PARTITIONS OF THE 8-DIMENSIONAL VECTOR SPACE OVER GF(2)

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Abstract. Let $V = V(n, q)$ denote the vector space of dimension n over GF(q). A set of subspaces of V is called a *partition* of V if every nonzero vector in V is contained in exactly one subspace of V. Given a partition $\mathcal{P}$ of V with exactly a_i subspaces of dimension i for $1 \le i \le n$, we have $\sum_{i=1}^{n} a_i(q^i - 1) = q^n - 1$, and we call the n-tuple $(a_n, a_{n-1}, \ldots, a_1)$ the *type* of $\mathcal{P}$. In this paper we identify all 8-tuples $(a_8, a_7, \ldots, a_2, 0)$ that are the types of partitions of $V(8, 2)$.

1. Introduction

Let n be a positive integer, q a prime power, and $V = V(n, q)$ denote a vector space of dimension n over the finite field GF(q). A *partition* of V is a collection of subspaces $U_1, U_2, \ldots, U_t$ such that

$$V = \bigcup_{i=1}^{t} U_i \qquad \text{and} \qquad U_i \cap U_j = \{0\} \quad \text{for} \quad i \neq j.$$

Given a partition $\mathcal{P}$ of V with exactly a_i subspaces of dimension i for $1 \le i \le n$, the following condition holds

$$\sum_{i=1}^{n} a_i(q^i - 1) = q^n - 1, \tag{1}$$

and we call the n-tuple $(a_n, a_{n-1}, \ldots, a_1)$ the *type* of $\mathcal{P}$. More compactly, we call $\mathcal{P}$ an $n^{a_n}(n-1)^{a_{n-1}} \cdots 1^{a_1}$-*partition*, and we often omit factors in this formal product with exponent 0. For example, we

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could call a partition of $V(8,2)$ into 17 subspaces of dimension 4 a 4^{17}-partition.

A second necessary condition for an n-tuple to be a partition type comes from dimension considerations.

$$\text{If } a_i \geq 2, \text{ then } i \leq n/2, \text{ and} \\ \text{if } i \neq j \text{ and } a_i a_j > 0, \text{ then } i + j \leq n. \tag{2}$$

A third condition involves the number of subspaces of least dimension in the partition $\mathcal{P}$ of $V(n,q)$. (See Theorem 3.2 of [1], and for a stronger condition, see [11].)

$$\text{If } i \text{ is the smallest integer such that } a_i > 0, \text{ and } i < n, \\ \text{then } a_i \geq qi + 1. \tag{3}$$

In general, necessary and sufficient conditions for the existence of an $n^{a_n} \cdots 1^{a_1}$-partition of $V(n,q)$ are not known. Of course, any collection of pairwise trivially intersecting subspaces can be expanded to a partition by including subspaces of dimension 1.

There are a few results that characterize vector space partitions of $V(n,q)$. O. Heden characterized the $(n-3)^1 3^a 2^b$-partitions of $V(n,2)$ for all $n \geq 9$ (see [9]). More recently, he gave necessary and sufficient conditions for the existence of certain vector space partitions of $V(n,q)$ for all $n \geq 9$ (see [10]). S. El-Zanati et al. characterized the $n^{a_n} \cdots 1^{a_1}$-partitions of $V(n,q)$ for $n \leq 7$ and $q = 2$ (see [5]), and the $3^a 2^b$-partitions of $V(n,2)$ for all $n \geq 2$ (see [6]).

Vector space partitions have applications in design theory (in particular, uniformly resolvable designs [1]), coding theory (see [3, 12, 13, 14]), and orthogonal arrays (see [4, 7]). The study of vector space partitions of $V(n,q)$ for small n and q is important in providing a rich set of examples and in supporting more general results. For instance, the partitions in this paper establish the tightness of bounds, obtained by Heden [11] (see Remark 20). Moreover, the constructions of vector space partitions of $V(n,q)$ for small n provide base cases for recursive constructions with larger values of n (see [6, 9]). This is similar to the situation in t-(v,k,λ) designs where the designs with small parameters provide the building blocks for recursive constructions (e.g., [16]).

