On λ-fold Partitions of Finite Vector Spaces and Duality

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Abstract

Vector space partitions of an n-dimensional vector space V over a finite field are considered in [5], [10], and more recently in the series of papers [3], [8], and [9]. In this paper, we consider the generalization of a vector space partition which we call a λ-fold partition (or simply a λ-partition). In particular, for a given positive integer, λ, we define a λ-fold partition of V to be a multiset of subspaces of V such that every nonzero vector in V is contained in exactly λ subspaces in the given multiset. A λ-fold spread as defined in [12] is one example of a λ-fold partition. After establishing some definitions in the introduction, we state some necessary conditions for a λ-fold partition of V to exist, then introduce some general ways to construct such partitions. We also introduce the construction of a dual λ-partition as a way of generating λ'-partitions from a given λ-partition. One application of this construction is that the dual of a vector space partition will, in general, be a λ-partition for some $\lambda > 1$. In the last section, we discuss a connection between λ-partitions and some designs over finite fields.

We denote by $V_n(q)$ the vector space of dimension n over the field $\mathbb{F}_q$ with q elements, where q is a power of a prime. In a series of papers ([3], [8], [9]), we extended the results of T. Bu ([5]) and O. Heden ([10] and [11]) on partitioning V into subspaces. (More precisely, we considered finding a set of subspaces of $V = V_n(q)$ such that every nonzero vector is in exactly one subspace in this set.)

One natural extension of our previous work is to examine the idea of a λ-fold partition of V. As in the vector space partition, we define a λ-fold partition to be a multiset of subspaces such that every nonzero vector in V is contained in exactly λ subspaces in our multiset. A λ-fold partition generalizes the idea of a λ-fold spread defined in Section 4.2 of J.W.P. Hirschfeld's book on projective geometries over finite fields [12]. In fact, Corollary 8 of this paper extends Theorem 4.16 of [12]. The purpose of this note is to construct certain λ-fold partitions and consider some questions that naturally arise from our treatment of these partitions.

We start with a more precise definition of λ-fold partition which will be specially useful to prove our duality theorem (Theorem 15).

Definition 1 *Let λ be a positive integer. A λ-*fold partition *of the vector space V is an ordered pair (A, α) such that A is a set and α is a map from A to 2^V, the set of subsets of V, such that*

1. *if $a \in A$, then $\alpha(a)$ is a nonzero subspace of V,*
2. *if $0 \neq v \in V$, then the cardinality of the set $\{a \in A : v \in \alpha(a)\}$ is λ.*

We call the cardinality of A the size *of the partition and say two λ-partitions (A, α) and (B, β) are* equal *if there exists a bijection $\tau : A \to B$ such that $\alpha = \beta\tau$.*

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Note that using this definition, a 1-fold partition of V is just a vector space partition in the sense mentioned above. For brevity, we will henceforth refer to a λ-fold partition simply as a λ-partition. We will use the term 1-partition of V when we are referring to a standard vector space partition.

We also make the observation that two λ-partitions (A, α) and (B, β) are equivalent if and only if their multiset images $\{\alpha(a) : a \in A\}$ and $\{\beta(b) : b \in B\}$ are equal as multisets. As a result, sometimes we will identify a λ-partition with its multiset image.

Given a 1-partition of V, one easy way to construct a λ-partition of V is to replicate the 1-partition λ times. If one has λ different 1-partitions, then we could also take the union (as multisets) of these 1-partitions to form another λ-partition of V. The λ-partitions generated in this way do not add much to our knowledge, but there are more interesting λ-partitions that do not come from 1-partitions in this way. One such example is the q-Grassmanian $G(n, n-1)$ consisting of the set of all $(n-1)$-dimensional subspaces of V when $n \geq 3$, which forms a $\left(\frac{q^{n-1}-1}{q-1}\right)$-partition. More generally, we can consider the q-Grassmanian $G(n, r)$ consisting of all r-dimensional subspaces of the n-dimensional vector space V. In this case $G(n, r)$ consists of $\left(\begin{array}{c} n \\ r \end{array}\right)_q = \frac{(q^n-1)(q^n-q)\cdots(q^n-q^{r-1})}{(q^r-1)(q^r-q)\cdots(q^r-q^{r-1})}$ subspaces of dimension r, each containing q^r-1 nonzero vectors, so that each of the q^n-1 nonzero vectors in V are included in $\left(\begin{array}{c} n-1 \\ r-1 \end{array}\right)_q$ of these subspaces. Therefore, $G(n, r)$ forms a $\left(\begin{array}{c} n-1 \\ r-1 \end{array}\right)_q$-partition of V.

If (A, α) is a λ-partition, then we define a λ_0-*subpartition* of (A, α) to be a λ_0-partition (B, β) of V where $B \subseteq A$, $\beta = \alpha|_B$, and $0 < \lambda_0 \leq \lambda$. We say that the λ_0-subpartition (B, β) is *proper* if $0 < \lambda_0 < \lambda$. Note that if (B, β) is a proper λ_0-partition of (A, α), then the complement of (B, β), or $(A \setminus B, \alpha|_{A \setminus B})$, also forms a $(\lambda - \lambda_0)$-subpartition of (A, α). We say a λ-partition (A, α) is *irreducible* if it has no proper λ_0-subpartitions for any $0 < \lambda_0 < \lambda$ and *reducible* otherwise. Note that a 1-partition is always irreducible. Clearly, the λ-partitions built as unions of 1-partitions are reducible.

Note that not all irreducible λ-partitions are 1-partitions. For example, consider the 2-partition of $V = V_3(2)$ given by $\alpha : \{1, 2, 3, 4, 5, 6\} \to 2^V$, where the nonzero vectors of $\alpha(i)$ for $1 \leq i \leq 6$ are

$$\begin{array}{llllll} \alpha(1) = \{100, 011, 111\}, & \alpha(2) = \{010, 001, 011\}, & \alpha(3) = \{001, 110, 111\}, \\ \alpha(4) = \{110, 010, 100\}, & \alpha(5) = \{101\}, & \alpha(6) = \{101\}. \end{array}$$

(Here we abbreviate the nonzero vector (a, b, c) by the string of digits abc, where $a, b, c \in \{0, 1\}$.) Since a 1-partition of $V_3(2)$ can contain at most one 2-dimensional subspace, this 2-partition cannot be written as the union of two 1-partitions since it contains more than two 2-dimensional subspaces. Therefore, this 2-partition must be irreducible. This turns out to be a special case of Corollary 3 in the next section.

One goal would be to classify all irreducible λ-partitions for a given V. We note that the problem of classifying all irreducible λ-partitions includes the classification of all vector space partitions as a subproblem. To aid us in classifying λ-partitions, we introduce the following terminology. Let (A, α) be a λ-partition of V, where V has dimension n. We say the λ-partition (A, α) is of *type*

$[(t_1, n_1), \dots, (t_s, n_s)]$ if for all $1 \le k \le n$ we have

$$|\{a : \dim(\alpha(a)) = k\}| = \sum_{n_i = k} t_i.$$

Note that this notation does not exclude $t_i = 0$ for some i nor do the n_i need to be distinct. We will consider two partition types $[(t_s, n_s), \dots, (t_1, n_1)]$ and $[(c_r, m_r), \dots, (c_1, m_1)]$ to be the same if for all $1 \le k \le n$ we have

$$\sum_{n_i = k} t_i = \sum_{m_j = k} c_j.$$

Sometimes it will be convenient to use the more compact notation $n_s^{t_s} \cdots n_2^{t_2} n_1^{t_1}$ for the type $[(t_s, n_s), \dots, (t_2, n_2), (t_1, n_1)]$.

Before continuing, we prove the following analogy to [5, Lemma 1].

