GAUSSIAN PARTITIONS

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ABSTRACT. Let $V = V(n, q)$ denote the finite vector space of dimension n over the finite field with q elements. A *subspace partition* of V is a collection Π of subspaces of V such that each 1-dimensional subspace of V is in exactly one subspace of Π. In a recent paper, we proved some strong connections between the lattice of the subspace partitions of V and the lattice of the set partitions of $\mathbf{n} = \{1, \ldots, n\}$. We now define a *Gaussian partition* of $[n]_q = (q^n - 1)/(q - 1)$ to be a nonincreasing sequence of positive integers formed by ordering all elements of some multiset $\{\dim(W) : W \in \Pi\}$, where Π is a subspace partition of V. The *Gaussian partition function* $\mathrm{gp}(n, q)$ is then the number of all Gaussian partitions of $[n]_q$, and is naturally analogous to the classical *partition function* $\mathrm{p}(n)$. In this paper, we initiate the study of $\mathrm{gp}(n, q)$ by exhibiting all Gaussian partitions for small n. In particular, we determine $\mathrm{gp}(n, q)$ as a polynomial in q for $n \leq 5$, and find a lower bound for $\mathrm{gp}(6, q)$.

1. INTRODUCTION AND BACKGROUND

Let $V = V(n, q)$ denote the finite vector space of dimension n over the finite field with q elements. A *subspace partition*[1] of V is a collection Π of subspaces of V such that each 1-dimensional subspace of V is in exactly one subspace of Π. Subspace partitions is a rich area of research [5, 7, 9, 15, 21] with applications in the constructions of translation planes and nets [2, 12], designs [8, 33], and codes [11, 25, 27, 29].

Let a be a positive integer. We adopt the notation

$$[a]_q = \frac{q^a - 1}{q - 1},$$

which is the number of 1-dimensional subspaces in $V(a, q)$. For positive integers $x_1, \ldots, x_s, d_1, \ldots, d_s$ with $d_1 < \ldots < d_s$, we let $d_s^{x_s} \ldots d_1^{x_1}$ denote the nonincreasing sequence containing x_i integers d_i for $1 \leq i \leq s$. Suppose that there exists a subspace partition Π of V such that the

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[1] A subspace partition is also known as a *vector space partition* in the literature.

nonincreasing sequence of positive integers obtained from the multiset $\{\dim(W) : W \in \Pi\}$ is equal to $T = d_s^{x_s} \ldots d_1^{x_1}$; then we say that T is a *Gaussian partition*[2] of $[n]_q$. We also say that T is the Gaussian partition *associated with* Π. In this case, the following condition holds:

$$[d_1]_q\, x_1 + \ldots + [d_s]_q\, x_s = [n]_q. \tag{1}$$

The above equation is obtained by adding up the numbers of distinct 1-dimensional subspaces in each subspace of the partition. Another well-known necessary condition for T to be a Gaussian partition of $[n]_q$ is as follows:

$$\begin{cases} 2d_i \le n, & \text{if } x_i \ge 2,\\ d_i + d_j \le n, & \text{if } i \ne j,\ x_i, x_j \ge 1. \end{cases} \tag{2}$$

This condition tells us that since the sum of any two subspaces of the partition is direct, their total dimension cannot exceed n. However, the converse is not true. If $x_1, \ldots, x_s$ and $d_1 < \ldots < d_s$ are positive integers such that conditions (1) and (2) hold, then $d_s^{x_s} \ldots d_1^{x_1}$ is not necessarily a Gaussian partition of $[n]_q$, since there may not exist a subspace partition of V with x_i subspaces of dimension d_i, $1 \le i \le s$. For instance, if $n = 5$ and $q = 2$, then $x_1 = 1$, $d_1 = 1$, $x_2 = 10$, and $d_2 = 2$ satisfy (1) and (2), but there does not exist a subspace partition of $V(5, 2)$ with 10 subspaces of dimension 2 and one subspace of dimension 1 (e.g., see [21]). In fact, the problem of finding necessary and sufficient conditions for $d_s^{x_s} \ldots d_1^{x_1}$ to be a Gaussian partition of $[n]_q$ is wide open for general n and q. A few special cases were solved by Beutelspacher [5] (and rediscovered by Bu [9]) for $s = 2$ and $d_1+d_2 = n$; by Heden for $q = 2$, $s = 3$, $n \ge 9$, $d_3 = n - 3$, $d_2 = 3$, and $d_1 = 2$ (see [23]); for $s \ge q + 1$ and $d_{p+1} = d_p = \ldots = d_s$ (see [24]); by Blinco et al. [8] for $s = 2$, d_1 dividing n, and d_2 dividing n; by El-Zanati et al. for $q = 2$, $s = 2$, $d_2 = 3$, and $d_1 = 2$ (see [15]); for $q = 2$ and $n \le 7$ (see [16]); and for $q = 2$, $n = 8$, and $d_i > 0$ with $1 \le i \le s$ (see [17]).

