THE SUPERTAIL OF A SUBSPACE PARTITION

O. HEDEN, J. LEHMANN, E. NĂSTASE, AND P. SISSOKHO

Abstract. Let $V = V(n, q)$ be a vector space of dimension n over the finite field with q elements, and let $d_1 < d_2 < \ldots < d_m$ be the dimensions that occur in a subspace partition $\mathcal{P}$ of V. Let $\sigma_q(n, t)$ denote the minimum size of a subspace partition of V, in which t is the largest dimension of a subspace. For any integer s, with $1 < s \le m$, the set of subspaces in $\mathcal{P}$ of dimension less than d_s is called the s-supertail of $\mathcal{P}$. The main result is that the number of spaces in an s-supertail is at least $\sigma_q(d_s, d_{s-1})$.

1. Introduction

Let $V = V(n, q)$ denote a vector space of dimension n over the finite field with q elements. Motivated by the correspondence to projective spaces, we call a 1-dimensional subspace of V, a *point* of V. A *subspace partition* $\mathcal{P}$ of V, also known as a *vector space partition*, is a collection of nonzero subspaces of V such that each point of V is in exactly one subspace of $\mathcal{P}$.

Throughout this paper we let n_i denote the number of spaces in $\mathcal{P}$ of dimension d_i and we assume that

$$d_1 < d_2 < \cdots < d_m \,, \tag{1}$$

where m is the number of distinct dimensions that occur in $\mathcal{P}$. The expression $[d_1^{n_1} \ldots d_m^{n_m}]$ is called the *type* of the subspace partition. The problem of classifying the spectrum of the different types of subspace partitions is far from solved and seems to be a most difficult problem, see e.g. [6].

For any integer s with $1 < s \le m$, the set of subspaces in $\mathcal{P}$ of dimension less than d_s is called the *s-supertail* of $\mathcal{P}$. In the following, we sometimes simply say *supertail* and denote it by ST. The union of all the subspaces in ST is a subset of the vector space; by a *point in the supertail* we mean a point contained in that union and we write P_{ST} for the set of such points.

The *size* of a subspace partition $\mathcal{P}$ is the number of subspaces in $\mathcal{P}$. Let $\sigma_q(n, t)$ denote the *minimum size* of a subspace partition of V in which the largest subspace has dimension t. The following theorem was proved in [7] and [9].

Supported by grant KAW 2005.0098 from the Knut and Alice Wallenberg Foundation.

Theorem 1. *Let $n, k, t,$ and r be integers such that $1 \leq r < t$, $k \geq 1$, and $n = kt + r$. Then*

$$\sigma_q(n,t) = q^t + 1 \;\; \textit{for } n < 2t \;,$$

and

$$\sigma_q(n,t) = q^{t+r} \sum_{i=0}^{k-2} q^{it} + q^{\lceil \frac{t+r}{2} \rceil} + 1 \; \textit{for } n \geq 2t \;.$$

Here we prove the following theorem.

Theorem 2. *For every n and q and every s-supertail ST of a subspace partition of $V(n,q)$,*

$$|ST| \geq \sigma_q(d_s, d_{s-1}) \;. \tag{2}$$

In the last section, we give examples of partitions with supertails for which we have equality in Theorem 2. This shows that the lower bound given above cannot be improved in general. In Section 5, we also give an example that shows how Theorem 2 can be used to prove non-existence of certain subspace partitions.

Theorem 2 extends the following theorem due to Heden [5].

Theorem 3. *Let $\mathcal{P}$ be a partition of $V(n,q)$ of type $[d_1^{n_1} \dots d_m^{n_m}]$, where $d_1 < \dots < d_m$ and $n_i > 0$ for $1 \leq i \leq m$. Then*

(i) *if $q^{d_2-d_1}$ does not divide n_1 and if $d_2 < 2d_1$, then $n_1 \geq q^{d_1} + 1$.*
(ii) *if $q^{d_2-d_1}$ does not divide n_1 and $d_2 \geq 2d_1$, then either $n_1 = (q^{d_2} - 1)/(q^{d_1} - 1)$ or $n_1 > 2q^{d_2-d_1}$.*
(iii) *if $q^{d_2-d_1}$ divides n_1 and $d_2 < 2d_1$, then $n_1 \geq q^{d_2} - q^{d_1} + q^{d_2-d_1}$.*
(iv) *if $q^{d_2-d_1}$ divides n_1 and $d_2 \geq 2d_1$, then $n_1 \geq q^{d_2}$.*

