that

$$\theta_n \geq |\Pi|\, \theta_t \geq (\ell q^t + q^t + 1)\theta_t = \theta_n - \theta_{t+r} + \theta_{2t}\ ,$$

which is a contradiction as $\theta_{2t} > \theta_{t+r}$. Thus $|\Pi| \leq \ell\, q^t + 1$. Since Π is an arbitrary partition, we obtain

$$\rho_q(n,t) \leq \ell\, q^t + 1. \tag{15}$$

Hence, from Proposition 1 now follows that

$$\rho_q(n,t) = \ell\, q^t + 1.$$

□

5 Application to maximal partial t-spreads

A *partial t-spread* of $V = V(n,q)$ is a collection $\mathcal{S} = \{W_1, \ldots, W_k\}$ of t-dimensional subspaces of V such that $W_i \cap W_j = \{0\}$ for $i \neq j$. The *size* of $\mathcal{S}$ is its cardinality $|\mathcal{S}|$. If $V = \bigcup_{W \in \mathcal{S}} W$, then $\mathcal{S}$ is called a *t-spread.* A partial t-spread is called *maximal* if it cannot be extended to a larger one. Maximal partial t-spreads have been extensively studied, see e.g. [4, 9, 12, 14, 16, 20, 21]. They can be used to construct error-correcting codes [6, 8], orthogonal arrays [7, 10], and recently factorial designs [23].

We let $\tau_q(n,t)$ denote the *minimum number* of subspaces in any maximal partial t-spread of $V(n,q)$. A maximal partial t-spread $\mathcal{S}$ of $V(n,q)$ such that $|\mathcal{S}| = \tau_q(n,t)$, is called a *minimum size* maximal partial t-spread. Let n and t be fixed integers and let k and r be the unique integers defined by $n = kt + r$ and $0 \leq r < t$. Beutelspacher [1] showed that if $r = 0$ and $k \geq 2$, then

$$\tau_q(n+t-1,t) = \sigma_q(n,t) = \frac{q^{kt}-1}{q^t-1}.$$

For $r > 0$, P. Govaerts [14] proved several results related to the number $\tau_q(n+t-1,t)$. In particular, he provided the following upper bound for $\tau_q(n+t-1,t)$.

Lemma 5 (Govaerts [14]) *Let n and $t > 1$ be integers such that $n \geq 2t$. Then there exist (see page 610 in [14] for a construction) maximal partial t-spreads of $V(n+t-1, q)$ of size $\sigma_q(n, t)$. Consequently, $\tau_q(n+t-1, t) \leq \sigma_q(n, t)$.*

We will prove the following theorem.

Theorem 2 *Let n and $t > 1$ be integers such that $n \geq 2t$. Then $\tau_q(n+t-1, t) = \sigma_q(n, t)$.*

The method employed to prove Theorem 2 will be the same as was used in [22] to prove $\tau_q(3t, t) = \sigma_q(2t+1, t)$. In particular, we will use Theorem 1 in Section 1. We first introduce the relevant definitions and a useful Lemma due to Govaerts [14]. A set of points B, i.e., 1-spaces of V, is called a *blocking set* with respect to the t-spaces of V if $W \cap B \neq \{0\}$ for any t-space W in V. Note that any $(n-t+1)$-dimensional subspace of V is a blocking set with respect to the t-spaces of V. Such blocking sets are called *trivial.* The following lemma follows from the results of Govaerts (see Case 2, page 612 in [14]).

Lemma 6 (Govaerts [14]) *Let n and $t > 1$ be integers such that $n \geq 2t$. If $\mathcal{S}$ is a minimum size maximal partial t-spread of $V(n, q)$, then $\bigcup_{W \in \mathcal{S}} W$ contains a trivial blocking set.*

In the proof of Theorem 2 we will also use the following proposition.

Proposition 2 *Let d, d', and n be integers such that $0 < d' < d \leq n/2$. Then*

$$\sigma_q(n, d) < \sigma_q(n, d') \ .$$

Proof. We will prove that $\sigma_q(n, t) < \sigma_q(n, t-1)$ holds, for $1 < t \leq n/2$.