In [1], we studied the lattice structure on the set of subspace partitions of $V = V(n, q)$ and proved some strong connections between this lattice and the lattice of set partitions of $\mathbf{n} = \{1, \ldots, n\}$. Using an order-preserving *Galois map* between the two lattices, we proved the following results (among others; see [1]):

Proposition 1. *The number of subspace partitions of $V(n, q)$ is congruent to the number of set partitions of $\mathbf{n}$ modulo $q - 1$.*

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[2] This is also called the *type* of a vector space partition in the literature; we have introduced this new term in light of our results in this paper.

Proposition 2. *Let $T = d_s^{x_s} \dots d_1^{x_1}$ be a Gaussian partition of $[n]_q$ such that*

$$\sum_{d_i \geq 2} x_i d_i > n.$$

Then the number of all subspace partitions of $V(n,q)$ with associated Gaussian partition T is congruent to zero modulo $q-1$.

Since the number of set partitions of $\mathbf{n}$ is B_n (the nth *Bell number*), we define the q-*Bell number* $B_n^{(q)}$ to be the number of vector space partitions of $V(n,q)$. Then Proposition 1 simply says that

$$B_n^{(q)} \equiv B_n \pmod{q-1}. \tag{3}$$

Let us emphasize that in addition to being a "numerical" q-analoque of B_n that is transformed into B_n when q is replaced by 1, $B_n^{(q)}$ is also a "combinatorial" q-analoque of B_n that *counts interesting objects* (subspace partitions of $V(n,q)$). In this respect, $B_n^{(q)}$ seems to be the most natural q-extension of the number of set partitions of $\mathbf{n}$. For other q-analogues of B_n in the literature, see, for example, [4, 10, 13, 19, 20, 28, 31, 34, 35, 36].

Just as the subspace partitions of $V(n,q)$ are counterparts of the set partitions of $\mathbf{n}$, the Gaussian partitions of $[n]_q$ are counterparts of the integer partitions of n. The number of integer partitions of n is the well-studied *partition function* $\mathrm{p}(n)$, which is equal to the number of nonnegative solutions of the linear Diophantine equation

$$x_1 + 2x_2 + \cdots + nx_n = n. \tag{4}$$

In contrast, enumerating the Gaussian partitions of $[n]_q$, even for specific n, is a highly non-trivial task. Although the Gaussian partitions of $[n]_q$ are among the nonnegative solutions of the Diophantine equation

$$x_1 + [2]_q x_2 + \cdots + [n]_q x_n = [n]_q, \tag{5}$$

not every solution qualifies. For instance, the problem of determining whether or not $3^{34}1^{17}$ is a Gaussian partition of $[8]_2$ was the first open case of a 1972 conjecture by Hong and Patel [27] (also see [14]) about the maximum size of a *partial t-spread*[3] of $V(n,q)$. This has been settled recently by El-Zanati et al. [18], who constructed a subspace partition of $V(8,2)$ with 34 subspaces of dimension 3 and 17 subspaces of dimension 1. Their result disproved the aforementioned conjecture, which was generally believed to be true.

[3] A partial t-spread of $V = (n,q)$ is a collection of t-dimensional subspaces of V with mutually zero intersections.

We define the *Gaussian partition function*, $\mathrm{gp}(n, q)$, to be the number of distinct Gaussian partitions of $[n]_q$. In this paper, we initiate the study of $\mathrm{gp}(n, q)$ by exhibiting all Gaussian partitions for small n. In particular, we determine $\mathrm{gp}(n, q)$ as a polynomial in q for $n \leq 5$, and find a lower bound for $\mathrm{gp}(6, q)$. We show that $\mathrm{gp}(n, q)$ has the same value as $\mathrm{p}(n)$ when q is set equal to 1 in these cases. We also conjecture that $\mathrm{gp}(n, q)$ is a polynomial in q for fixed n, and that the above-mentioned relationship between $\mathrm{gp}(n, q)$ and $\mathrm{p}(n)$ holds for all positive integers n.

2. Main results

2.1. Gaussian partition function $\mathrm{gp}(n, q)$ for $n \leq 6$. It is easy to compute the three smallest values of the Gaussian partition function $\mathrm{gp}(n, q)$:

$$\mathrm{gp}(1, q) = 1 = \mathrm{p}(1),\ \mathrm{gp}(2, q) = 2 = \mathrm{p}(2), \text{ and } \mathrm{gp}(3, q) = 3 = \mathrm{p}(3).$$

In this section, we will determine $\mathrm{gp}(n, q)$ for $n = 4$ and 5 as polynomials in q, and find a similar polynomial as a lower bound for $\mathrm{gp}(6, q)$. In all of our examples, setting $q = 1$ in the formula for $\mathrm{gp}(n, q)$ will yield the corresponding integer partition function $\mathrm{p}(n)$ (Theorem 14). We will be using the following results.