In this paper, we prove the following theorem.

Theorem 1. *With one exception, an* $8^{a_8} 7^{a_7} \cdots 2^{a_2} 1^0$*-partition of* $V(8,2)$ *exists if and only if the feasibility conditions* (1), (2), *and* (3) *are satisfied. The exception is that there is no* $4^{13} 3^6 2^6$*-partition of* $V(8,2)$.

This paper is organized as follows. In Section 2.1, we characterize the $8^{a_8} 7^{a_7} \cdots 2^{a_2} 1^0$-partitions of $V(8,2)$ with $a_5 \geq 1$. In Section 2.2

and Section 3, we construct all feasible $4^a3^b2^c$-partitions of $V(8,2)$ with $(a,b,c) \neq (13,6,6)$. We then show in Section 2.3 that there is no $4^{13}3^62^6$-partition of $V(8,2)$. Finally, we discuss in Section 4 the computational aspects of our constructions.

2. Proof of Theorem 1

The proof of Theorem 1 has three main parts, which we consider in the following sections.

2.1. Partitions with a subspace of dimension at least 5.

We begin with two known results.

Lemma 2 ([2], Lemma 4)**.** *Let n and d be integers such that $1 \leq d \leq n/2$. Then $V(n,q)$ can be partitioned into one subspace of dimension $n-d$ and q^{n-d} subspaces of dimension d.*

Lemma 3 ([6], Theorem 6.1)**.** *Suppose that a, b, and c are nonnegative integers that satisfy $31a+7b+3c = 2^8-1$. Then $V(8,2)$ has a $5^a3^b2^c$-partition unless such a partition is ruled out by conditions* (2) *or* (3)*.*

The following lemma constitutes the first part of the proof of Theorem 1.

Lemma 4. *Let $a_2,\dots,a_8$ be nonnegative integers such that $a_5+a_6+a_7+a_8 \geq 1$. Then there exists an $8^{a_8}7^{a_7}\cdots 2^{a_2}1^0$-partition of $V(8,2)$ if and only if conditions* (1)*,* (2)*, and* (3) *hold.*

Proof. From condition (2), it follows that $a_5+a_6+a_7+a_8 = 1$. There is the trivial 8^1-partition, but $a_7 = 1$ is ruled out by conditions (2) and (3). If $a_6 = 1$, then only a 6^12^{64}-partition is possible by the three conditions, and this exists by Lemma 2. Finally, if $a_5 = 1$, then $a_4 = 0$ by condition (2) and $31a_5 + 7a_3 + 3a_2 = 2^8-1$ by condition (1). Partitions of all such types exist by Lemma 3. □

2.2. Partitions with subspaces of dimension at most 4.

The second piece in the proof of Theorem 1 deals with the $4^a3^b2^c$-partitions of $V = V(8,2)$. To explain their constructions, we start with the following setup. Let $F = \mathrm{GF}(256)$ be the field with 256 elements generated by the irreducible polynomial $x^8+x^4+x^3+x^2+1$. Take α to be x, and let $G = \{\alpha^i : 0 \leq i \leq 254\}$ be the multiplicative group of nonzero elements of F. Consider the multiplicative subgroup $H = \{1, \alpha^{85}, \alpha^{2\cdot 85}\}$ of G. Then $H \cup \{0\} = \mathrm{GF}(4)$ is a 2-dimensional subspace of V. It is also easy to see that the cosets of H, namely $H, \alpha H, \dots, \alpha^{84}H$, form (with 0 added to each) a 2^{85}-partition of V. For any integer r not divisible by 255, define r' as that integer j, $0 \leq$