The paper is organized in the following way. In Section 2, we give some elementary definitions and results that will be used later. In Section 3, we show that if the number of points in the s-supertail is not too large, then the set of vectors in the supertail must constitute a subspace of V of dimension s. As will be observed in Proposition 4, in Section 4, this proves Theorem 2 in the case where the number of points in the supertail is not too large. If $d_s > 2d_{s-1}$ and the number of points in the s-supertail ST is large, then the size of ST can easily be bounded from below. These calculations are done in the proof of Proposition 5, which verifies Inequality (2) for this case. Finally, we prove Proposition 6 where we show that Inequality (2) is satisfied when $d_{s-1} < d_s \leq 2d_{s-1}$.

In the case $s = 2$, that is, when a supertail of a subspace partition $\mathcal{P}$ is the so called *tail* of $\mathcal{P}$, easy verifications show that Theorem 3 implies Theorem 2. This fact is an ingredient in the proofs of Propositions 5 and 6, which together with Proposition 4 prove Theorem 2.

For historical remarks, applications and a survey of known results in this area, see for example [6].

2. Preliminaries

Every subspace partition $\mathcal{P}$ of $V = V(n,q)$ satisfies the so-called *packing condition*:

$$\sum_{i=1}^{m} n_i(q^{d_i} - 1) = q^n - 1 \ ,$$

and the *dimension condition*:

$$U, U' \in \mathcal{P}, \quad U \neq U' \qquad \Longrightarrow \qquad \dim(U) + \dim(U') \leq n \ ,$$

see e.g. [2].

A *hyperplane* is a subspace of V of dimension $n-1$. The set of all hyperplanes of V is throughout this paper denoted by $\mathcal{H}$.

Proposition 1. *For any subspace U of V, the number of hyperplanes containing U is*

$$\frac{q^{n-\dim(U)} - 1}{q-1} \ . \tag{3}$$

Proposition 2. *If U is a subset of V containing $q^{d-1} + \cdots + q + 1$ points and contained in precisely*

$$\frac{q^{n-d} - 1}{q-1} \tag{4}$$

hyperplanes, then U is a subspace of V of dimension d.

Proof. The points of U span a subspace $\langle U \rangle$ of V of dimension $d' \geq d$. The intersection of a family of hyperplanes is a subspace of V, and so the intersection of all the hyperplanes that contain U is exactly equal to $\langle U \rangle$. The proposition now follows from Proposition 1. □

Let $t_i(H)$ denote the number of spaces of dimension d_i, in a given subspace partition $\mathcal{P}$, that are contained in the hyperplane H.

Lemma 1. *For every s-supertail of every subspace partition $\mathcal{P}$ of V and every hyperplane H in V, there is an integer k such that*

$$\sum_{i=1}^{s-1} (n_i - t_i(H)) q^{d_i - 1} = k \cdot q^{d_s - 1} \ .$$

Proof. The number of vectors not contained in a hyperplane H is $q^n - q^{n-1}$. Hence, the number of points of V not contained in H is $(q^n - q^{n-1})/(q-1) = q^{n-1}$. The intersection of H with a d-dimensional space U, not contained in H, is a subspace of dimension $d-1$, and hence, the number of points in U that are not contained in H is equal to q^{d-1}.

The spaces in a subspace partition $\mathcal{P}$ of V cover all points in V. Now compare the number of points in V that are not contained in H with the number of points in each space in $\mathcal{P}$ that are not contained in H. From that we get

$$\sum_{i=1}^{m}(n_i - t_i(H))q^{d_i-1} = q^{n-1} \ ,$$

and the lemma follows. □

This lemma has two immediate corollaries:

Corollary 1. *For every $H \in \mathcal{H}$, the number of points of the s-supertail ST that do not belong to H is a multiple of q^{d_s-1}.*

Corollary 2. *For any two hyperplanes H and H' such that $t_i(H) = t_i(H')$ for $i = 2, 3, \ldots, s-1$, it is true that*

$$t_1(H') = t_1(H) + kq^{d_s-d_1} \ ,$$

for some integer k.