Lemma 3 (André [2] and Bu [9]). *Let n, d be positive integers such that d divides n. Then there exists a partition of $V(n, q)$ consisting of $(q^n - 1)/(q^d - 1)$ subspaces of dimension d, i.e., a full d-spread.*

Lemma 4 (Beutelspacher [6]). *Let n, d be positive integers such that $1 \leq d < n/2$. Then $V(n, q)$ can be partitioned into one subspace of dimension $n - d$ and q^{n-d} subspaces dimension of d.*

Lemma 5 (Blinco et al. [8]). *Let r and t be positive integers with $rt = n$, and let x and y be nonnegative integers such that*

$$[r]_q\, x + [t]_q\, y = [n]_q.$$

Then there exists a partition of $V(n, q)$ into x subspaces of dimension r and y subspaces of dimension t.

Lemma 6 (Beutelspacher[7]). *Let $n \geq 3$ be an odd integer, and let Π be a partition of $V(n, q)$ with x_2 subspaces of dimension 2, x_1 subspaces of dimension 1, and no other subspaces. Then*

$$x_2 \leq \frac{q^n - q}{q^2 - 1} - q + 1.$$

The generalization of Lemma 6, where 2 is replaced by t, has long been an open problem for all prime powers q and for t, n with $t > 2$ and $n \not\equiv 0 \pmod{t}$. However, as mentioned in the introduction, the case $q = 2$ and $t = 3$ has recently been settled by El-Zanati et al. [18].

Lemma 7 (Heden [23]). *Let Π be a partition of the finite vector space $V(n,q)$ of type $d_1^{x_1} d_2^{x_2} \cdots d_s^{x_s}$, with $s \geq 2$, $1 \leq d_1 < d_2 < \cdots < d_s$, and $x_i > 0$ for all $1 \leq i \leq s$. Then the following hold:*

(i) *If $q^{d_2-d_1}$ does not divide x_1, and $d_2 < 2d_1$, then $x_1 \geq q^{d_1} + 1$.*
(ii) *If $q^{d_2-d_1}$ does not divide x_1, and $d_2 \geq 2d_1$, then either d_1 divides d_2 and $x_1 = (q^{d_2} - 1)/(q^{d_1} - 1)$, or $x_1 > 2q^{d_2-d_1}$.*
(iii) *If $q^{d_2-d_1}$ divides x_1, and $d_2 < 2d_1$, then $x_1 \geq q^{d_2} - q^{d_1} + q^{d_2-d_1}$.*
(iv) *If $q^{d_2-d_1}$ divides x_1, and $d_2 \geq 2d_1$, then $x_1 \geq q^{d_2}$.*

We now enumerate the Gaussian partitions of $[4]_q$ and $[5]_q$, and compute a lower bound for $[6]_q$.

Proposition 8. *The distinct Gaussian partitions of $[4]_q$ are given by*

$$4^1,\ 3^1 1^{q^3}, \tag{6}$$

and

$$2^{(q^2+1)-i} 1^{i(q+1)},\ 0 \leq i \leq q^2 + 1. \tag{7}$$

Proof. Among the solutions of Eq. (5) for $n = 4$, we have legitimate Gaussian partitions of $[4]_q$, as well as solutions that do not correspond to actual partitions of $V(4,q)$. An example of the latter is the "type" $3^1 2^{q^2-q} 1^q$, which does not satisfy condition (2). It is easy to verify that the Gaussian partitions of $[4]_q$ are given by Eqs. (6) and (7): these are indeed all solutions of (5) that also satisfy (2). The second partition exists because of Lemma 4. The full 2-spread is of the form 2^{q^2+1} by Lemma 3, and subsequent partitions of its two-dimensional subspaces are obtained by complete refinement into one-dimensional subspaces, again by Lemmas 3 and 4. □