Lemma 1 *Let (A, α) be a λ-partition of V and let W be a subspace of V. Define $A_W = \{a \in A : \alpha(a) \cap W \ne \{0\}\}$ and $\alpha_W : A_W \to 2^W$ by $\alpha_W(a) = \alpha(a) \cap W$. Then (A_W, α_W) is a λ-partition of W.*

Proof. We verify the two conditions for (A_W, α_W) to be a λ-partition of W. Indeed, for every $a \in A_W$ we have $\alpha_W(a) = \alpha(a) \cap W$, which is a nonzero subspace. Also, for any $0 \ne w \in W$ we have $\{a \in A : w \in \alpha(a)\} = \{a \in A : w \in \alpha(a) \cap W\} = \{a \in A_W : w \in \alpha_W(a)\}$, where the last equality follows because if $0 \ne w \in \alpha(a) \cap W$ then $a \in A_W$. Hence, $|\{a \in A_W : w \in \alpha_W(a)\}| = |\{a \in A : w \in \alpha(a)\}| = \lambda$. Therefore, (A_W, α_W) is a λ-partition of W as claimed. ∎

Note, when $\dim(W) = \dim(V) - 1$, we have for any $a \in A$ either $\dim(\alpha(a) \cap W) = \dim(\alpha(a))$ or $\dim(\alpha(a) \cap W) = \dim(\alpha(a)) - 1$, hence we can use this observation to determine the type of (A_W, α_W) from (A, α).

For example, this lemma can be applied to the $\left(\frac{q^{n-1}-1}{q-1}\right)$-partition of V consisting of all the $(n-1)$-dimensional subspaces by intersecting with one of those $(n-1)$-dimensional subspaces W to get a $\left(\frac{q^{n-1}-1}{q-1}\right)$-partition of type $\left[(1, n-1), \left(\frac{q^n-q}{q-1}, n-2\right)\right]$.

In Section 1, we first discuss some necessary conditions for a λ-partition to exist. In Section 2, we create some further examples. In Section 3, we introduce the concept of a dual λ-partition. This allows us to construct λ-partitions from known 1-partitions in a nontrivial way as well as to create new λ-partitions from those constructed in Section 2.

1 Necessary conditions

In this section, we prove a series of necessary conditions for λ-partitions to exist. For 1-partitions, there are two immediate necessary conditions. The first of these is the usual diophantine equation counting the nonzero vectors. So for a 1-partition of $V_n(q)$ of type $[(a_1, n_1), \dots, (a_t, n_t)]$ to exist, we must have

$$\sum_{i=1}^{t} a_i(q^{n_i} - 1) = q^n - 1.$$

The second condition is a simple dimension consideration that can be stated as follows:

$$\text{if } a_i \neq 0 \neq a_j \text{ with } i \neq j, \text{ then } n_i + n_j \leq n \text{ and if } a_i \geq 2, \text{ then } n_i \leq n/2.$$

The diophantine equation for 1-partitions has an easy generalization to λ-partitions. In particular, if (A, α) is a λ-partition of $V_n(q)$ and $n_a = \dim \alpha(a)$, then

$$\sum_{a \in A} (q^{n_a} - 1) = \lambda(q^n - 1). \tag{1}$$

Therefore, if (A, α) is a λ-partition of type $n_1^{c_1} \cdots n_t^{c_t}$, we must have

$$\sum_{i=1}^{t} c_i(q^{n_i} - 1) = \lambda(q^n - 1). \tag{2}$$

The next theorem is a generalization of the dimension condition for 1-partitions.

Theorem 2 *Let (A, α) be a λ-partition of the n-dimensional vector space V over $\mathbb{F}_q$, and suppose that $a_1, a_2, \ldots, a_{\lambda+1} \in A$ are distinct elements of A. Then*

$$\sum_{i=1}^{\lambda+1} \dim \alpha(a_i) \leq \lambda n.$$

Proof. Let $W_j = \alpha(a_1) \cap \alpha(a_2) \cap \cdots \cap \alpha(a_j)$ for $1 \leq j \leq \lambda + 1$. We will prove by induction that

$$\dim W_j \geq \left(\sum_{i=1}^{j} \dim \alpha(a_i) \right) - (j-1)n, \qquad 1 \leq j \leq \lambda + 1.$$

This is trivial for $j = 1$. Assume it holds for j. Then

$$\dim W_{j+1} = \dim(W_j \cap \alpha(a_{j+1})) = \dim W_j + \dim \alpha(a_{j+1}) - \dim(W_j + \alpha(a_{j+1}))$$

$$\geq \left(\sum_{i=1}^{j} \dim \alpha(a_i) \right) - (j-1)n + \dim \alpha(a_{j+1}) - n = \left(\sum_{i=1}^{j+1} \dim \alpha(a_i) \right) - jn.$$

Therefore, the $j+1$ case is established, hence $\dim W_{\lambda+1} \geq \left(\sum_{i=1}^{\lambda+1} \dim \alpha(a_i) \right) - \lambda n$.

Now if $\sum_{i=1}^{\lambda+1} \dim \alpha(a_i) > \lambda n$, then $\dim W_{\lambda+1} > 0$ and hence $W_{\lambda+1}$ contains a nonzero vector w. Since w is in each set $\alpha(a_i)$ for all $1 \leq i \leq \lambda + 1$, the set $\{a \in A : w \in \alpha(a)\}$ has cardinality at least $\lambda + 1$. This contradicts the assumption that (A, α) is a λ-partition of V. ∎

We can use the above theorem to determine some irreducible λ-partitions, as pointed out by a referee for this paper. We are grateful for this observation.

Corollary 3 *Suppose (A, α) is a λ-partition of $V = V_n(q)$ and $n > \lambda$. If there exists an integer $0 < k < n/\lambda$ such that $|\{a \in A : \dim \alpha(a) = n - k\}| > \lambda$, then (A, α) is irreducible.*

Proof. Let k be as in the statement of the Corollary and assume (A, α) is reducible. Let (A_1, α_1) be a proper λ_1-subpartition and let (A_2, α_2) be its complement, which is a λ_2-partition. By the Pigeonhole principle, for either $i = 1$ or $i = 2$ we know (A_i, α_i) must contain at least $\lambda_i + 1$ subspaces of dimension $n - k$. By Theorem 2

$$\lambda_i n \geq (\lambda_i + 1)(n - k) = (\lambda_i + 1)n - (\lambda_i + 1)k > (\lambda_i + 1)n - n = \lambda_i n,$$

which is a contradiction. Therefore, (A, α) must be irreducible. ∎

Theorem 4 *Let (A, α) be a λ-partition of $V = V_n(q)$. Assume $r = \max\{\dim \alpha(a) : a \in A\} < n$ and $\dim \alpha(a) \geq n - r$ for all $a \in A$. Then*

$$|A| \geq \lambda + q^r.$$

Proof. We have the usual diophantine equation

$$\sum_{a \in A} (q^{\dim \alpha(a)} - 1) = \lambda(q^n - 1),$$

and so

$$\sum_{a \in A} q^{\dim \alpha(a)} = \lambda(q^n - 1) + |A|.$$

Choose $a_0 \in A$ with $\dim \alpha(a_0) = r$. Taking W_1 to be $\alpha(a_0)$, we note for $a \neq a_0$ we have

$$\dim(\alpha(a_0) \cap \alpha(a)) = \dim(\alpha(a_0)) + \dim(\alpha(a)) - \dim(\alpha(a_0) + \alpha(a)) \geq \dim(\alpha(a_0)) + \dim(\alpha(a)) - n.$$

Let t count the elements v of $\alpha(a_0) \setminus \{0\}$, each counted as many times as there exists an $a \in A \setminus \{a_0\}$ such that $v \in \alpha(a)$. Then

$$t = \sum_{a_0 \neq a \in A} |(\alpha(a_0) \cap \alpha(a)) \setminus \{0\}| \geq \sum_{a_0 \neq a \in A} (q^{\max(0, \dim(\alpha(a)) + r - n)} - 1).$$

But each element of $\alpha(a_0) \setminus \{0\}$ must be in $\alpha(a)$ for $\lambda - 1$ elements of $A \setminus \{a_0\}$, so $t = (\lambda - 1)(q^r - 1)$. Hence we get

$$\sum_{a \in A \setminus \{a_0\}} (q^{\dim \alpha(a) + r - n} - 1) + q^r - 1 \leq (q^r - 1)\lambda.$$

The left side is

$$\begin{aligned}
\sum_{a \in A} (q^{\dim \alpha(a) + r - n} - 1) &- (q^{2r-n} - 1) + q^r - 1 \\
&= q^{r-n} \sum_{a \in A} q^{\dim \alpha(a)} - |A| - q^r(q^{r-n} - 1) \\
&= q^{r-n}[\lambda(q^n - 1) + |A|] - |A| - q^r(q^{r-n} - 1) \\
&= \lambda q^{r-n}(q^n - 1) + (q^{r-n} - 1)|A| - q^r(q^{r-n} - 1).
\end{aligned}$$