3. Small tails

Lemma 2. *Let ST be an s-supertail of a subspace partition $\mathcal{P}$ of V, and let P_{ST} be the set of all points in ST. Then we have the following possibilities for the cardinality of P_{ST}.*
(i) $|P_{ST}| \geq 2(q^{d_s-1} + q^{d_s-2})$,
(ii) $d_{s-1} = 1$ and $|P_{ST}| = 2q^{d_s-1}$,[1]
(iii) $|P_{ST}| = 2q^{d_s-1} + (q^{d_s-1} - 1)/(q-1)$, or
(iv) $|P_{ST}| = (q^{d_s} - 1)/(q-1)$ and P_{ST} constitutes a d_s-dimensional subspace.

Proof. Assume that $|P_{ST}| < 2(q^{d_s-1} + q^{d_s-2})$ and that either $d_{s-1} \neq 1$ or $|P_{ST}| \neq 2q^{d_s-1}$. Let H be any hyperplane. From Corollary 1 and since $|P_{ST}| < 2(q^{d_s-1} + q^{d_s-2})$, it follows that the number of points of ST that are not contained in H is equal to $\delta \cdot q^{d_s-1}$, where $\delta \in \{0, 1, 2\}$.

Suppose that there is a hyperplane H containing all but $2q^{d_s-1}$ points of P_{ST}. Derive a partition $\mathcal{Q}$ from $\mathcal{P}$ by splitting up all the spaces in the supertail into points. Then $\mathcal{Q}$ induces a partition $\mathcal{Q}_H$ of H: Intersect the spaces in $\mathcal{Q}$ with H and keep those intersections that are not the zero space. Since ST either contains a subspace of dimension at least 2 or $|P_{ST}| \neq 2q^{d_s-1}$, it follows that the tail T of $\mathcal{Q}_H$ contains at least one point. We now apply Theorem 3 to the partition $\mathcal{Q}_H$. Then, using the assumptions above, straightforward verifications show that the only possibility for the size of the tail T of $\mathcal{Q}_H$ is $|T| = (q^{d_s-1} - 1)/(q-1)$. This shows that if we assume that neither (i) nor (ii) occurs, then either *(iii)* holds or the following property holds: *"every hyperplane contains*

[1]It can be shown that this case occurs only if $q = 2$. For example, $V(3, 2)$ has a subspace partition consisting of one 2-dimensional space and four 1-dimensional spaces.

all points in the supertail or all but q^{d_s-1} points". We now analyze the latter case.

Let d denote the dimension of the subspace of V spanned by the points of ST. From Proposition 1, we obtain that the number of hyperplanes that do not contain the complete supertail and thus avoid exactly q^{d_s-1} supertail points, is equal to

$$\frac{q^n-1}{q-1} - \frac{q^{n-d}-1}{q-1} = q^{n-d}\frac{q^d-1}{q-1}\ . \tag{5}$$

We now count the number of incidences (H, P) where H is a hyperplane not containing the point P in the supertail. As the number of hyperplanes that do *not* contain a given point P is equal to q^{n-1} by Proposition 1, we then obtain from Equation (5) that

$$|P_{ST}| \cdot q^{n-1} = q^{n-d}\frac{q^d-1}{q-1}q^{d_s-1}\ . \tag{6}$$

One evident solution to this equation in d and the size of P_{ST} is

$$d = d_s \qquad \text{and} \qquad |P_{ST}| = \frac{q^{d_s}-1}{q-1}\ .$$

If this case occurs, then, by Proposition 2, the points of the supertail constitute a subspace of dimension d_s. It remains to show that there are no other solutions.

If $d > d_s$, then from Equation (6) follows that q has to divide $(q^d - 1)/(q-1)$, which is impossible.

If $d < d_s$, then the subspace U spanned by the points of the tail has at most dimension $d_s - 1$. Such a subspace contains less than q^{d_s-1} points, implying by Corollary 1 that all hyperplanes contain all points of the supertail, which is impossible, as the supertail is not empty. □

One consequence of Lemma 2 is that if the supertail is "small" then the points in the supertail constitute a subspace of V. This fact will be used in the proof of our main theorem. In general, collecting points together to get subspaces of higher dimensions is a useful technique in the study of subspace partitions. We thus think that the following remark is interesting.