Proposition 9. *The distinct Gaussian partitions of $[5]_q$ are given by*

$$5^1,\ 4^1 1^{q^4}, \tag{8}$$

$$3^1 2^{q^3-i} 1^{i(q+1)},\ 0 \leq i \leq q^3, \tag{9}$$

and

$$2^{q^3+1-i} 1^{q^2+i(q+1)},\ 0 \leq i \leq q^3 + 1. \tag{10}$$

Proof. We recursively apply Lemma 4 to $V(5,q)$ and obtain the distinct Gaussian partitions (8)-(10). Because of the necessary condition (2), at most one subspace of dimension greater than or equal to 3 can appear. Moreover, all nonnegative solutions of the Diophantine equation

$$x_1 + [2]_q x_2 + [3]_q = [5]_q$$

are represented in Eq. (9). Hence, the only other potential Gaussian partitions of $[5]_q$ not shown above would be those of the form $2^x 1^y$ where $x > q^3+1$. However, Lemma 6 yields

$$x \leq \frac{q^5-q}{q^2-1} - q + 1 = q^3+1,$$

which shows that such Gaussian partitions cannot exist. □

Proposition 10. *The distinct Gaussian partitions of* $[6]_q$ *containing one of 6, 5, or 4 are given by*

$$6^1,\ 5^1 1^{q^5}, \tag{11}$$

$$4^1 2^{q^4},\ 4^1 2^{q^4-i} 1^{i(q+1)}\ (1 \leq i \leq q^4-1),\ \textit{and}\ 4^1 1^{q^5+q^4}. \tag{12}$$

Proof. Once again, because of Eq. (2), at most one subspace of dimension 4, 5, or 6 may appear in a partition of $V(6,q)$, and never with a 3-dimensional space. The above Gaussian partitions exist because of Lemma 4. The Gaussian partitions in Eq. (12) represent the whole set of nonnegative solutions of the Diophantine equation

$$x_1 + [2]_q x_2 + [4]_q = [6]_q,$$

hence the list of Gaussian partitions of $[6]_q$ containing 4^1 is complete. □

Proposition 11. *The distinct Gaussian partitions of* $[6]_q$ *of the form* $3^x 2^y 1^z$ *with* $x, y, z \geq 0$ *that can be obtained from the nonnegative solutions of the Diophantine equation*

$$[3]_q x + [2]_q y = [6]_q$$

via Lemmas 3–5 are completely determined by the pairs (x, y) *in the following list:*

$$\begin{array}{l}
0 \leq x \leq x_0, \quad 0 \leq y \leq -x + x_0 + y_0 \\
0 \leq x \leq x_1, \quad -x + x_0 + y_0 + 1 \leq y \leq -x + x_1 + y_1 \\
0 \leq x \leq x_2, \quad -x + x_1 + y_1 + 1 \leq y \leq -x + x_2 + y_2 \\
\vdots \\
0 \leq x \leq x_{q^2-q}, \quad -x + x_{q^2-q-1} + y_{q^2-q-1} + 1 \leq y \leq -x + x_{q^2-q} + y_{q^2-q} \\
x = 0, \quad x_{q^2-q} + y_{q^2-q} + 1 \leq y \leq x_{q^2-q+1} + y_{q^2-q+1},
\end{array}$$

where

$$(x_i, y_i) = ((q^3+1) - i(q+1), i(q^2+q+1)), \quad 0 \le i \le q^2 - q + 1.$$

Proof. All nonnegative solutions

$$3^{(q^3+1)-i(q+1)} 2^{i(q^2+q+1)} \ (0 \le i \le q^2 - q + 1) \tag{13}$$

of the Diophantine equation

$$[3]_q \, x + [2]_q \, y = [6]_q$$

represent Gaussian partitions of $[6]_q$ by Lemma 5. We start with these solutions (*base pairs*),

$$(x_i, y_i) = ((q^3+1) - i(q+1), i(q^2+q+1)), \quad 0 \le i \le q^2 - q + 1,$$

and obtain new Gaussian partitions by finding all refinements of a representative vector space partition of $V(6, q)$ of the given type. We can also talk about refinements of a Gaussian partition in the obvious sense. Hence, we consider refinements of (i) 3^1 into $2^1 1^{q^2}$, (ii) 3^1 into 1^{q^2+q+1}, and (iii) 2^1 into 1^{q+1} (these are supported by Lemma 3 and Lemma 4). Let (x_i, y_i) be a base pair with nonnegative integer coordinates, and (x^*, y^*) be any lattice point obtained from it by applying some of the refinements (i)–(iii). Let us consider the effects of these refinements on (x^*, y^*). In case (i), we obtain the nonnegative integer lattice point $(x^* - 1, y^* + 1)$ on the line $x + y = x^* + y^*$, above and to the left of (x^*, y^*). In case (ii), we obtain the point $(x^* - 1, y^*)$ to the left of (x^*, y^*), and in case (iii), the point $(x^*, y^* - 1)$ below (x^*, y^*). Clearly, we can only move on and inside the trapezoid with vertices

$$(0,0), \ (x_i, 0), \ (x_i, y_i), \ \text{and} \ (0, x_i + y_i),$$

with the exceptions of $i = 0$ and $i = q^2 - q + 1$. In these last two cases, the trajectories lie inside a right triangle and on a line segment respectively. We note for later use that $(x_{i+1} + y_{i+1}) - (x_i + y_i) = q^2$. These possibilities are depicted in Figure 1.