Since this is less than or equal to the right hand side, $(q^r - 1)\lambda$, we have

$$(q^{r-n} - 1)|A| - q^r(q^{r-n} - 1) \leq \lambda[q^r - 1 - q^{r-n}(q^n - 1)] = \lambda(q^{r-n} - 1).$$

Dividing by the negative number $q^{r-n} - 1$ reverses the sense of the inequality, and the theorem follows. ∎

Lemma 5 *Let (A, α) be a λ-partition of $V = V_n(q)$ such that $n > m = \min\{\dim \alpha(a) : a \in A\}$. Let $W \subseteq V$ be a subspace of dimension $n - 1$. If $k = |\{a \in A : \alpha(a) \not\subseteq W$ and $\dim \alpha(a) = m\}|$, then q divides k.*

Proof. First suppose that (B, β) is a λ-partition of $V_N(q)$ where the minimum dimension of any subspace in the partition is M. Let $B' = \{b \in B : \dim \beta(b) = M\}$, and suppose $|B'| = R$. Then by Equation (1) we have

$$\lambda(q^N - 1) = R(q^M - 1) + \sum_{b \in B \setminus B'} (q^{\dim \beta(b)} - 1) = Rq^M + \sum_{b \in B \setminus B'} q^{\dim \beta(b)} - |B|,$$

and so

$$|B| = \lambda - \lambda q^N + Rq^M + \sum_{b \in B \setminus B'} q^{\dim \beta(b)}. \quad (*)$$

Thus

$$|B| \equiv \lambda \pmod{q^M} \quad \text{and} \quad |B| \equiv \lambda \pmod{q}.$$

Applying this to (A, α) gives $|A| \equiv \lambda \pmod{q^m}$ and $|A| \equiv \lambda \pmod{q}$.

Let (A_W, α_W) be the λ-partition induced by (A, α) on W. If $m = 1$, then $|A_W| = |A| - k$. Since $|A_W| \equiv \lambda \pmod{q}$ also, we see that q divides k.

Now assume $m > 1$ and $k > 0$. Then $A = A_W$ and the minimum dimension of a subspace of (A_W, α_W) is $m - 1$. Applying (*) to (A_W, α_W) gives

$$|A| = |A_W| = \lambda - \lambda q^{n-1} + kq^{m-1} + \sum_{\substack{a \in A_W \\ \dim \alpha_W(a) \geq m}} q^{\dim \alpha_W(a)}.$$

Since $|A| \equiv \lambda \pmod{q^m}$ and $n - 1 \geq m$, we see that q divides k. ∎

For any λ-partition $\mathcal{P}$ of $V_n(q)$, let $\dim_{\min}(\mathcal{P})$ be the minimum dimension that occurs in $\mathcal{P}$. Define

$$S(\mathcal{P}) = \{U \in \mathcal{P} : \dim(U) = \dim_{\min}(\mathcal{P})\},$$

and let $\tau(\mathcal{P})$ denote the number of subspaces of dimension $\dim_{\min}(\mathcal{P})$ in $\mathcal{P}$ (counting duplications).

Corollary 6 *Let $\mathcal{P}$ be a λ-partition of $V = V_n(q)$, and let $m = \dim_{\min}(\mathcal{P}) < n$ and $|S(\mathcal{P})| = 1$. Then q divides $\tau(\mathcal{P})$.*

Proof. If $|S(\mathcal{P})| = 1$, then $S(\mathcal{P}) = \{U\}$ for some subspace $U \subseteq V$. Let $W \subseteq V$ be an $(n-1)$-dimensional subspace not containing the subspace U. Then none of the $k = \tau(\mathcal{P})$ subspaces of dimension m in $\mathcal{P}$ is contained in W (since they are all identical to U). Thus, it follows from Lemma 5 that q divides $\tau(\mathcal{P})$ and our conclusion holds. ∎

2 Some Initial Constructions

We start this section with a well-known example.

Example 1
Let V be an n-dimensional vector space over $F = \mathbb{F}_q$ and identify V with $\mathbb{F}_{q^n}$. Then V can be partitioned into 1-dimensional $\mathbb{F}_q$ subspaces to form the projective space $\mathbb{P}_F(V)$. Let $J \subseteq V$ be a subset consisting of one nonzero element from each one-dimensional subspace. Note $|J| = \frac{q^n-1}{q-1}$.

If U is a k-dimensional subspace of V, then the multiset $\mathcal{P}(U) = \{\alpha U : \alpha \in J\}$ will have $|J|$ elements and so $\mathcal{P}(U)$ will form a $\left(\frac{q^k-1}{q-1}\right)$-partition of V of type $\left[\left(\dfrac{q^n-1}{q-1}, k\right)\right]$. Indeed, note that for any nonzero $v \in V$ we have $v \in \alpha U \Leftrightarrow \alpha^{-1} v \in U$, hence there are exactly $\dfrac{q^k-1}{q-1}$ subspaces in our set that contain v.

Next, we generalize the above example to examine *homogeneous* λ*-partitions*, i.e., λ-partitions of type $n_1^{t_1}$.

Proposition 7 *Let $1 \le k \le n = \dim V$ and let $r = \gcd(k,n)$. There exists a $\left(\dfrac{q^k-1}{q^r-1}\right)$-partition of V of type $\left[\left(\dfrac{q^n-1}{q^r-1}, k\right)\right]$.*

Proof. If $k \mid n$, we get the 1-partition given in [5, Lemma 2]. So assume k does not divide n. Let $r = \gcd(k,n)$ and $V = V_{n/r}(q^r)$, hence V is an n-dimensional vector space over $\mathbb{F}_q$. Then we can choose U to be a (k/r)-dimensional $\mathbb{F}_{q^r}$-subspace of V. Using Example 1, we can use U to create a $\lambda = \left(\dfrac{(q^r)^{k/r}-1}{q^r-1}\right)$-partition of V of type $(k/r)^t$ of $\mathbb{F}_{q^r}$ subspaces where

$$t = \left(\frac{(q^r)^{n/r}-1}{q^r-1}\right) = \frac{q^n-1}{q^r-1}.$$

Since each $\mathbb{F}_{q^r}$-subspace of V of dimension k/r is also a k-dimensional $\mathbb{F}_q$-subspace of V, this gives us the desired $\left(\dfrac{q^k-1}{q^r-1}\right)$-partition of V of type $\left[\left(\dfrac{q^n-1}{q^r-1}, k\right)\right]$. ∎

Corollary 8 *Let $1 \le k \le n = \dim V$ and $r = \gcd(k,n)$. Then there exists a λ-partition of V of type k^t if and only if*

$$\left(\frac{q^k-1}{q^r-1}\right) \Bigg| \ \lambda.$$

Proof. Let $\tau = \dfrac{q^k - 1}{q^r - 1}$ and $m = \dfrac{q^n - 1}{q^r - 1}$. If $\tau \mid \lambda$, we can just take λ/τ copies of the τ-partition of V from Proposition 7 to get the corresponding λ-partition.

Conversely, assume that there exists a λ-partition of type k^t. Then it follows from Equation (2) that

$$t(q^k - 1) = \lambda(q^n - 1) \Rightarrow t\tau = \lambda m \Rightarrow \tau \mid \lambda m.$$

Therefore, since $\gcd(\tau, m) = 1$, we see that $\tau \mid \lambda$. ∎

Next, we describe two methods that allow us to construct λ-partitions from 1-partitions. First, we introduce a technique for generating some q^m-partitions of V.

Proposition 9 *Let (A, α) be a λ-partition of $V = V_n(q)$, and let U, W be subspaces such that $V = U \oplus W$. If $\pi : V \to U$ is the projection onto U associated with the above direct sum decomposition of V, then π induces a λq^m-partition (B, β) of U where $m = \dim(W)$, $B = \{(a, w) : a \in A, w \in W \cap \alpha(a) \neq \alpha(a)\}$, and $\beta : B \to 2^U$ is given by $\beta(a, w) = \pi(\alpha(a))$.*

Proof. Note that for any $a \in A$, $\pi(\alpha(a))$ is a subspace of U, so it is clear that $\beta(a, w) = \pi(\alpha(a))$ is a subspace of U for all $(a, w) \in B$. Since $W \cap \alpha(a) \neq \alpha(a)$, we get $\beta(a, w) = \pi(\alpha(a)) \neq \{0\}$.