Remark 1. Let ST be an s-supertail. We claim that if $|P_{ST}| < 3q^{d_s-1} + (q^{d_s-1}-1)/(q-1)$ and $|P_{ST}| \neq \delta(q^{d_s}-1)/(q-1)$ for $\delta \in \{1, 2\}$, then no subset of P_{ST} forms a d_s-dimensional space. Indeed, suppose some of the points of ST form a d_s-dimensional space W. Derive a partition $\mathcal{P}'$ from $\mathcal{P}$ by replacing all the points of ST that form a d_s-dimensional space with the space W, and split all the remaining spaces in the supertail into points. Then $\mathcal{P}'$ has subspaces of dimension 1 and of dimension greater than or equal to d_s. Let S' be the tail of $\mathcal{P}'$. If $|P_{ST}| < 3q^{d_s-1} + (q^{d_s-1}-1)/(q-1)$, then

$$|S'| < 2q^{d_s-1},$$

and since $|P_{ST}| \neq 2(q^{d_s}-1)/(q-1)$, the size of S' contradicts Theorem 3 which states that $|S'| \geq$ [2] $2q^{d_s-1}$ or $|S'| = (q^{d_s}-1)/(q-1)$.

4. Proof of Theorem 2

Trivial verifications show

Proposition 3. *The bounds provided by Theorem 3 are all at least* $\sigma_q(d_2, d_1)$.

An immediate consequence of Lemma 2 together with Proposition 3 is the following proposition:

Proposition 4. *If the number of points in the s-supertail ST is less than* $2 \cdot q^{d_s-1} + (q^{d_s-1}-1)/(q-1)$, *then*

$$|ST| \geq \sigma_q(d_s, d_{s-1}) \ .$$

By simply counting the number of points in the members of a supertail, we obtain

Proposition 5. *If* $d_s \geq 2d_{s-1}$ *and the number of points in the s-supertail ST is at least equal to* $2 \cdot q^{d_s-1} + (q^{d_s-1}-1)/(q-1)$, *then*

$$|ST| > \sigma_q(d_s, d_{s-1}).$$

Proof. Let k and r be integers with $d_s = kd_{s-1} + r$ such that $0 \leq r < d_{s-1}$. Suppose that $|P_{ST}| \geq 2 \cdot q^{d_s-1} + (q^{d_s-1}-1)/(q-1)$.

Since the largest subspace in ST has dimension d_{s-1}, we can, by using the packing condition, estimate the size of the supertail as follows:

$$|ST| \geq \frac{2 \cdot q^{d_s-1} + (q^{d_s-1}-1)/(q-1)}{(q^{d_{s-1}}-1)/(q-1)} = \frac{2 \cdot q^{d_s} - q^{d_s-1} - 1}{q^{d_{s-1}}-1} \ . \tag{7}$$

If $r = 0$, then $\sigma_q(d_s, d_{s-1}) = (q^{d_s}-1)/(q^{d_{s-1}}-1)$ and so from the inequality above follows that $|ST| > \sigma_q(d_s, d_{s-1})$.

If $r \geq 1$, then from Theorem 1 we deduce that

$$\sigma_q(d_s, d_{s-1}) = \frac{q^{d_s} - q^{d_{s-1}+r}}{q^{d_{s-1}}-1} + q^{\lceil \frac{d_{s-1}+r}{2} \rceil} + 1 \ . \tag{8}$$

Hence, from Equation (7) it follows that $|ST| > \sigma_q(d_s, d_{s-1})$ if

$$2q^{d_s} - q^{d_s-1} - 1 > q^{d_s} - q^{d_{s-1}+r} + (q^{\lceil \frac{d_{s-1}+r}{2} \rceil} + 1)(q^{d_{s-1}} - 1)$$

or equivalently,

$$q^{d_s-1}(q-1) + q^{d_{s-1}+r} > q^{\lceil \frac{d_{s-1}+r}{2} \rceil}(q^{d_{s-1}}-1) + q^{d_{s-1}}.$$

Since $d_s \geq 2d_{s-1}+1$, $r \geq 1$, and $q \geq 2$, the inequality above holds, and the proof of the proposition is complete. □

It remains to prove

[2] Equality can occur only if $q = 2$.