The base pair $(x_0, y_0) = (q^3 + 1, 0)$ generates a right triangle with vertices

$$(0,0), \ (x_0, 0), \ \text{and} \ (0, x_0).$$

The lattice points on and inside the triangle are of the form (x, y) with

$$0 \le x \le x_0, \quad 0 \le y \le -x + x_0 + y_0 \ \ (y_0 = 0).$$

Next, we have the trapezoid with vertices

$$(0,0), \ (x_1, 0), \ (x_1, y_1), \ \text{and} \ (0, x_1 + y_1).$$

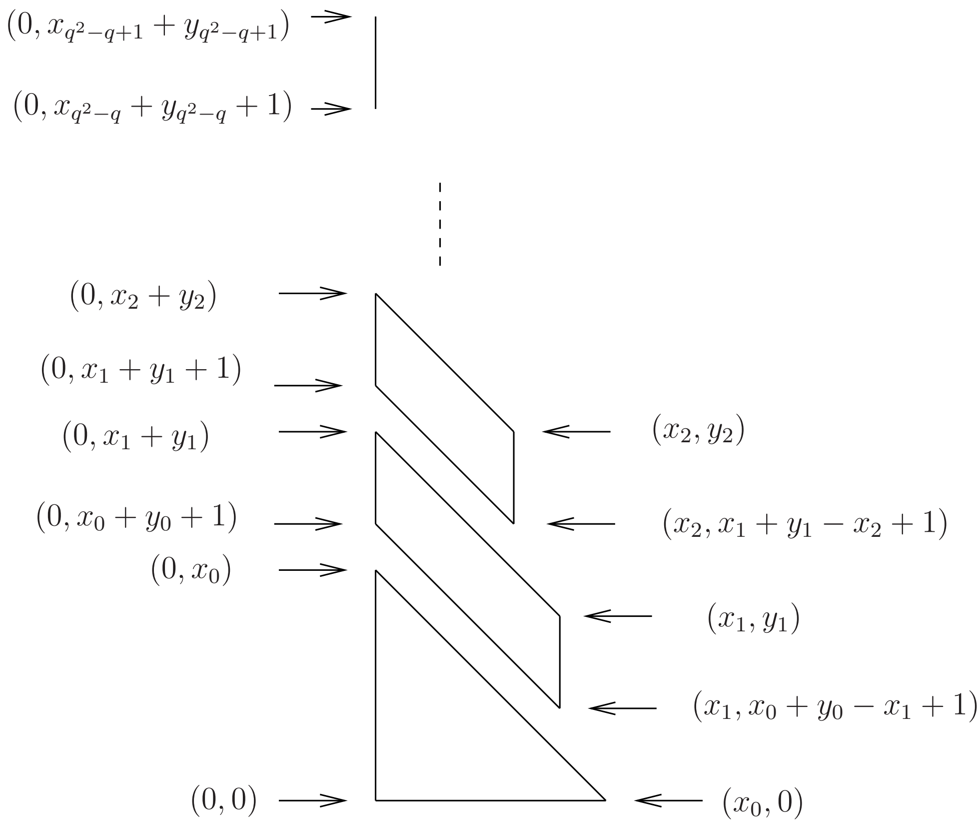

FIGURE 1. The solution pairs (x, y) representing $3^x 2^y 1^z$.

The trapezoid is thinner and taller than the previous triangle and the lattice points in the part that is not covered by the triangle lie on and inside the parallelogram with vertices

$$(0, x_0 + y_0 + 1), \ (0, x_1 + y_1), \ (x_1, y_1), \ \text{and} \ (x_1, x_0 + y_0 - x_1 + 1).$$

This set of points (x, y) is characterized by

$$0 \le x \le x_1, \quad -x + x_0 + y_0 + 1 \le y \le -x + x_1 + y_1.$$

As we argued above, we only need to count the new points on and inside the parallelogram with vertices

$$(0, x_1 + y_1 + 1), \ (0, x_2 + y_2), \ (x_2, y_2), \ \text{and} \ (x_2, x_1 + y_1 - x_2 + 1).$$

These points are characterized by

$$0 \le x \le x_2, \quad -x + x_1 + y_1 + 1 \le y \le -x + x_2 + y_2,$$

and so on. The final base point, $(x_{q^2-q+1}, y_{q^2-q+1}) = (0, q^4 + q^2 + 1)$, is on the y-axis and we only need to add the points $(0, y)$ with

$$x_{q^2-q} + y_{q^2-q} + 1 \le y \le x_{q^2-q+1} + y_{q^2-q+1}$$

that lie outside the last trapezoid. □

We can now establish a lower bound for $\mathrm{gp}(n,q)$ using Propositions 10 and 11.