Let $u \in U^* = U \setminus \{0\}$ and let $B_u = \{(a, w) \in B : u \in \beta(a, w)\}$. We now show that $|B_u| = \lambda q^m$ by counting in two ways the cardinality of the set

$$S = \{(u, w) : \ u \in U^*, \ w \in W, \ \text{ and } u \in \beta(a, w) \text{ for some } a \in A\}.$$

For each $u \in U^*$, there are exactly $|B_u|$ subspaces $\beta(a, w) \in B$ that that contain u. So $|S| = |U^*|\,|B_u|$. On the other hand, for each of the $|U^*|\,|W|$ pairs (u, w) with $u \in U^*$ and $w \in W$, the number of $a \in A$ such that $u \in \beta(a, w)$ is the same as the number of $a \in A$ such that the vector $v = u + w$ is in the subspace $\alpha(a)$. Since this latter number is λ, we also have $|S| = \lambda|U^*|\,|W|$. Combining these two counts of $|S|$ yields

$$|U^*|\,|B_u| = |S| = \lambda|U^*|\,|W| \Rightarrow |B_u| = \lambda|W| = \lambda q^m,$$

which concludes the proof. ∎

It follows from the above construction that the type of the λq^m-partition will depend on the relationship between the subspaces $\alpha(a)$ and the subspace W. In particular, if $n_a = \dim \alpha(a)$ and $r_a = \dim(\alpha(a) \cap W)$, then this subspace will contribute q^{r_a} copies of a subspace of dimension $n_a - r_a$ in the new partition (B, β). In this way, we can decompose every subspace $\alpha(a)$ of (A, α) to determine a λq^m-partition of U.

Example 2

Consider $V_5(2)$. We can identify $V_5(2)$ with a 5-dimensional subspace V of $V_6(2)$ and let W be a one-dimensional complement of V. Let (A, α) be a partition of $V_6(2)$ of type $[(21, 2)]$. Since W is one-dimensional, it is contained in exactly one of the two-dimensional subspaces. Hence the

2-partition induced on V is of type $[(20,2),(2,1)]$. Similarly, we can see that a $[(9,3)]$ partition of $V_6(2)$ induces a 2-partition of V of type $[(8,3),(2,2)]$.

One important special case of the above is when (A,α) is a 1-partition and $W = \bigcup_{a\in C} \alpha(a)$ for some proper subset $C \subset A$. If this is the case, we can take $B = A \setminus C$ and get a q^m-partition of V.

A second technique for generating λ-partitions from 1-partitions is given in the theorem below.

Theorem 10 *Let $V = V_n(q)$ and let (A,α) be a 1-partition of type $n^{t_n}\cdots 2^{t_2}1^{t_1}$. (Here we allow the possibility that $t_j = 0$ if $j > 1$.) Then for any integer $1 < k \leq n$, there exists a λ-partition (B,β) of type*

$$n^{\lambda t_n}\cdots(k+1)^{\lambda t_{k+1}}k^{\lambda t_k + t_1}(k-1)^{\lambda t_{k-1}}\cdots 2^{\lambda t_2}$$

where $\lambda = \dfrac{q^k-1}{q-1}$.

Proof. Let us identify V with the field $\mathbb{F}_{q^n}$ and let W be a subspace of V of dimension k. Define $A_1 = \{a \in A : \dim\alpha(a) = 1\}$ and $A_+ = A \setminus A_1$. Furthermore, let (C,γ) be a 1-partition of W of type 1^λ where $\lambda = \dfrac{q^k-1}{q-1}$ and let $B = (A_+ \times C) \cup A_1$.

Then we can define a function $\beta : B \to 2^V$ as follows. If $y = (a,c) \in A_+ \times C$, define $\beta(y) = \beta(a,c) = \{x\cdot w : x \in \alpha(a), w \in \gamma(c)\}$. If $y \in A_1$, define $\beta(y) = \{x \cdot w : x \in \alpha(y), w \in W\}$.

We claim the pair (B,β) is a λ-partition of V. Indeed, if $y = (a,c) \in A_+ \times C$, for any nonzero $v_1, v_2 \in \beta(y)$ there exist $x_1, x_2 \in \alpha(a)$, $w_1, w_2 \in \gamma(c)$ such that $v_1 = x_1w_1$ and $v_2 = x_2w_2$. Since $\gamma(c)$ is one-dimensional, there exists $d \in \mathbb{F}_q \setminus \{0\}$ such that $w_2 = dw_1$, so $v_2 = (dx_2)w_1$. Hence, for any $d' \in \mathbb{F}_q \setminus \{0\}$, we have $v_1 + d'v_2 = x_1w_1 + d'dx_2w_1 = (x_1 + d'dx_2)w_1 \in \beta(y)$. Therefore $\beta(y)$ is a subspace of V. The proof that $\beta(y)$ is a subspace of V when $y \in A_1$ is similar.

Note that for any $x \in \mathbb{F}_{q^n}^\times$ the function $\phi_x : V \to V$ defined by $\phi_x(v) = xv$ is a vector space automorphism. If $y = (a,c) \in A_+ \times C$, then $\gamma(c)$ is one-dimensional so for any nonzero $w \in \gamma(c)$ we have $\phi_w(\alpha(a)) = \{xw : x \in \alpha(a)\} = \{xw' : x \in \alpha(a), w' \in \gamma(c)\} = \beta(y)$. Hence $\dim\beta(y) = \dim\alpha(a)$. Also, if $y \in A_1$, then $\alpha(y)$ is one-dimensional so for any nonzero $x \in \alpha(y)$ we have $\phi_x(W) = \{xw : w \in W\} = \{x'w : w \in W, x' \in \alpha(y)\} = \beta(y)$. Therefore, $\dim(\beta(y)) = \dim(W) = k$.

Next, we need to show that for any $0 \neq v \in V$ we have $|\{y \in B : v \in \beta(y)\}| = \lambda$. But if $y = (a,c) \in A_+ \times C$, we have $v \in \beta(y) \Leftrightarrow \mathbb{F}_q w^{-1}v \subseteq \alpha(a)$ for some $0 \neq w \in \gamma(c)$. If $y \in A_1$, then $v \in \beta(y) \Leftrightarrow \mathbb{F}_q w^{-1}v \subseteq \alpha(y)$ for some $0 \neq w \in \gamma(c)$. Therefore, since (A,α) is a 1-partition, $|\{y \in B : v \in \beta(y)\}| = |\{\mathbb{F}_q w^{-1}v : 0 \neq w \in W\}| = \lambda$ since $\dim(W) = k$. ∎

Next, we use Theorem 10 to make an observation about the existence of a λ-partition of type $[(t_2,s),(t_1,r)]$ where r and s are distinct.

Corollary 11 *Let $1 < r \leq n$, $1 \leq s \leq n$ where $r \neq s$. Then there exists a $\left(\dfrac{q^s-1}{q-1}\right)$-partition of type $\left[\left(\dfrac{q^s-1}{q-1}, r\right), \left(\dfrac{q^n-q^r}{q-1}, s\right)\right]$.*

Proof. Let U be an r-dimensional subspace of V. Let $\mathcal{P}$ be a 1-partition consisting of U and all the one-dimensional subspaces not contained in U. Then $\mathcal{P}$ is a 1-partition of type $r^1 1^t$, where

$t = \dfrac{q^n - q^r}{q-1}$. Now we can apply Theorem 10 to this 1-partition to get a $\left(\dfrac{q^s-1}{q-1}\right)$-partition of V of type $\left[\left(\dfrac{q^s-1}{q-1}, r\right), \left(\dfrac{q^n-q^r}{q-1}, s\right)\right]$ ∎

Next, we note that if we are given a λ-partition (A, α), we can also take "multiples" of (A, α) as follows. For each positive integer k, let kA be the set $A \times \{1, 2, \ldots, k\}$ and define the function $k\alpha : kA \to 2^V$ by $(k\alpha)(x, i) = \alpha(x)$ for all $x \in A$ and $1 \leq i \leq k$. Then $(kA, k\alpha)$ is a $k\lambda$-partition of V. If $\mathcal{P} = (A, \alpha)$, the we write $k\mathcal{P}$ to indicate $(kA, k\alpha)$. Note that if $\mathcal{P} = (A, \alpha)$ is of type $n_1^{t_1} n_2^{t_2} \cdots n_s^{t_s}$, then $k\mathcal{P} = (kA, k\alpha)$ is of type $n_1^{kt_1} n_2^{kt_2} \cdots n_s^{kt_s}$.