Proposition 6. *If* $d_{s-1} < d_s < 2d_{s-1}$ *then for the* s*-supertail* ST *we have that* $|ST| \geq q^{d_{s-1}} + 1$*, that is,*

$$|ST| \geq \sigma_q(d_s, d_{s-1}).$$

Proof. The case $s = 2$ is covered by Theorem 3, so we may assume that $s \geq 3$. Let ℓ_0 denote the smallest non negative integer ℓ that satisfies the equation

$$\ell + kq^{d_s - d_1} = \sum_{i=1}^{s-1} n_i q^{d_i - d_1}$$

for some integer k. Let k_0 denote the corresponding value of k. It is clear that $0 \leq \ell_0 < q^{d_s - d_1}$, and that

$$(n_1 - \ell_0)q^{d_1 - 1} + \sum_{i=2}^{s-1} n_i q^{d_i - 1} = k_0 q^{d_s - 1} \tag{9}$$

holds.

We consider two cases.

Case 1: $\ell_0 \geq q^{d_{s-1} - d_1}$. For $j = 2, 3, \ldots, s-1$ and $0 \leq i_j \leq n_j$, let $b_{(i_2, i_3, \ldots, i_{s-1})}$ denote the number of hyperplanes that contain exactly i_j spaces of dimension d_j from the supertail. For any non negative integer n, let $[n]$ denote the set $\{0, 1, 2, \ldots, n\}$ and write N for $[n_2] \times \ldots \times [n_{s-2}] \times [n_{s-1}]$, write N_0 for $[n_2] \times \ldots \times [n_{s-2}] \times [0]$ and $N_{>0}$ for $[n_2] \times \ldots \times [n_{s-2}] \times \{1, 2, \ldots, n_{s-1}\}$.

Define the integer L_{s-1} by

$$L_{s-1} = \sum_{i \in N} i_{s-1} \cdot b_i = \sum_{i \in N_{>0}} i_{s-1} \cdot b_i \ ,$$

the integers L_j, for $j = 2, 3 \ldots, s-2$, by

$$L_j = \sum_{i \in N_0} i_j \cdot b_i \ ,$$

and the integer L_1 by

$$L_1 = \sum_{i \in N_0} b_i \left(q^{d_{s-1} - d_1} - \sum_{j=2}^{s-2} i_j \cdot q^{d_j - d_1} \right) \ .$$

The above equality can be expressed as

$$\sum_{j=1}^{s-2} q^{d_j} L_j = q^{d_{s-1}} \sum_{i \in N_0} b_i \ . \tag{10}$$

We also have that

$$\sum_{i \in N} b_i = |\mathcal{H}| = \frac{q^n - 1}{q - 1} \ . \tag{11}$$

If we determine the number of incidences (H, W) where H is a hyperplane and W a subspace of dimension d_{s-1} in the supertail, we then get, from Proposition 1 and the definition of L_{s-1}, that

$$L_{s-1} = n_{s-1}\frac{q^{n-d_{s-1}}-1}{q-1} . \tag{12}$$

Doing the same with $\dim W = d_j$ instead yields for $j = 2, 3, \ldots, s-2$

$$L_j \leq n_j\frac{q^{n-d_j}-1}{q-1} . \tag{13}$$

We observe that by Equation (9) the equation below for t

$$(n_1 - \ell_0 + t)q^{d_1-1} + \sum_{j=2}^{s-2}(n_j - i_j)q^{d_j-1} + n_{s-1}q^{d_{s-1}-1} = k_0 q^{d_s-1}$$

has the unique solution

$$t = t_0 = \sum_{j=2}^{s-2} i_j \cdot q^{d_j-d_1} .$$

Let $\mathcal{H}_0$ be the set of all hyperplanes that contain no space of dimension d_{s-1} from the supertail and exactly i_j spaces of dimension d_j, for $j = 2, 3, \ldots, s-2$, from it. As $\ell_0 - t_0 < q^{d_s-d_1}$, it follows from Corollary 2 that every hyperplane in $\mathcal{H}_0$ contains at least $\ell_0 - t_0$ spaces of dimension d_1 from the supertail. Since $\ell_0 \geq q^{d_{s-1}-d_1}$ in the case under consideration, it follows that every hyperplane in $\mathcal{H}_0$ contains at least