Proposition 12. *Let* $\mathrm{gp}(n,q)$ *denote the Gaussian partition function. Then*

$$\mathrm{gp}(6,q) \geq \frac{q^7}{2} + \frac{5q^4}{2} + q^3 + q^2 + 6.$$

Moreover, within the list of the Gaussian partitions of $[6]_q$ *that are enumerated above, there exist:*

- *One Gaussian partition of each of the forms* 6^1, $5^1 1^{q^5}$, $4^1 2^{q^4}$, $4^1 1^{q^5+q^4}$, 3^{q^3+1}, $2^{q^4+q^2+1}$, *and* $1^{[6]_q}$*;*
- $q^4 - 1$ *of the form* $4^1 2^x 1^y$ *with* $x, y > 0$*;*
- $q^2 - q$ *of the form* $3^x 2^y$ *with* $x, y > 0$*;*
- q^3 *of the form* $3^x 1^z$ *with* $x, z > 0$*;*
- $q^4 + q^2$ *of the form* $2^y 1^z$ *with* $y, z > 0$*; and*
- $\frac{q^7}{2} + \frac{q^4}{2} - q^2 + q$ *of the form* $3^x 2^y 1^z$ *with* $x, y, z > 0$.

Proof. The total number of Gaussian partitions of $[6]_q$ depicted in Proposition 10 is

$$(q^4 - 1) + 4 = q^4 + 3.$$

The number of Gaussian partitions in Proposition 11 that come from the "triangle" is

$$1 + 2 + \cdots + (x_0 + 1) = \frac{1}{2}(x_0 + 1)(x_0 + 2) = \frac{1}{2}(q^3 + 2)(q^3 + 3),$$

and the total number of Gaussian partitions from the $q^2 - q$ parallelograms is

$$\begin{aligned} & q^2(x_1 + 1) + \cdots + q^2(x_{q^2-q} + 1) \\ = \; & q^2[(q^3 + 1) - (q + 1) + \cdots + (q^3 + 1) - (q^2 - q)(q + 1)] + q^2(q^2 - q) \\ = \; & q^2(q^3 + 1)(q^2 - q) - q^2(q + 1)\frac{1}{2}(q^2 - q)(q^2 - q + 1) + q^2(q^2 - q) \\ = \; & \frac{q^7}{2} - \frac{q^6}{2} + \frac{3q^4}{2} - \frac{3q^3}{2}. \end{aligned}$$

Finally, there are q^2 Gaussian partitions in Proposition 11 that come from the q^2 points on the y-axis. By adding up the numbers of all Gaussian partitions of $[6]_q$ from Propositions 10 and 11, we obtain (after simplifications):

$$\mathrm{gp}(6,q) \geq q^7/2 + 5q^4/2 + q^3 + q^2 + 6.$$

The distribution of the various forms of Gaussian partitions can be deduced from this number and Propositions 10 and 11. □

Proposition 13. *Let the notation and terminology be as in Proposition 11. The positive solutions of the Diophantine equation*

$$[3]_q\, x + [2]_q\, y + [1]_q\, z = [6]_q$$

that are not listed in Proposition 11 are lattice points inside the triangle defined by the lines $x = 0$, $y = 0$, and $[3]_q\, x + [2]_q\, y = [6]_q$ and outside the boundaries of the regions described in the Proposition. Then none of the unlisted lattice points (which we call exceptional types) is a Gaussian partition of $[6]_q$ if and only if none of the following exceptional types is a Gaussian partition:

$$3^{\,q^3-(i+1)q-i+1}\,2^{\,iq^2+(i+1)(q+1)}\,1^{q^3-q-1}, \quad 0 \le i \le q^2 - q.$$

Proof. The $q^2 - q + 1$ points (x, y) that correspond to the types $3^x 2^y 1^z$ listed above fall immediately to the right of the intersections of the right edges of trapezoids (or in the last case, the y-axis) with the previous structure (triangle or trapezoid) as described in Proposition 11. If we show that these points do not correspond to Gaussian partitions, then neither will any other unlisted point in the big triangular region, because of the admissible moves (or refinements) (i) 3^1 into $2^1 1^{q^2}$, (ii) 3^1 into 1^{q^2+q+1}, and (iii) 2^1 into 1^{q+1}, which are supported by Lemma 3 and Lemma 4. □

El-Zanati et al. [16] have shown that among all the exceptional types defined in Proposition 13, $3^7 2^3 1^5$ is the only one which does not correspond to a Gaussian partition of $[6]_2$. For instance, the exceptional types $3^4 2^{10} 1^5$ and $3^1 2^{17} 1^5$ are Gaussian partitions of $[6]_2$. However, these examples do not seem to indicate a generalization to $q > 2$, and the problem of determining the exceptional types of the form $3^x 2^y 1^z$ remains open in general.

The results in Propositions 8–13 directly yield the following theorem.