If t divides n, then $\sigma_q(n, t) = \theta_n/\theta_t$. Consequently, by Theorem 1 and with the use of Equation (4), we note that it is always true that

$$\frac{\theta_n}{\theta_t} \leq \sigma_q(n, t) < \frac{\theta_n}{\theta_t} + q^\beta \ ,$$

where $0 \leq r = n - kt < t$ and $\beta = \lceil (t+r)/2 \rceil$. As $\theta_t > q\theta_{t-1}$ and $q^\beta < \theta_n/\theta_t$, we thus get

$$\sigma_q(n, t) < 2\frac{\theta_n}{\theta_t} \leq q\frac{\theta_n}{\theta_t} < \frac{\theta_n}{\theta_{t-1}} \leq \sigma_q(n, t-1) \ .$$

$\square$

Proof. [**Theorem 2**] By Lemma 5, we have $\tau_q(n+t-1, t) \leq \sigma_q(n, t)$. So, it remains to show that

$$\tau_q(n+t-1, t) \geq \sigma_q(n, t). \qquad (16)$$

Let $\mathcal{S}$ be a minimum size maximal partial t-spread in $V(n+t-1, q)$. Then by Lemma 6, $A = \bigcup_{W \in \mathcal{S}} W$ contains a trivial blocking set. In other words, there exists an n-dimensional subspace $B \subseteq A$. Let

$$\Pi_S = \{W \cap B : \ W \in \mathcal{S}\}.$$

Since B is a blocking set with respect to t-spaces, we have $W \cap B \neq \{0\}$ for any $W \in \mathcal{S}$. Thus, $\Pi_{\mathcal{S}}$ is a subspace partition of $B \cong V(n,q)$ containing subspaces of dimensions at most t. If $\Pi_{\mathcal{S}}$ contains a t-subspace, then it follows from Theorem 1 and the minimality of $\mathcal{S}$ that

$$\tau_q(n+t-1,t) = |\mathcal{S}| = |\Pi_{\mathcal{S}}| \geq \sigma_q(n,t).$$

If $\Pi_{\mathcal{S}}$ does not contain any t-subspace, then each subspace in $\Pi_{\mathcal{S}}$ has dimension at most $t-1$ (and contains at most θ_{t-1} 1-dimensional subspaces). So the theorem now follows from the fact that the function $\sigma_q(n,t)$ is antimonotone in t by Proposition 2. □

6 Some remarks

Let Π be a subspace partition of $V = V(n,q)$ consisting of n_i subspaces of dimension d_i, for $1 \leq i \leq k$. Let us assume that $d_1 < d_2 < \ldots < d_k$ (and $n_1 n_2 \cdots n_k \neq 0$). In [17] a lower bound on n_1 was given as a function of q, d_1 and d_2, and, as easily verified from that result, it is always true that $n_1 \geq \sigma_q(d_2, d_1)$. Working on the results of this paper has given us many indications that the following conjecture holds.

Conjecture 1 *Let Π be a subspace partition of $V(n,q)$ with $n_i > 0$ subspaces of dimension d_i, $1 \leq i \leq k$, and where $d_1 < \ldots < d_k$. Then, for any integer j, $1 \leq j < k$, we have*

$$n_1 + \ldots + n_j \geq \sigma_q(d_{j+1}, d_j).$$

Let us also remark that for $n \leq 2t-1$, the problem of determining the minimum size $\tau_q(n+t-1,t)$ of a maximal partial t-spread in $V(n+t-1,q)$ is still open. For $t=2$ and $n=3$, the following lower bound was achieved by Glynn [13]:

$$\tau_q(4,2) \geq 2q,$$

while the following two upper bounds are due to Gács and Szönyi [12]:

$$\tau_q(4,2) \leq (2\log_2 q + 1)q + 1, \quad \text{if } q \text{ odd},$$

and

$$\tau_q(4,2) \leq (6.1 \ln q + 1)q + 1, \quad \text{if } q > q_0 \text{ even}.$$

Finally, let us remark that one of the reviewers of this paper suggests the following research problem: *Is the type of a subspace partition that has the maximum or minimum size (as found in Theorem 1) unique?* Since we do not have any answer to this question, we take this opportunity to forward it to the public.

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