In some sense, we can reverse the above process using the concept of multiplicity. We define the *multiplicity* of the λ-partition $\mathcal{P} = (A, \alpha)$ as the greatest common divisor of the set $\{|\alpha^{-1}(\alpha(a))| : a \in A\}$.

Lemma 12 *Let (A, α) be a λ-partition of multiplicity $m > 1$. Then there exists a (λ/m)-partition (B, β) such that (A, α) is equivalent to $(mB, m\beta)$.*

Proof. Let (A, α) be a λ-partition of V of multiplicity m. Therefore, for every subspace $W \in \{\alpha(a) : a \in A\}$ there exists a positive integer k_W such that W occurs $k_W m$ times in the multiset image of α. Now let (B, β) be the (λ/m)-partition corresponding to the multiset where every $W \in \{\alpha(a) : a \in A\}$ occurs k_W times. Then it is straightforward to check (A, α) is equivalent to $(mB, m\beta)$ since they have the same multiset image. ∎

3 Dual λ-Partitions

In this section, we use vector space duals to define the dual of a λ-partition. This is slightly more complicated than taking the dual of each subspace in a λ-partition since we can increase multiplicities when doing this. Therefore, to get the dual of a λ-partition, we take the vector space duals of each subspace and then adjust the multiplicity of the resulting λ'-partition to match that of the original λ-partition. In the lemma below, we state some basic results about vector spaces and their duals using non-degenerate symmetric bilinear forms. Refer to [1, Chapter 3] or [6, Chapter 8, §27] for proofs of these results.

Let $\langle\ ,\ \rangle : V \times V \to \mathbb{F}_q$ be a non-degenerate symmetric bilinear form. For example, we could use the standard dot product when $V = \mathbb{F}_q^n$. Then $\langle\ ,\ \rangle$ induces an isomorphism between V and its dual, $V^* = \mathrm{Hom}(V, \mathbb{F}_q)$. For any subset $S \subseteq V$, we define $S^\perp = \{v \in V : \langle v, x \rangle = 0 \text{ for every } x \in S\}$. When $x \in V$, we denote $\{x\}^\perp$ by writing $x^\perp$.

Lemma 13 *Let S, T be subsets of a finite-dimensional vector space V over F and let $\langle\ ,\ \rangle : V \times V \to F$ be a symmetric non-degenerate bilinear form on V. Then we have the following properties:*

1. *$S^\perp$ is a subspace of V.*
2. *$S \subseteq T \Rightarrow T^\perp \subseteq S^\perp$.*
3. *$S^\perp = \mathrm{span}(S)^\perp$.*

4. $\dim(S^{\perp}) = n - \dim(\text{span}(S))$.

5. $(S^{\perp})^{\perp} = \text{span}(S)$.

6. $(S \cup T)^{\perp} = S^{\perp} \cap T^{\perp}$.

7. $(\text{span}(S) \cap \text{span}(T))^{\perp} = S^{\perp} + T^{\perp}$.

In the proofs below, we will use some of these standard properties of $S^{\perp}$. We start with an important example that we will use to build dual λ-partitions.

Example 3

Let $J \subseteq V$ be a set of nonzero vectors representing the one-dimensional subspaces of V. So if $J = \{x_1, x_2, \ldots, x_k\}$, we have the following properties:

1. $\bigcup_{i=1}^{k} \mathbb{F}_q x_i = V$,
2. for any $x, y \in J$, we have $\mathbb{F}_q x \cap \mathbb{F}_q y \neq \{0\} \Rightarrow x = y$.

Note here that $k = |J| = \dfrac{q^n - 1}{q - 1}$.

Next, define a function $\alpha : J \to 2^V$ by $\alpha(x) = x^{\perp}$ for all $x \in J$. We claim that (J, α) forms a $\left(\dfrac{q^{n-1}-1}{q-1}\right)$-partition of V. It is clear that $\alpha(x) = x^{\perp}$ is a subspace for every $x \in J$. Also, for any $0 \neq v \in V$, we have $v \in x^{\perp} = \alpha(x) \Leftrightarrow x \in v^{\perp}$. So, since $\dim v^{\perp} = n - 1$, there are exactly $\left(\dfrac{q^{n-1}-1}{q-1}\right)$ elements $x \in J$ such that $v \in \alpha(x)$. Hence (J, α) is the claimed $\left(\dfrac{q^{n-1}-1}{q-1}\right)$-partition of V of type $\left[\left(\dfrac{q^n-1}{q-1}\right), n-1\right]$. Indeed, (J, α) is just the q-Grassmanian $G(n, n-1)$ mentioned in our introduction.

Given a λ'-partition of V, we use Proposition 14 as a first step in accomplishing our goal of defining a λ-partition that is dual to the initial λ'-partition. We will then create such a dual through a series of reductions starting from the above example.

Proposition 14 *Let $U \subseteq V = V_n(q)$ be a subspace of dimension r. Let $Q \subseteq U$ consist of one nonzero vector representative for each one-dimensional subspace of U. (So for each $0 \neq u \in U$ there exists $x \in Q$ such that $\mathbb{F}_q u = \mathbb{F}_q x$; and for any $x, y \in Q$, if $\mathbb{F}_q x = \mathbb{F}_q y$, then $x = y$.) Then the following hold:*

1. If $r = \dim(U) \geq 2$, then $\bigcup_{x \in Q} x^{\perp} = V$.

2. If $w \in U^{\perp}$, then the set $\{x \in Q : w \in x^{\perp}\}$ has order $\dfrac{q^r - 1}{q - 1}$.

3. If $w \notin U^{\perp}$, then the set $\{x \in Q : w \in x^{\perp}\}$ has order $\dfrac{q^{r-1} - 1}{q - 1}$.

Proof. Choose $x_1, \ldots, x_r \in Q$ so that $\{x_1, \ldots, x_r\}$ is a basis of U. Let $0 \neq v \in V$ and for each $1 \leq i \leq r$ define $\gamma_i = \langle x_i, v\rangle$. If $\gamma_j = 0$ for any j, then $v \in x_j^\perp \subseteq \bigcup_{i=1}^r x_i^\perp$. If $\gamma_j \neq 0$ for all j, then the vector

$$y = \left(\sum_{i=2}^r \gamma_i\right) x_1 - \gamma_1 \left(\sum_{i=2}^r x_i\right) \in U \setminus \{0\}$$

satisfies

$$\begin{aligned} \langle y, v\rangle &= \left(\sum_{i=2}^r \gamma_i\right) \langle x_1, v\rangle - \gamma_1 \left(\sum_{i=2}^r \langle x_i, v\rangle\right) \\ &= \left(\sum_{i=2}^r \gamma_i\right) \gamma_1 - \gamma_1 \left(\sum_{i=2}^r \gamma_i\right) \\ &= 0. \end{aligned}$$

So $v \in y^\perp$. Since $y \neq 0$, there exists $z \in Q$ such that $\mathbb{F}_q y = \mathbb{F}_q z$. Therefore, $v \in z^\perp \subseteq \bigcup_{x\in Q} x^\perp$. So we have established that $\bigcup_{x\in Q} x^\perp = V$.

Next, since $Q \subseteq U$, for every $x \in Q$ we have $U^\perp \subseteq x^\perp$; so for any $w \in U^\perp$, the set $\{x \in Q : w \in x^\perp\} = Q$, hence has order $\dfrac{q^r - 1}{q-1}$ as claimed.