$$\ell_0 - t_0 \geq q^{d_{s-1}-d_1} - \sum_{j=2}^{s-2} i_j \cdot q^{d_j-d_1}$$

spaces of dimension d_1 from the supertail. Hence, by counting the number of incidences (H, W), where H is a hyperplane and W is a space of dimension d_1 in the supertail, we get

$$L_1 \leq n_1\frac{q^{n-d_1}-1}{q-1} . \tag{14}$$

Now, from Equations (10) and (11), we get that

$$q^{d_{s-1}} \cdot |\mathcal{H}| = \sum_{j=1}^{s-2} q^{d_j} L_j + q^{d_{s-1}} \sum_{i \in N_{>0}} b_i ,$$

and thus from the definition of the integer L_{s-1} that

$$q^{d_{s-1}} \cdot |\mathcal{H}| \leq \sum_{j=1}^{s-2} q^{d_j} L_j + q^{d_{s-1}} L_{s-1} . \tag{15}$$

Trivially, for any positive integer d,

$$q^d\frac{q^{n-d}-1}{q-1} < \frac{q^n-1}{q-1} , \tag{16}$$

holds. Thus, combining the Equations (11), (13), (14) and (15) gives

$$q^{d_{s-1}}\frac{q^n-1}{q-1} < (\sum_{j=1}^{s-1} n_j)\frac{q^n-1}{q-1} .$$

An immediate consequence of the inequality above is that $n_1 + n_2 + \ldots + n_{s-1} > q^{d_{s-1}}$, that is, that $|ST| \geq q^{d_{s-1}} + 1$.

Case 2: $0 \leq \ell_0 < q^{d_{s-1}-d_1}$. Let U' be one of the spaces of dimension d_{s-1} in the supertail and let $\mathcal{H}'$ denote the set of all hyperplanes that contain U'. Let $n' = n - d_{s-1}$. Let $b'_{(i_2,i_3,\ldots i_{s-1})}$ denote that number of hyperplanes in $\mathcal{H}'$ that beside U' contain exactly i_j, for $j = 2, 3, \ldots, s-1$, of the spaces of dimension d_j in the supertail. Let $n'_{s-1} = n_{s-1} - 1$ and let $n'_i = n_i$ for $i = 1, 2, \ldots, s-2$. Define N', N'_0 and $N'_{>0}$ by $N' = [n'_1] \times \ldots \times [n'_{s-2}] \times [n'_{s-1}]$, $N'_0 = [n'_1] \times \ldots \times [n'_{s-2}] \times [0]$ and $N' = [n'_1] \times \ldots \times [n'_{s-2}] \times \{1, 2, \ldots n'_{s-1}\}$.

Let ℓ'_0 denote the smallest non-negative integer value of ℓ satisfying the equation

$$(n'_1 - \ell)q^{d_1-1} + n'_2 q^{d_2-1} + \ldots + n'_{s-1} q^{d_{s-1}-1} = kq^{d_s-1} . \tag{17}$$

for some integer k, so $\ell'_0 < q^{d_s-d_1}$.

With ℓ_0 and k_0 defined as above, we have that

$$(n'_1-\ell_0+q^{d_{s-1}-d_1}-q^{d_s-d_1})q^{d_1-1}+n'_2q^{d_2-1}+\ldots+n'_{s-1}q^{d_{s-1}-1} = (k_0-1)q^{d_s-1}$$

In the case under consideration, it is true that

$$0 \leq \ell_0 - q^{d_{s-1}-d_1} + q^{d_s-d_1} < q^{d_s-d_1} ,$$

and hence we may deduce that

$$\ell'_0 = \ell_0 - q^{d_{s-1}-d_1} + q^{d_s-d_1}$$

and

$$k'_0 = k_0 - 1$$

for the solution k'_0 of Equation (17) corresponding to ℓ'_0. Finally, since $\ell_0 \geq 0$, we obtain

$$\ell'_0 \geq q^{d_s-d_1} - q^{d_{s-1}-d_1} \geq q^{d_{s-1}-d_1} .$$

We can now proceed as in Case 1, but considering only hyperplanes in $\mathcal{H}'$. First, note that

$$\sum_{i\in N'} b'_i = |\mathcal{H}'| = \frac{q^{n'}-1}{q-1}, \tag{18}$$

and that the number of hyperplanes containing both U' and some other subspace W in the supertail is