$j \leq 254$, such that $1 + \alpha^r = \alpha^j$. (Note that $1 + \alpha^0 = 0$, which is not a power of α, or even in G.) Then for any integers i and j not congruent modulo 255, $\alpha^i + \alpha^j = \alpha^i(1 + \alpha^{j-i}) = \alpha^i \alpha^{(j-i)'} = \alpha^{i+(j-i)'}$. For instance $1' = 25$ and $2' = 50$. Moreover, since $1 + \alpha^r = \alpha^{r'}$ yields $1 + \alpha^{r'} = \alpha^r$, we have $(r')' = r$. Table 1 shows the pairs (r, r'), $1 \leq r \leq r' \leq 254$; and it can be easily generated using mathematical software (e.g., Maple).

(1, 25)	(2, 50)	(3, 223)	(4, 100)	(5, 138)	(6, 191)	(7, 112)
(8, 200)	(9, 120)	(10, 21)	(11, 245)	(12, 127)	(13, 99)	(14, 224)
(15, 33)	(16, 145)	(17, 68)	(18, 240)	(19, 92)	(20, 42)	(21, 10)
(22, 235)	(23, 196)	(24, 254)	(25, 1)	(26, 198)	(27, 104)	(28, 193)
(29, 181)	(30, 66)	(31, 45)	(32, 35)	(33, 15)	(34, 136)	(35, 32)
(36, 225)	(37, 179)	(38, 184)	(39, 106)	(40, 84)	(41, 157)	(42, 20)
(43, 121)	(44, 215)	(45, 31)	(46, 137)	(47, 101)	(48, 253)	(49, 197)
(50, 2)	(51, 238)	(52, 141)	(53, 147)	(54, 208)	(55, 63)	(56, 131)
(57, 83)	(58, 107)	(59, 82)	(60, 132)	(61, 186)	(62, 90)	(63, 55)
(64, 70)	(65, 162)	(66, 30)	(67, 216)	(68, 17)	(69, 130)	(70, 64)
(71, 109)	(72, 195)	(73, 236)	(74, 103)	(75, 199)	(76, 113)	(77, 228)
(78, 212)	(79, 174)	(80, 168)	(81, 160)	(82, 59)	(83, 57)	(84, 40)
(85, 170)	(86, 242)	(87, 167)	(88, 175)	(89, 203)	(90, 62)	(91, 209)
(92, 19)	(93, 158)	(94, 202)	(95, 176)	(96, 251)	(97, 190)	(98, 139)
(99, 13)	(100, 4)	(101, 47)	(102, 221)	(103, 74)	(104, 27)	(105, 248)
(106, 39)	(107, 58)	(108, 161)	(109, 71)	(110, 126)	(111, 246)	(112, 7)
(113, 76)	(114, 166)	(115, 243)	(116, 214)	(117, 122)	(118, 164)	(119, 153)
(120, 9)	(121, 43)	(122, 117)	(123, 183)	(124, 180)	(125, 194)	(126, 110)
(127, 12)	(128, 140)	(129, 239)	(130, 69)	(131, 56)	(132, 60)	(133, 250)
(134, 177)	(135, 144)	(136, 34)	(137, 46)	(138, 5)	(139, 98)	(140, 128)
(141, 52)	(142, 218)	(143, 150)	(144, 135)	(145, 16)	(146, 217)	(147, 53)
(148, 206)	(149, 188)	(150, 143)	(151, 178)	(152, 226)	(153, 119)	(154, 201)
(155, 159)	(156, 169)	(157, 41)	(158, 93)	(159, 155)	(160, 81)	(161, 108)
(162, 65)	(163, 182)	(164, 118)	(165, 227)	(166, 114)	(167, 87)	(168, 80)
(169, 156)	(170, 85)	(171, 211)	(172, 229)	(173, 232)	(174, 79)	(175, 88)
(176, 95)	(177, 134)	(178, 151)	(179, 37)	(180, 124)	(181, 29)	(182, 163)
(183, 123)	(184, 38)	(185, 249)	(186, 61)	(187, 204)	(188, 149)	(189, 219)
(190, 97)	(191, 6)	(192, 247)	(193, 28)	(194, 125)	(195, 72)	(196, 23)
(197, 49)	(198, 26)	(199, 75)	(200, 8)	(201, 154)	(202, 94)	(203, 89)
(204, 187)	(205, 207)	(206, 148)	(207, 205)	(208, 54)	(209, 91)	(210, 241)
(211, 171)	(212, 78)	(213, 233)	(214, 116)	(215, 44)	(216, 67)	(217, 146)
(218, 142)	(219, 189)	(220, 252)	(221, 102)	(222, 237)	(223, 3)	(224, 14)
(225, 36)	(226, 152)	(227, 165)	(228, 77)	(229, 172)	(230, 231)	(231, 230)
(232, 173)	(233, 213)	(234, 244)	(235, 22)	(236, 73)	(237, 222)	(238, 51)
(239, 129)	(240, 18)	(241, 210)	(242, 86)	(243, 115)	(244, 234)	(245, 11)
(246, 111)	(247, 192)	(248, 105)	(249, 185)	(250, 133)	(251, 96)	(252, 220)
(253, 48)	(254, 24)					