Finally, if $w \notin U^\perp$, then for any $x \in Q \subseteq U$ we have $w \in x^\perp \Leftrightarrow x \in w^\perp \cap U$. But $\dim(w^\perp \cap U) = r - 1$ since $\dim w^\perp = n - 1$ and $U \not\subseteq w^\perp$. Hence, there are exactly $\dfrac{q^{r-1} - 1}{q-1}$ one-dimensional subspaces of $w^\perp \cap U$. So it follows that the order of the set $\{x \in Q : w \in x^\perp\}$ is $\dfrac{q^{r-1} - 1}{q-1}$. ∎

We can use the above observations to make a "reduction" in the λ-partition $\mathcal{P}$ given in Example 3. In particular, based on the above proposition, if we are given an r-dimensional subspace $U \subseteq V$, we can reduce λ by $\dfrac{q^{r-1} - 1}{q-1}$ by eliminating $\dfrac{q^r - 1}{q-1}$ subspaces of dimension $n-1$ (corresponding to the $x \in J \cap U$, where J is the set defined in Example 3) and replacing them with $\left(\dfrac{q^r - 1}{q-1}\right) - \left(\dfrac{q^{r-1} - 1}{q-1}\right) = q^{r-1}$ copies of the $(n-r)$-dimensional subspace $U^\perp$.

Using the technique described above, given a λ'-partition (A, α) of V, if we naively try to define $\alpha^\perp : A \to 2^V$ by $\alpha^\perp(a) = (\alpha(a))^\perp$ for all $a \in A$, we will not in general get a λ''-partition for some λ''. Proposition 14 suggests a minor modification to this strategy to create such a λ''-partition. We first demonstrate this technique through an example.

Example 4

Let $V = V_6(2)$. For convenience, we can view the vectors of $V_6(2)$ as a binary representation of an integer and then convert this to decimal form to represent this vector. Hence we use decimal notation to represent the nonzero vectors in $V_6(2)$ in this example. For example, the vector $(1, 1, 0, 1, 0, 1)$ would be represented by $1 \cdot 2^5 + 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 53$.

Now consider the following subspaces of $V_6(2)$, where we give only the nonzero vectors in each subspace:
$U_1 = \{1, 2, 3, 4, 5, 6, 7\}$, $U_2 = \{8, 16, 24, 32, 40, 48, 56\}$,

$U_3 = \{9, 18, 36, 27, 54, 63, 45\}$, $U_4 = \{20, 35, 30, 55, 61, 41, 10\}$,
$U_5 = \{38, 31, 53, 57, 42, 12, 19\}$, $U_6 = \{29, 49, 58, 44, 11, 22, 39\}$,
$U_7 = \{28, 46, 50\}$, $U_8 = \{15, 51, 60\}$, $U_9 = \{21, 43, 62\}$, $U_{10} = \{14, 33, 47\}$,
$U_{11} = \{13, 23, 26\}$, $U_{12} = \{17, 37, 52\}$, $U_{13} = \{25, 34, 59\}$.
Then $\{U_1, U_2, \ldots, U_{13}\}$ is a 1-partition of $V_6(2)$ of type $[(6, 3), (7, 2)]$.

Next, we consider the following subspaces (again we only indicate the nonzero vectors in each subspace), where we use the standard dot product to define $S^\perp$ for any subset $S \subseteq V_6(2)$:
$U_1^\perp = \{8, 16, 24, 32, 40, 48, 56\}$, $U_2^\perp = \{1, 2, 3, 4, 5, 6, 7\}$,
$U_3^\perp = \{9, 18, 36, 27, 54, 63, 45\}$, $U_4^\perp = \{11, 20, 31, 33, 42, 53, 62\}$,
$U_5^\perp = \{15, 17, 30, 35, 44, 50, 61\}$, $U_6^\perp = \{14, 19, 29, 39, 41, 52, 58\}$,
$U_7^\perp = \{1, 12, 13, 22, 23, 26, 27, 34, 35, 46, 47, 52, 53, 56, 57\}$,
$U_8^\perp = \{3, 12, 15, 21, 22, 25, 26, 37, 38, 41, 42, 48, 51, 60, 63\}$,
$U_9^\perp = \{7, 10, 13, 19, 20, 25, 30, 34, 37, 40, 47, 49, 54, 59, 60\}$,
$U_{10}^\perp = \{6, 10, 12, 16, 22, 26, 28, 33, 39, 43, 45, 49, 55, 59, 61\}$,
$U_{11}^\perp = \{5, 11, 14, 18, 23, 25, 28, 32, 37, 43, 46, 50, 55, 57, 60\}$,
$U_{12}^\perp = \{2, 8, 10, 21, 23, 29, 31, 36, 38, 44, 46, 49, 51, 57, 59\}$,
$U_{13}^\perp = \{4, 9, 13, 17, 21, 24, 28, 34, 38, 43, 47, 51, 55, 58, 62\}$.
It is straightforward to check that $\{U_7^\perp, U_8^\perp, \ldots, U_{13}^\perp, 2U_1^\perp, 2U_2^\perp, \ldots, 2U_6^\perp\}$ is a 3-partition of $V_6(2)$ of type $[(7, 4), (12, 3)]$, where we use $2U_j^\perp$ to denote two copies of $U_j^\perp$. Note that here we needed two copies of the $U_j^\perp$ of smallest dimension in order to make this a 3-partition.

Moreover, if we repeat this procedure for this new 3-partition (doubling $U_i = (U_i^\perp)^\perp$ for $7 \leq i \leq 13$), we get a 2-partition of type $[(12, 3), (14, 2)]$, which consists of two copies of the original 1-partition $\{U_1, U_2, \ldots, U_{13}\}$, hence has multiplicity 2.

Theorem 15 takes into account the multiplicities that can occur and uses Lemma 12 to give us a range of possible candidates for a dual partition. We then identify the candidate with the same multiplicity as the original λ'-partition to be the dual partition.

Before stating Theorem 15, we will need to introduce the concept of d-multiplicity. Given a λ'-partition $\mathcal{P} = (Y, \omega)$ of V, let $D = \{\dim \omega(y) : y \in Y\}$. For each $d \in D$ define the *d-multiplicity* μ_d of $\mathcal{P}$ to be the greatest common divisor of the set $\{|\omega^{-1}(\omega(y))| : \ y \in Y \text{ and } \dim \omega(y) = d\}$. (If $d \notin D$, we can define μ_d to be 0.) It follows from the definitions that the multiplicity of $\mathcal{P}$ is the greatest common divisor of $\{\mu_d : \ d \in D\}$.

Theorem 15 *Let $\mathcal{P} = (Y, \omega)$ be a λ-partition of $V = V_n(q)$ of type $[(a_k, k), (a_{k-1}, k-1), \ldots, (a_{s+1}, s+1), (a_s, s)]$, where $a_k a_s \neq 0$. For each $s \leq d \leq k$, let μ_d denote the d-multiplicity of $\mathcal{P}$. Then for every $\ell \geq 1$ such that ℓ is a common divisor of the set $\{\mu_k q^k, \mu_{k-1} q^{k-1}, \ldots, \mu_s q^s\}$, there exists a λ_ℓ-partition $\mathcal{P}^{(\ell)} = (C_\ell, \gamma_\ell)$ of V such that:*

1. $\lambda_\ell = \dfrac{1}{\ell}\left[\left(\displaystyle\sum_{i=s}^{k} a_i\right) - \lambda\right] = \dfrac{1}{\ell}\left(|Y| - \lambda\right).$

2. *$\mathcal{P}^{(\ell)}$ is of type*

$$\left[\left(\frac{a_s q^s}{\ell}, n-s\right), \left(\frac{a_{s+1} q^{s+1}}{\ell}, n-s-1\right), \ldots, \left(\frac{a_k q^k}{\ell}, n-k\right)\right].$$

3. $\{\gamma_\ell(c) : c \in C_\ell\} = \{\omega(y)^\perp : y \in Y\}$ *as sets.*

4. $\left|\gamma_\ell^{-1}\left(\omega(y)^\perp\right)\right| = \dfrac{q^{r_y}}{\ell}\left|\omega^{-1}(\omega(y))\right|$ *where* $r_y = \dim \omega(y)$.

Proof. Let (J, α) be the $\left(\dfrac{q^{n-1}-1}{q-1}\right)$-partition of $(n-1)$-dimensional subspaces of V defined in Example 3, where $\alpha(x) = x^\perp$ for all $x \in J$. Let (Y, ω) be a λ-partition of V of type $[(a_k, k), \ldots, (a_s, s)]$, where $a_k a_s \neq 0$ and $m = \sum_{i=s}^{k} a_i$ is the size of (Y, ω). For each $y \in Y$, let $r_y = \dim(\omega(y))$.