$$\frac{q^{n'-\dim W}-1}{q-1}.$$

Observe that by Equation (17) for all $i = (i_2, \ldots, i_{s-1}) \in N'$

$$(n'_1 - \ell'_0 + t')q^{d_1-1} + \sum_{j=2}^{s-2}(n'_j - i_j)q^{d_j-1} + n'_{s-1}q^{d_{s-1}-1} = k'_0 q^{d_s-1}$$

has the unique solution

$$t' = t'_0 = \sum_{j=2}^{s-2} i_j q^{d_j-d_1}.$$

As in Case 1, each hyperplane in $\mathcal{H}'$ containing no d_{s-1}-dimensional supertail space except U' and i_j supertail spaces of dimension d_j for all $j \in \{2, 3, \ldots, s-2\}$ contains at least

$$\ell'_0 - t'_0 \geq q^{d_{s-1}-d_1} - \sum_{j=2}^{s-2} i_j q^{d_j-d_1}$$

supertail spaces of dimension d_1. Thus, double counting incidences (H, U) with $H \in \mathcal{H}'$, $U \in ST$ and $\dim U = d_1$ yields

$$\text{(19)} \qquad L'_1 = \sum_{i \in N'_0} b'_i(q^{d_{s-1}-d_1} - \sum_{j=2}^{s-2} i_j q^{d_j-d_1}) \leq n_1 \frac{q^{n'-d_1} - 1}{q-1}.$$

Double counting incidences (H, U) with $H \in \mathcal{H}'$ and U a d_{s-1}-dimensional supertail space distinct from U' yields

$$\text{(20)} \qquad L'_{s-1} = \sum_{i \in N'} i_{s-1} b'_i = n'_{s-1} \frac{q^{n'-d_{s-1}} - 1}{q-1},$$

and considering incidences (H, U) with $H \in \mathcal{H}'$ and U a d_j-dimensional supertail space gives

$$\text{(21)} \qquad L'_j = \sum_{i \in N'_0} i_j b'_i \leq n'_j \frac{q^{n'-d_j} - 1}{q-1}.$$

With these definitions, again the definition of L'_1 can be rewritten to

$$\text{(22)} \qquad q^{d_{s-1}} \sum_{i \in N'_0} b'_i = \sum_{j=1}^{s-2} q^{d_j} L'_j.$$

Together, we obtain the following chain of inequalities:

$$\begin{aligned} q^{d_{s-1}} \cdot \frac{q^{n'}-1}{q-1} &\overset{(18),(22)}{=} \sum_{j=1}^{s-2} q^{d_j} L'_j + q^{d_{s-1}} \sum_{i \in N'_{>0}} b'_i \\ &\overset{(20)}{\leq} \sum_{j=1}^{s-1} q^{d_j} L'_j \\ &\overset{(19),(20),(21)}{\leq} \sum_{j=1}^{s-1} n'_j q^{d_j} \frac{q^{n'-d_j}-1}{q-1} \\ &< \sum_{j=1}^{s-1} n'_j \frac{q^{n'}-1}{q-1}, \end{aligned}$$

where the last inequality follows trivially as in Case 1. Hence, we have

$$q^{d_{s-1}} < \sum_{j=1}^{s-1} n'_j = \sum_{j=1}^{s-1} n_j - 1,$$

meaning that in this case, the size of the supertail is greater than $q^{d_{s-1}} + 1$. □

5. Examples and Conluding Remarks

We first show that the bound in Theorem 2 cannot be improved in general. We use a general construction of a special class of subspace partitions that originates from a paper by André [1]:

Let F be any field and let K be any subfield of F. Choose field elements α_i, for $i \in I$, such that they constitute a set of coset representatives of the multiplicative group of K in the multiplicative group of F. If we consider F as a vector space over its prime field, then the sets $\alpha_i K$, for $i \in I$, constitutes a subspace partition of F.

Example 1. Consider the finite field $F = \mathrm{GF}(64)$ and the subfield K of F with 8 elements. Let $\mathcal{P}$ denote the subspace partition obtained by the recipe given above. The subspaces in the subspace partition all have dimension 3; let U be one of the members of $\mathcal{P}$.