TABLE 1. All pairs (r, r') with $1 + \alpha^r = \alpha^{r'}$ in GF(256).

For any integer i, let S_i be the 3-tuple $(\alpha^i, \alpha^{i+85}, \alpha^{i+2\cdot 85})$, where the powers of α are taken modulo 255. For example, $S_0 = (1, \alpha^{85}, \alpha^{170})$,

$S_1 = (\alpha, \alpha^{86}, \alpha^{171})$, and $S_{25} = (\alpha^{25}, \alpha^{110}, \alpha^{195})$. We define the addition of these 3-tuples to be coordinatewise addition, so that, for $1 \le t \le 3$, entry t of $S_i + S_j$ is

$$(4) \qquad \alpha^{i+85(t-1)} + \alpha^{j+85(t-1)} = \alpha^{i+85(t-1)+(j-i)'} = \alpha^{i+(j-i)'+85(t-1)}.$$

Thus $S_i + S_j = S_{i+(j-i)'}$ and this addition is associative. Observe that if $i \equiv j \pmod{85}$, then S_i and S_j contain the same entries (those of $\alpha^i H$), but in a different cyclic order unless $i \equiv j \pmod{255}$. However, if $i \not\equiv j \pmod{85}$, we use the following well-known lemma to guarantee that the subspaces generated by combining entries from the S_i's have intersection $\{0\}$.

Lemma 5. *Let i and j be two integers such that $i \not\equiv j \pmod{85}$. Then the entries in S_i and S_j are pairwise distinct. That is*

$$\alpha^i H \cap \alpha^j H = \{\alpha^i, \alpha^{i+85}, \alpha^{i+2\cdot 85}\} \cap \{\alpha^j, \alpha^{j+85}, \alpha^{j+2\cdot 85}\} = \emptyset.$$

For any i and j such that $i \not\equiv j \pmod{85}$, it follows from Lemma 5 that the entries of S_i and S_j span 2-dimensional subspaces of V with intersection $\{0\}$. In the following, we provide a construction that reconfigures the entries of five S_i's into one 4-dimensional subspace and another one that reconfigures the entries of seven S_i's into three 3-dimensional subspaces.

Construction 1: Let W denote the 4-dimensional subspace of V spanned by the entries of S_i and S_j, and suppose that α^k is a nonzero vector in W that is not an entry of S_i or S_j. Then there exist integers p and q such that α^p is an entry of S_i, α^q is an entry of S_j, and $\alpha^p + \alpha^q = \alpha^k$. Because of equation (4), each of the entries of S_k is a vector in W. Thus the nonzero vectors in W are the entries of five of the 3-tuples S_r. Specifically, the nonzero vectors in W are the entries of S_i, S_j, S_b, S_c, and S_d, where