Theorem 14. *Let* $\mathrm{gp}(n, q)$ *denote the Gaussian partition function. Then*
(i) $\mathrm{gp}(4, q) = q^2 + 4$ *and* $\mathrm{gp}(4, q) = 5 = p(4)$ *when q is set equal to* 1.
(ii) $\mathrm{gp}(5, q) = 2q^3 + 5$ *and* $\mathrm{gp}(5, q) = 7 = p(5)$ *when q is set equal to* 1.
(iii) $\mathrm{gp}(6, q) \ge \frac{q^7}{2} + \frac{5q^4}{2} + q^3 + q^2 + 6$.

It is interesting that for $n = 4$ and 5, the stronger condition $\mathrm{gp}(n, q) \equiv \mathrm{p}(n) \pmod{q-1}$ holds. Moreover, the correspondence between the Gaussian partitions of $[6]_q$ depicted in Proposition 12 and the integer partitions of 6 is striking: the number of Gaussian partitions containing certain parts (with positive coefficients) becomes the number of integer partitions of 6 with the same parts when q is set equal to 1.

For example, there are $q^2 - q$ Gaussian partitions containing only the parts 3 and 2, and no integer partitions of 6 only with parts 3 and 2. Furthermore, the lower bound in Proposition 12 is equal to $\mathrm{p}(6) = 11$ after replacing q by 1. We actually believe that the number of the exceptional types (as explained in Proposition 13) should be zero when q is replaced by 1. If this claim could be confirmed, then the exact value of the function $\mathrm{gp}(6, q)$ would be $\mathrm{p}(6)$ when q is set equal to 1.

The following conjecture is motivated by the above theorem, as well as by our results in [1] (see Propositions 1 and 2 in Section 1) that show strong connections between the lattice of subspace partitions of $V(n, q)$ and the lattice of set partitions of $\mathbf{n} = \{1, 2, \ldots, n\}$.

Conjecture 15. *The Gaussian partition function* $\mathrm{gp}(n, q)$ *is a polynomial in* q *with rational coefficients for fixed* $n \geq 1$. *Moreover,* $\mathrm{gp}(n, q)$ *becomes* $\mathrm{p}(n)$ *when* q *is replaced by* 1.

2.2. **Another q-analogue.** In this section, we consider a traditional method of obtaining q-analogues, i.e., replacing n with $[n]_q$. The formulas for $\mathrm{p}(n)$ and the resulting q-analogue $\mathrm{P}_n^{(q)}$ given below in Propositions 17 and 18 respectively may be known, but we have not seen them published elsewhere. These propositions follow directly from Lemma 16 (below), which is a minor variation of a result recently proved by Mahmoudvand et al. [30].

Before stating these formulas, recall that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}$$

is the *Gaussian coefficient*, an integer equal to the number of k-dimensional subspaces of $V(n, q)$. We have seen in Section 1 that a necessary condition for the existence of a Gaussian partition $n^{x_n} \cdots 1^{x_1}$ $(x_i \geq 0)$ of $[n]_q$ is

$$[1]_q \, x_1 + [2]_q \, x_2 + \cdots + [n]_q \, x_n = [n]_q, \tag{14}$$

where we use the simplified notation $[a]_q$ for $\begin{bmatrix} a \\ 1 \end{bmatrix}_q$. The number of nonnegative solutions $\mathrm{P}^{(q)}(n)$ of the above equation (an upper bound for $\mathrm{gp}(n, q)$) is –in a strictly numerical sense– a q-analogue of $\mathrm{p}(n)$, which, in turn, is the number of nonnegative solutions of

$$x_1 + 2x_2 + \cdots + nx_n = n. \tag{15}$$

Consider the generic linear Diophantine equation

$$a_1 x_1 + \cdots + a_r x_r = n \tag{16}$$

with positive coefficients. Then it is a simple matter to count the number of nonnegative solutions in the case $a_1 = 1$ by induction.

Lemma 16 (Mahmoudvand et al. [30]). *For $r \geq 3$, the number of non-negative solutions of the linear Diophantine equation (16) with positive coefficients and $a_1 = 1$ is given by*

$$\sum_{i_r=0}^{\left\lfloor \frac{n}{a_r} \right\rfloor} \sum_{i_{r-1}=0}^{\left\lfloor \frac{n-a_r i_r}{a_{r-1}} \right\rfloor} \sum_{i_{r-2}=0}^{\left\lfloor \frac{n-a_r i_r - a_{r-1} i_{r-1}}{a_{r-2}} \right\rfloor} \cdots \sum_{i_3=0}^{\left\lfloor \frac{n-a_r i_r - a_{r-1} i_{r-1} - \cdots - a_4 i_4}{a_3} \right\rfloor}$$
$$\left(\left\lfloor \frac{n - a_r i_r - a_{r-1} i_{r-1} - \cdots - a_4 i_4 - a_3 i_3}{a_2} \right\rfloor + 1 \right).$$