By substituting U in $\mathcal{P}$ with one hyperplane H in U and the four points in U that are not contained in H, we get a derived partition $\mathcal{P}'$ from $\mathcal{P}$ with $n_3 = 8$ subspaces of dimension $d_3 = 3$, $n_2 = 1$ subspace of dimension $d_2 = 2$ and $n_1 = 4$ subspaces of dimension $d_1 = 1$. With $s = 3$ we then get an s-supertail ST consisting of the five subspaces of dimension 2 or 1, respectively. The size of this supertail ST attains the bound given in Theorem 2.

Similarly, as in the example above, we can for every prime power q produce an infinite sequence of subspace partitions of a vector space

$V(n, q)$ with supertails reaching the bound of Theorem 2, since the supertail is obtained by splitting up one of the d_s-dimensional spaces into smaller ones.

In the next example, we give an application of Lemma 2.

Example 2. We show the nonexistence of a subspace partition $\mathcal{P}$ of $V(7, 2)$ consisting of 17 subspaces of dimension 3, one subspace of dimension 2, and five subspaces of dimension 1.

Let ST denote the supertail formed by the subspaces of $\mathcal{P}$ of dimensions 2 and 1. Then the number of points of ST is $1 \cdot 3 + 5 = 8 < (2 \cdot 2^{3-1} + (2^{3-1} - 1) - 1)$. So by Lemma 2, the points of ST must form a 3-dimensional subspace, which is clearly impossible since such a subspace would contain exactly 7 points. So there is no subspace partition of this type.

The nonexistence of the subspace partition in the example above can also be verified by the results given in [8], where some new necessary conditions for the existence of subspace partitions were derived by exploiting methods similar to those used in the proof of Proposition 6.

The next example shows that the supertail conditions in Theorem 2 together with the tail conditions in Theorem 3, are not sufficient for the existence of a subspace partition.

Example 3. From the results of [8], see also [4], it follows that there is no subspace partition of the type $[2^6 3^6 4^{13}]$ in $V(8, 2)$ (although this type of subspace partition satisfies both the packing condition, the dimension condition, and the tail conditions). As a supertail with $s = 3$ in this type of partition consists of 12 spaces, and $\sigma_2(4, 3) = 9$, the existence of this type of subspace partition can not be excluded by our Theorem 2.

We also ran a computer search for a type of subspace partition, the existence of which can be excluded by the supertail conditions in Theorem 2, but not by the necessary conditions in [8]. However, we have not yet found any such type of subspace partition.

Finally, we have the following corollary of Proposition 5 and Lemma 2.

Corollary 3. *Let ST be an s-supertail of the subspace partition $\mathcal{P}$ of V, and let P_{ST} be the set of all points in ST. If $|ST| = \sigma(d_s, d_{s-1})$ and $d_s \geq 2d_{s-1}$, then P_{ST} constitutes a d_s-dimensional subspace.*

Proof. If $|P_{ST}| \geq 2q^{d_s-1} + (q^{d_s-1} - 1)/(q - 1)$, then it follows from Proposition 5 that $|ST| > \sigma(d_s, d_{s-1})$. Thus, if $|ST| = \sigma(d_s, d_{s-1})$, then we must have $|P_{ST}| < 2q^{d_s-1} + (q^{d_s-1} - 1)/(q - 1)$. Consequently, Lemma 2 implies that $|P_{ST}| = (q^{d_s} - 1)/(q - 1)$ and P_{ST} constitutes a d_s-dimensional subspace. □

We believe that Corollary 3 also holds for $d_{s-1} < d_s < 2d_{s-1}$. In particular, we conjecture that the bound in Theorem 2 holds with equality if and only if ST is a subspace.

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O. Heden (olohed@math.kth.se), Department of Mathematics, KTH, S-100 44 Stockholm, Sweden.

J. Lehmann (jlehmann@math.uni-bremen.de), Department of Mathematics, Bremen University, Bibliothekstrasse 1 - MZH, 28359 Bremen, Germany.

E. Năstase (nastasee@xavier.edu): Department of Mathematics and Computer Science, Xavier University, 3800 Victory Parkway, Cincinnati, Ohio 45207.

P. Sissokho (psissok@ilstu.edu): Mathematics Department, Illinois State University, Normal, Illinois 61790.