$$b \equiv i + (j-i)' \pmod{85},$$
$$c \equiv i + (j-i+85)' \pmod{85},$$

and

$$d \equiv i + (j-i+170)' \pmod{85}.$$

In Examples 11–19 in Section 3, each of the sets in $\mathcal{A}$ consists of five of the S_r obtained in this manner. For instance, if $i = 7$ and $j = 29$, then

$$b = 7 + (29-7)' = 7 + 22' = 7 + 235 = 242 \equiv 72 \pmod{85},$$
$$c = 7 + (29-7+85)' = 7 + 107' = 7 + 58 = 65 \pmod{85},$$

and

$$d = 7 + (29 - 7 + 170)' = 7 + 192' = 7 + 247 \equiv 84 \pmod{85}.$$

Thus the set

$$\{0\} \cup \left\{\alpha^{i+85t} : \ i = 7, 29, 65, 72, 84 \text{ and } 0 \le t \le 2\right\}$$

is a 4-dimensional subspace of V.

Construction 2**:** We explain a method for constructing 3 disjoint 3-dimensional subspaces by using the entries of triplets S_{i_j}, $1 \le j \le 7$. Let i_1, i_2, and i_3 be integers such that no two of them are congruent modulo 85, and $i_3 \not\equiv i_1 + (i_2 - i_1)' \pmod{255}$. Let i_4, i_5, i_6, and i_7 be such that:

$$\begin{aligned} i_4 &\equiv i_1 + (i_2 - i_1)' \pmod{255}, \\ i_5 &\equiv i_1 + (i_3 - i_1)' \pmod{255}, \\ i_6 &\equiv i_2 + (i_3 - i_2)' \pmod{255}, \end{aligned}$$

and

$$i_7 \equiv i_3 + (i_4 - i_3)' \pmod{255}.$$

Then the first entries of $S_{i_1}, S_{i_2}, \ldots, S_{i_7}$ are the nonzero vectors in a 3-dimensional subspace of V. Similarly, for any $0 \le t \le 2$,

$$\{0\} \cup \{\alpha^{i_j+85t} : \ 1 \le j \le 7\}$$

is a 3-dimensional subspace of V.

In Examples 11–19 in Section 3, each of the sets in $\mathcal{B}$ consists of 7 triplets S_{i_j} obtained in this manner. For instance, if $i_1 = 0$, $i_2 = 1$, and $i_3 = 2$, then

$$\begin{aligned} i_4 &= 0 + (1 - 0)' = 25, \\ i_5 &= 0 + (2 - 0)' = 50, \\ i_6 &= 1 + (2 - 1)' = 1 + 25 = 26, \end{aligned}$$

and

$$i_7 = 2 + (25 - 2)' = 2 + 196 = 198.$$

So the respective entries of $S_0, S_1, S_2, S_{25}, S_{26}, S_{50}$, and S_{198} are the nonzero vectors in a 3-dimensional subspace of V, and thus

$$\left\{0, \alpha^{85k}, \alpha^{1+85k}, \alpha^{2+85k}, \alpha^{25+85k}, \alpha^{26+85k}, \alpha^{50+85k}, \alpha^{198+85k}\right\}$$

is a 3-dimensional subspace of V for $0 \le k \le 2$.

We now prove the following lemma, which relies on Examples 11–19 in Section 3.

Lemma 6. *Let a, b, and c be nonnegative integers such that $(a, b, c) \neq (13, 6, 6)$. Then there exists a $4^a3^b2^c$-partition of $V = V(8, 2)$ if and only if conditions* (1), (2), *and* (3) *hold.*

Proof. The triplets (a, b, c) for which there exist $4^a3^b2^c$-partitions of V satisfying the hypothesis of Theorem 1 are arranged in Table 2.