Combining Lemma 16 with Equation (15), we obtain the following formula for $\mathrm{p}(n)$:

Proposition 17. *A formula for the partition function $\mathrm{p}(n)$ is given by*

$$\mathrm{p}(n) = \sum_{i_n=0}^{\left\lfloor \frac{n}{n} \right\rfloor} \sum_{i_{n-1}=0}^{\left\lfloor \frac{n-n i_n}{n-1} \right\rfloor} \sum_{i_{n-2}=0}^{\left\lfloor \frac{n-n i_n-(n-1) i_{n-1}}{n-2} \right\rfloor} \cdots \sum_{i_3=0}^{\left\lfloor \frac{n-n i_n-(n-1) i_{n-1}-\cdots-4 i_4}{3} \right\rfloor}$$
$$\left(\left\lfloor \frac{n - n i_n - (n-1) i_{n-1} - \cdots - 4 i_4 - 3 i_3}{2} \right\rfloor + 1 \right).$$

Similarly, combining Lemma 16 with Equation (14), we obtain the following formula for $\mathrm{P}^{(q)}(n)$:

Proposition 18. *A formula for $\mathrm{P}^{(q)}(n)$ is given by*

$$\mathrm{P}^{(q)}(n)$$
$$= \sum_{i_n=0}^{\left\lfloor \frac{[n]_q}{[n]_q} \right\rfloor} \sum_{i_{n-1}=0}^{\left\lfloor \frac{[n]_q-[n]_q i_n}{[n-1]_q} \right\rfloor} \sum_{i_{n-2}=0}^{\left\lfloor \frac{[n]_q-[n]_q i_n-[n-1]_q i_{n-1}}{[n-2]_q} \right\rfloor} \cdots \sum_{i_3=0}^{\left\lfloor \frac{[n]_q-[n]_q i_n-[n-1]_q i_{n-1}-\cdots-[4]_q i_4}{[3]_q} \right\rfloor}$$
$$\left(\left\lfloor \frac{[n]_q - [n]_q i_n - [n-1]_q i_{n-1} - \cdots - [4]_q i_4 - [3]_q i_3}{[2]_q} \right\rfloor + 1 \right).$$

A direct application of Proposition 18 yields

$$\mathrm{P}^{(q)}(4) = q^3/2 + q^2/2 + q + 3,$$

which is equal to $p(4) = 5$ when q is replaced by 1, and

$$\mathrm{P}^{(q)}(5) = q^6/6 + q^5/4 + q^4/2 + 11q^3/12 + 4q^2/3 + 11q/6 + 2,$$

which is equal to $p(5) = 7$ when q is replaced by 1. The analogies break down modulo $q - 1$.

Some results about the properties of $\mathrm{P}^{(q)}(n)$ in the literature are worth mentioning. For example, Stanton [32] recently proved two q-analogues of Euler's Theorem, which states that "the number of integer partitions of n with all parts odd is equal to the number of integer partitions of n with all parts distinct". The first of analogue is

Proposition 19 (Stanton)**.** *Let N, q be positive integers. Then the number of integer partitions of N into q-odd parts, i.e., parts of the form $[2m+1]_q$, is equal to the number of integer partitions of N into parts $[n]_q$ of multiplicity at most q^n.*

An older result by Hickerson [26], also about Euler-type identities, can be written in our notation as follows:

Proposition 20 (Hickerson)**.** *Let $f^{(q)}(n)$ be the number of partitions of $[n]_q$ of the form*

$$[n]_q = b_0 + b_1 + \cdots + b_s, \quad \textit{where } b_i \geq q\, b_{i+1} \textit{ for } 0 \leq i \leq s-1.$$

Then

$$P^{(q)}(n) = f^{(q)}(n).$$

3. Concluding Remarks and Open Problems

If Conjecture 15 were true, then we would have some handle on the *existence* of Gaussian partitions of $[n]_q$: by using known values of $\mathrm{p}(n)$, or congruences involving $\mathrm{p}(n)$, we might be able to derive new necessary conditions for the existence of certain types of subspace partitions of $V(n,q)$.

But whether Conjecture 15 is true or false, determining the Gaussian partition functions $\mathrm{gp}(n,q)$ is an interesting open problem (for $n \geq 6$) by virtue of the rich combinatorial objects (subspace partitions of $V(n,q)$) associated with them. To our knowledge, there is no other q-analogue of the partition function $\mathrm{p}(n)$ in the literature which is based on counting combinatorial objects. Moreover, the computation of the polynomials $\mathrm{gp}(q,n)$, as well as attaching other meanings to them, are interesting problems.

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