Next, consider the Cartesian product $J \times Y$ and the canonical projection $\pi : J \times Y \to J$ onto J defined by $\pi(x, y) = x$ for all $(x, y) \in J \times Y$. Define

$$A = \{(x, y) |\, y \in Y, x \in \omega(y)\} \subseteq J \times Y.$$

We claim that $(A, \alpha\pi)$ is a $\lambda\left(\dfrac{q^{n-1}-1}{q-1}\right)$-partition of V. Clearly $\alpha\pi(x, y) = \alpha(x) = x^\perp$ is a subspace for all $(x, y) \in A$. Let $0 \neq v \in V$. Then

$$v \in \alpha\pi(x, y) \Leftrightarrow v \in x^\perp \text{ and } x \in \omega(y) \Leftrightarrow x \in v^\perp \cap \omega(y).$$

So

$$|\{(x, y) \in A : v \in \alpha\pi(x, y)\}| = \sum_{y \in Y} \frac{1}{q-1}|v^\perp \cap \omega(y)| = \lambda\left(\frac{q^{n-1}-1}{q-1}\right),$$

where the last equality follows because (Y_W, ω_W) is a λ-partition of $W = v^\perp$ by Lemma 1.

Now, for each $y \in Y$, let $A_y = \{(x, y) \in A : x \in \omega(y)\}$, and define $\alpha_y : A_y \to 2^V$ to be the restriction of $\alpha\pi$ to A_y. Then $(A, \alpha\pi) = \left(\bigcup_{y \in Y} A_y, \bigcup_{y \in Y} \alpha_y\right)$. For each $y \in Y$, choose a subset $B_y \subseteq A_y$ of cardinality $q^{r_y - 1}$, let $B = \bigcup_{y \in Y} B_y$, and define a function $\beta : A \to 2^V$ by

$$\beta(x, y) = \begin{cases} \omega(y)^\perp & \text{if } (x, y) \in B \\ V & \text{if } (x, y) \in A \setminus B \end{cases}$$

for all $(x, y) \in A$.

We claim that (A, β) is a $\lambda\left(\dfrac{q^{n-1}-1}{q-1}\right)$-partition of V.

Proof of Claim: It is clear that $\beta(x, y)$ is a subspace of V for all $(x, y) \in A$. Next, for any $0 \neq v \in V$, we let $S_v = \{(x, y) \in A : v \in \alpha\pi(x, y)\}$ and $T_v = \{(x, y) \in A : v \in \beta(x, y)\}$. We prove that $|T_v| = |S_v|$ and we know $|S_v|$ has the required cardinality since $(A, \alpha\pi)$ is a $\lambda\left(\dfrac{q^{n-1}-1}{q-1}\right)$-partition of V.

Note that since A is the disjoint union of the A_y for $y \in Y$, it suffices to show that $|T_v \cap A_y| = |S_v \cap A_y|$ for all $y \in Y$. So fix $y \in Y$. If $v \in \omega(y)^\perp$, then $A_y \cap T_v = A_y = A_y \cap S_v$, where the last equality follows from Proposition 14(2). If $v \notin \omega(y)^\perp$, then $|A_y \cap T_v| = |A_y| - |B_y| = \dfrac{q^{r_y - 1} - 1}{q-1}$ and, it follows from Proposition 14(3) that $|A_y \cap T_v| = |A_y \cap S_v|$. Therefore, our claim is established.

Now consider the pair (B, β_0), where β_0 is the restriction of β to B. By definition, it follows that $\{\beta_0(x,y) : (x,y) \in B\} = \{\omega(y)^\perp : y \in Y\}$ as sets. Furthermore, (B, β_0) is also a λ_0-partition of V for some λ_0 since for all $(x,y) \in A \setminus B$, $\beta(x,y) = V$. We can compute λ_0 as follows.

$$\lambda_0 = \lambda \left(\frac{q^{n-1}-1}{q-1} \right) - \sum_{y \in Y} \left(\frac{q^{r_y-1}-1}{q-1} \right) = \lambda \left(\frac{q^{n-1}-1}{q-1} \right) - \sum_{i=s}^{k} a_i \left(\frac{q^{i-1}-1}{q-1} \right).$$

But, since (Y, ω) is a λ-partition, we know

$$\sum_{i=s}^{k} a_i(q^i - 1) = \lambda(q^n - 1) \quad \Rightarrow \quad \lambda q^{n-1} - \left(\sum_{i=s}^{k} a_i q^{i-1} \right) = \frac{1}{q} \left(\lambda - \left(\sum_{i=s}^{k} a_i \right) \right).$$

Hence we see that

$$\begin{aligned}
\lambda_0 &= \frac{1}{q-1} \left[\left(\lambda q^{n-1} - \sum_{i=s}^{k} a_i q^{i-1} \right) - \left(\lambda - \sum_{i=s}^{k} a_i \right) \right] \\
&= \frac{1}{q-1} \left[\frac{1}{q} \left(\lambda - \sum_{i=s}^{k} a_i \right) - \left(\lambda - \sum_{i=s}^{k} a_i \right) \right] \\
&= \frac{1}{q-1} \left(\frac{1-q}{q} \right) \left(\lambda - \sum_{i=s}^{k} a_i \right) \\
&= \frac{1}{q} \left[\left(\sum_{i=s}^{k} a_i \right) - \lambda \right] \\
&= \frac{1}{q} \left(|Y| - \lambda \right).
\end{aligned}$$

Furthermore, (B, β_0) is of type

$$\left[(a_s q^{s-1}, n-s), (a_{s+1} q^{s}, n-s-1), \ldots, (a_k q^{k-1}, n-k) \right].$$

Because β_0 is constant when restricted to $B_y = A_y \cap B$, in (B, β_0) we have $|\beta_0^{-1}(\omega(y)^\perp)| = |\beta_0^{-1}(\beta_0(x,y))| = |B_y||\omega^{-1}(\omega(y))| = q^{r_y-1}|\omega^{-1}(\omega(y))|$, where $(x,y) \in B$. Therefore, for any $s \leq d \leq k$, the $(n-d)$-multiplicity of (B, β_0) is $\mu_d q^{d-1}$. Hence the multiplicity of (B, β_0) is the greatest common divisor g of the set $\{\mu_s q^{s-1}, \mu_{s-1} q^{s-1}, \ldots, \mu_k q^{k-1}\}$. So by Lemma 12, there exists a λ'-subpartition (C, γ) of (B, β_0) of multiplicity 1 of type

$$\left[\left(\frac{a_s q^s}{qg}, n-s \right), \left(\frac{a_{s+1} q^{s+1}}{qg}, n-s-1 \right), \ldots, \left(\frac{a_k q^k}{qg}, n-k \right) \right],$$

where

$$\lambda' = \frac{\lambda_0}{g} = \frac{1}{qg} \left(|Y| - \lambda \right).$$

Furthermore, for every $(x,y) \in B$, there exists a $c \in C$ such that $\gamma(c) = \beta_0(x,y) = \omega(y)^\perp$.

Finally, to get the partition $\mathcal{P}^{(\ell)} = (C_\ell, \gamma_\ell)$, we take the $(gq)/\ell$ multiple of (C, γ) as described in Lemma 12 and the discussion immediately preceding it. Then $\mathcal{P}^{(\ell)}$ satisfies the conclusion of the theorem. $\blacksquare$

Given a λ'-partition $\mathcal{P}$ of V, in Theorem 15 there is a smallest partition $\mathcal{P}^{\min}$ of multiplicity 1 that occurs when ℓ is maximized.

Definition 2 *Let $\mathcal{P} = (Y, \omega)$ be a λ'-partition of a vector space V of multiplicity m. The* dual λ-partition *$\mathcal{P}^*$ of $\mathcal{P}$ is the λ-partition of multiplicity m given by $m\mathcal{P}^{\min}$.*

It follows from the definition of $\mathcal{P}^*$ that $(m\mathcal{P})^* = m(\mathcal{P}^*)$ for any $m \geq 1$.

Corollary 16 *Let $\mathcal{P}$ be a λ-partition. Then $(\mathcal{P}^*)^* = \mathcal{P}$.*

Proof. Note that since for any λ-partition we have $(m\mathcal{P})^* = m(\mathcal{P}^*)$, it suffices to assume the multiplicity of $\mathcal{P}$ is 1.