(a, b, c)	Range	Exp.
$T(1, i, j) = (17 - i, 0, 5i)$	$0 \leq i \leq 17$	11
$T(2, i, j) = (14 - i, 3 - 3j, 5i + 7j + 8)$	$0 \leq i \leq 14$, $0 \leq j \leq 1$	12
$T(3, i, j) = (12 - i, 6 - 3j, 5i + 7j + 11)$	$0 \leq i \leq 12$, $0 \leq j \leq 2$	13
$T(4, i, j) = (11 - i, 9 - 3j, 5i + 7j + 9)$	$0 \leq i \leq 11$, $0 \leq j \leq 3$	14
$T(5, i, j) = (10 - i, 15 - 3j, 5i + 7j)$	$0 \leq i \leq 10$, $0 \leq j \leq 5$	15
$T(6, i, j) = (7 - i, 18 - 3j, 5i + 7j + 8)$	$0 \leq i \leq 7$, $0 \leq j \leq 6$	16
$T(7, i, j) = (6 - i, 21 - 3j, 5i + 7j + 6)$	$0 \leq i \leq 6$, $0 \leq j \leq 7$	17
$T(8, i, j) = (4 - i, 24 - 3j, 5i + 7j + 9)$	$0 \leq i \leq 4$, $0 \leq j \leq 8$	18
$T(9, i, j) = (3 - i, 30 - 3j, 5i + 7j)$	$0 \leq i \leq 3$, $0 \leq j \leq 10$	19

TABLE 2. The potential $4^a3^b2^c$-partitions of V

All the partitions in Table 2 can be constructed from 9 special partitions: the $4^a3^b2^c$-partitions of V, where $(a, b, c) = T(k, 0, 0)$ and $1 \leq k \leq 9$. These partitions are given by Examples 11–19 in Section 3. □

2.3. **Nonexistence of a $4^{13}3^62^6$-partition of $V(8, 2)$.** The third and last piece in the proof of Theorem 1 is the proof of the following result.

Theorem 7. *There is no $4^{13}3^62^6$-partition of $V = V(8, 2)$.*

We require several lemmas for the proof. The family of all subspaces of V of dimension 7 plays a fundamental role and is denoted by $\mathcal{H}$.

Lemma 8. *Let $V = V(n, q)$.*
(i) The number of subspaces of dimension $n - 1$, as well as the number of subspaces of dimension 1, in V is $(q^n - 1)/(q - 1)$. In particular, $|\mathcal{H}| = 2^8 - 1 = 255$.
(ii) Any subspace U of V of dimension d is contained in exactly $2^{8-d} - 1$ members of $\mathcal{H}$.

Proof. To prove (i), note that in V, each subspace H of dimension $n - 1$ corresponds by duality to a unique subspace of dimension 1.

We now prove (ii). A subspace U of V is contained in H if and only if $H^{\perp} \subseteq U^{\perp}$. Because $\dim(U^{\perp}) = 8 - d$ and $\dim(H^{\perp}) = 1$ if and only if H belongs to $\mathcal{H}$, the result follows from part (i) of the lemma. □

Lemma 9. *To any two subspaces U' and U'' of V of dimensions d' and d'' satisfying $U' \cap U'' = \{0\}$, there are exactly $2^{8-d'-d''} - 1$ members H of $\mathcal{H}$ such that*

$$U' \subseteq H \qquad \text{and} \qquad U'' \subseteq H.$$

Proof. The dimension of U, the linear span of the elements of U' and U'', equals $d = d' + d''$. Because any vector space containing both U' and U'' must contain U, the result follows from Lemma 8 . □

Lemma 10. *Assume that $\mathcal{P}$ is a $4^{13}3^{6}2^{6}$-partition of $V = V(8,2)$ and H is any member of $\mathcal{H}$. If H contains exactly a, b, and c subspaces of dimensions 4, 3, and 2 from $\mathcal{P}$, respectively, then*

$$8a + 4b + 2c = 12.$$

Proof. For $H \in \mathcal{H}$, it is easily seen that $\mathcal{P}_H = \{U \cap H : U \in \mathcal{P}\}$ is a partition of H. Moreover, for any subspace U of V, either

$$U \subseteq H \qquad \text{or} \qquad \dim(U \cap H) = \dim(U) - 1.$$

Because every subspace of V of dimension d contains $2^d - 1$ nonzero elements and every nonzero element of H is contained in a unique subspace in $\mathcal{P}$, we see that $\mathcal{P}_H$ is of type $4^a 3^{b+(13-a)} 2^{c+(6-b)} 1^{6-c}$. Hence

$$127 = |H \setminus \{0\}| = a \cdot 15 + [b + (13-a)] \cdot 7 + [c + (6-b)] \cdot 3 + (6-c) \cdot 1,$$

which reduces to the equation of the lemma. □

Clearly the only possibilities for the triples (a,b,c) in Lemma 10 are

$$(1,1,0), \quad (1,0,2), \quad (0,3,0), \quad (0,2,2), \quad (0,1,4), \quad \text{and} \quad (0,0,6).$$

Let $s(a,b,c)$ denote the number of members of $\mathcal{H}$ that contain exactly a, b, and c subspaces of dimensions 4, 3, and 2 from $\mathcal{P}$, respectively. For example,

$$s(0,1,4) = 0, \tag{5}$$

because if $H \in \mathcal{H}$ contains exactly 0, 1, and 4 subspaces of dimensions 4, 3, and 2 from $\mathcal{P}$, respectively, then $\mathcal{P}_H$ would be of type $3^{14}2^{9}1^{2}$, in contradiction of condition (3). Furthermore,

$$s(0,0,6) \leq 7, \tag{6}$$

for the 18 nonzero vectors in the six 2-dimensional subspaces in $\mathcal{P}$ must span a subspace S with dimension at least 5, so that S is contained in at most 7 members of $\mathcal{H}$ by Lemma 8.

Proof of Theorem 7. Assume that there is a $4^{13}3^62^6$-partition $\mathcal{P}$ of $V(8,2)$. No two subspaces of dimension 4 in $\mathcal{P}$ can be subspaces of the same member of $\mathcal{H}$. Thus if we count the members of $\mathcal{H}$ that contain at least one (or, in fact, exactly one) subspace of dimension 4 in $\mathcal{P}$, we obtain by Lemma 8

$$s(1,1,0) + s(1,0,2) = 13(2^{8-4} - 1) = 195. \tag{7}$$

The number of triples (U_i, U_j, H), where U_i and U_j are distinct subspaces of dimension 3 in $\mathcal{P}$ and H is a member of $\mathcal{H}$ that contains both these subspaces, is

$$\binom{6}{2}(2^{8-3-3} - 1) = 45$$

by Lemma 9. Counting these same triples starting with a member of $\mathcal{H}$, we thus obtain

$$\binom{3}{2}s(0,3,0) + s(0,2,2) = 45,$$

so that

$$s(0,3,0) + s(0,2,2) \leq 45. \tag{8}$$

By combining the results of (7), (8), (5), and (6), we see that the number of elements of $\mathcal{H}$ is

$$(s(1,1,0) + s(1,0,2)) + (s(0,3,0) + s(0,2,2)) + s(0,1,4) + s(0,0,6)$$

$$\leq 195 + 45 + 0 + 7 = 247,$$

which is a contradiction of Lemma 8. $\square$

3. Examples of $4^a3^b2^c$-partitions

Let $V = V(8,2)$. In Examples 11–19 below, the given sets $\mathcal{A}$, $\mathcal{B}$, and C satisfy properties (i)–(v) below (see Section 2.2).

(i) The sets in $\{A : A \in \mathcal{A}\}$, $\{B : B \in \mathcal{B}\}$, and $\{C\}$ form a partition of $\mathbb{Z}_{85}$.

(ii) For each $A \in \mathcal{A}$,

$$W^{(A)} = \{0\} \cup \{\alpha^{i+85k} : i \in A, \ 0 \leq k \leq 2\}$$

is a 4-dimensional subspace of V.