Let $\mathcal{P} = (Y, \omega)$ be a partition of multiplicity 1 of type $[(a_k, k), \ldots, (a_s, s)]$, where $a_k a_s \neq 0$. Let μ_d denote the d-multiplicity of $\mathcal{P}$ for all $s \leq d \leq k$. Furthermore, let $\mathcal{P}^* = (C, \gamma)$ and $(\mathcal{P}^*)^* = (Z, \xi)$. Then it follows from Theorem 15(3) that

$$\{\xi(z) : z \in Z\} = \{\gamma(c)^\perp : c \in C\} = \left\{ \left(\omega(y)^\perp\right)^\perp : y \in Y \right\} = \{\omega(y) : y \in Y\}.$$

Let $y \in Y$ and $z \in Z$ such that $\xi(z) = \omega(y)$. It suffices to show $|\xi^{-1}(\xi(z))| = |\omega^{-1}(\omega(y))|$. Let $c \in C$ be such that $\gamma(c)^\perp = \omega(y) = \xi(z)$. By Theorem 15(4) it follows that the d-multiplicity of $\mathcal{P}^*$ is $(\mu_{n-d}\, q^{n-d})/g$ for $n - k \leq d \leq n - s$, so

$$|\xi^{-1}(\xi(z))| = \frac{q^{n-r_y}}{g'} |\gamma^{-1}(\omega(y)^\perp)| = \frac{q^{r_y} q^{n-r_y}}{g' g} |\omega^{-1}(\omega(y))|$$

where $r_y = \dim \omega(y)$, g is the gcd of $\{\mu_k q^k, \mu_{k-1} q^{k-1}, \ldots, \mu_s q^s\}$, and g' is the gcd of the set

$$\left\{ \frac{\mu_s q^s}{g} q^{n-s}, \frac{\mu_{s-1} q^{s-1}}{g} q^{n-s+1}, \ldots, \frac{\mu_k q^k}{g} q^{n-k} \right\}.$$

Therefore, $g'g$ is the gcd of the set $\{\mu_k q^n, \mu_{k-1} q^n, \ldots, \mu_s q^n\}$, hence $g'g = q^n$ since we assumed the multiplicity of $\mathcal{P}$ was 1. So it follows that $|\xi^{-1}(\xi(z))| = |\omega^{-1}(\omega(y))|$, hence $(\mathcal{P}^*)^* = \mathcal{P}$, as claimed. $\blacksquare$

Many of the λ-partition types that we have discussed above seem realizable to be duals of 1-partitions. An example of a minimal λ-partition that is not the dual of a 1-partition is the 7-partition of $V_8(2)$ of type 3^{255}. In order for this to have been a dual partition of a 1-partition, we would need a 1-partition of $V_8(2)$ of type 5^{255}, which is clearly impossible.

4 λ-partitions and Designs Over Finite Fields

A number of well-studied mathematical structures arise from certain partitions of finite vector spaces. For example, if $\mathcal{P}$ is the set of all subspaces of $V_n(q)$ (which is a λ-partition of $V_n(q)$), then the set of all cosets of the elements of $\mathcal{P}$, denoted by $AG(n,q)$, is what is known as the *affine geometry of dimension n over* $\mathbb{F}_q$ (see [2]). Similarly, the set of all subspaces of $V_{n+1}(q)$, denoted by $PG(n,q)$, is the *projective geometry of dimension n over* $\mathbb{F}_q$. Other designs arise similarly either from taking cosets of subspaces in a partition or from taking the subspaces themselves as blocks in the design. We will first define these terms.

A *design* is a pair $(X, \mathcal{A})$, where X is a set of elements called *points*, and $\mathcal{A}$ is a collection of nonempty subsets of X called *blocks*. Suppose $v \geq 2$, $\lambda \geq 1$, and $L \subseteq \{n \in \mathbb{Z} : n \geq 2\}$. A (v, L, λ)-*pairwise balanced design* (abbreviated (v, L, λ)-PBD) is a design $(X, \mathcal{A})$ where: (1) $|X| = v$, (2) $|A| \in L$ for all $A \in \mathcal{A}$, and (3) every pair of distinct points is contained in exactly λ blocks. It is easy to see that a (v, L, λ)-PBD is equivalent to a decomposition of the λ-fold complete multigraph ${}^{\lambda}K_v$ into complete subgraphs with orders in L. A $(v, \{k\}, \lambda)$-PBD is better known as a *balanced incomplete block design* and is denoted by (v, k, λ)-BIBD.

Suppose $(X, \mathcal{A})$ is a (v, L, λ)-PBD. A *parallel class* in $(X, \mathcal{A})$ is a subset of disjoint blocks from $\mathcal{A}$ whose union is X. A partition of $\mathcal{A}$ into r parallel classes is called a *resolution*, and $(X, \mathcal{A})$ is said to be a *resolvable* PBD if $\mathcal{A}$ has at least one resolution.

A parallel class in a (v, L, λ)-PBD is *uniform* if every block in the parallel class is of the same size. Let $L = \{\ell_1, \ell_2, \ldots, \ell_r\}$ be an ordered set of integers ≥ 2 and let $R = \{t_1, t_2, \ldots, t_r\}$ be an ordered multiset of positive integers. A *uniformly resolvable design*, denoted (v, L, λ, R)-URD, is a resolvable (v, L, λ)-PBD with t_i parallel classes with blocks of size ℓ_i for $1 \leq i \leq r$. It is easy to see that a $(v, \{\ell_1, \ldots, \ell_r\}, \lambda, \{t_1, \ldots, t_r\})$-URD is equivalent to a factorization of ${}^{\lambda}K_v$ into t_i K_{ℓ_i}-factors for $1 \leq i \leq r$. For some of the necessary conditions for the existence of URDs, we direct the reader to [7] and the references therein.

If W is a subset of $V_n(q)$, we denote the complete graph with vertices labeled with elements of W by $K(W)$. If W and X are subsets of $V_n(q)$ with $0 \notin X$, we define $G(W, X)$ to be the subgraph of $K(V_n(q))$ with edge set $\{\{w, w + x\} : w \in W, x \in X\}$. It is easy to see that if X is a subspace of $V_n(q)$ of dimension n_i, then $G(V_n(q), X \setminus \{0\})$ is a $K_{q^{n_i}}$-factor of K_{q^n}. Moreover, if X_1 and X_2 are disjoint subspaces, then the factors they induce are also disjoint. Thus a λ-partition $\mathcal{P}$ of $V_n(q)$ of type $[(t_1, n_1), \ldots, (t_k, n_k)]$ induces a factorization of ${}^{\lambda}K_{q^n}$ into t_i $K_{q^{n_i}}$-factors for $1 \leq i \leq k$. Equivalently, if we let $\mathcal{A}$ denote the subspaces in $\mathcal{P}$, along with all their cosets, then, $(V_n(q), \mathcal{A})$ is a $(q^n, \{q^{n_1}, \ldots, q^{n_k}\}, \lambda, \{t_1, \ldots, t_k\})$-URD. Thus we have the following result on URDs as a corollary to Corollary 11.

Corollary 17 *Let $1 < r \leq n$, $1 \leq s \leq n$ where $r \neq s$ and let q be a prime power. Then there exists a* $(q^n, \{q^r, q^s\}, \frac{q^s-1}{q-1}, \{\frac{q^s-1}{q-1}, \frac{q^n-q^r}{q-1}\})$-URD.

Similarly, we have the following result on resolvable designs as a corollary to Proposition 7.

Corollary 18 *Let q be a prime power and let k, n be positive integers with $k \leq n$. Let $r = \gcd(k, n)$. Then there exists a resolvable* $\left(q^n, q^k, \dfrac{q^k - 1}{q^r - 1}\right)$-BIBD.

Another related area with potential applications for λ-partitions with additional properties is the area of designs over finite fields (see [4], for example). A t-$(n, k, \lambda^*; q)$ *design* is a collection $\mathcal{B}$ of k-dimensional subspaces of an n-dimensional vector space over $\mathbb{F}_q$ with the property that any t-dimensional subspace is contained in exactly λ^* members of $\mathcal{B}$. It is also called a *design over a finite field* or a *q-analog of t-(n, k, λ) design*. The collection $\mathcal{B}$ is necessarily a λ-partition of $V_n(q)$.

The first nontrivial example for $t \geq 2$ was given by S. Thomas [13]. Namely, he constructed a series of 2-$(n,3,7;2)$ designs for all $n \geq 7$ satisfying $(n,6)=1$.

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