(iii) For each $B \in \mathcal{B}$ and each $0 \leq k \leq 2$,

$$W^{(B,k)} = \{0\} \cup \{\alpha^{i+85k} : i \in B\}$$

is a 3-dimensional subspace of V.

(iv) For each $t \in A \in \mathcal{A}$, $W_t^{(A)} = \{0\} \cup \alpha^t H$ is a 2-dimensional subspace of V. Similarly, for each $t \in B \in \mathcal{B}$, $W_t^{(B)} = \{0\} \cup \alpha^t H$ is a 2-dimensional subspace of V, and for each $t \in C$, $W_t^{(C)} = \{0\} \cup \alpha^t H$ is a 2-dimensional subspace of V.

(v) It follows from properties (i)–(iv) and Lemma 5 that the subspaces generated by $\mathcal{A}$, $\mathcal{B}$, and C (via (ii)–(iv)) in each of Examples 11–19 form a vector space partition of V.

By properties (i)-(v), it follows that

$$\{W^{(A)} : A \in \mathcal{A}\} \cup \{W^{(B,k)} : B \in \mathcal{B}, 0 \le k \le 2\} \cup \{W_j^{(C)} : j \in C\}$$

is a $4^a 3^b 2^c$-partition of V, where $a = |\mathcal{A}|$, $b = 3|\mathcal{B}|$, and $c = |C|$. Moreover, for each $A \in \mathcal{A}$, the 2-dimensional subspaces $W_t^{(A)}$, $t \in A$, form a vector space partition of the 4-dimensional subspace $W^{(A)}$. Similarly, for each $B \in \mathcal{B}$, the 2-dimensional subspaces $W_t^{(B)}$, $t \in B$, form a vector space partition of the three 3-dimensional subspaces $W^{(B,k)}$, $0 \le k \le 2$.

These observations yield a $4^{a-i} 3^{b-j} 2^{5i+7j+c}$-partition of V for any $0 \le i \le a$ and $0 \le j \le b$.

Example 11. *Partition of type* $T(1, 0, 0)$.

$$\begin{aligned}\mathcal{A} = \{&\{7,29,65,72,84\},\{22,31,54,57,59\},\{44,53,76,79,81\},\{1,32,45,46,70\},\\ &\{2,25,28,30,78\},\{3,19,50,63,64\},\{4,35,48,49,73\},\{5,8,10,58,67\},\\ &\{6,15,38,41,43\},\{9,12,14,62,71\},\{11,42,55,56,80\},\{13,16,18,66,75\},\\ &\{17,34,51,68,85\},\{20,27,39,47,69\},\{21,24,26,74,83\},\{23,36,37,61,77\},\\ &\{33,40,52,60,82\}\}.\end{aligned}$$

Example 12. *Partition of type* $T(2, 0, 0)$.

$$\begin{aligned}\mathcal{A} = \{&\{5,22,39,56,73\},\{6,29,32,34,82\},\{7,24,41,58,75\},\{8,16,38,74,81\},\\ &\{9,40,53,54,78\},\{10,19,42,45,47\},\{11,12,36,52,83\},\{13,59,65,69,80\},\\ &\{14,23,46,49,51\},\{15,18,20,68,77\},\{17,30,31,55,71\},\{21,57,64,76,84\},\\ &\{27,33,37,48,66\},\{35,44,67,70,72\}\}.\end{aligned}$$

$\mathcal{B} = \{\{0,1,2,25,26,50,198\}\}$.
$C = \{3,4,43,60,61,62,63,79\}$.

Example 13. *Partition of type* $T(3, 0, 0)$.

$$\begin{aligned}\mathcal{A} = \{&\{9,45,52,64,72\},\{4,11,23,31,53\},\{5,8,10,58,67\},\{6,15,38,41,43\},\\ &\{12,29,46,63,80\},\{13,16,18,66,75\},\{17,20,22,70,79\},\{21,37,68,81,82\},\\ &\{24,33,56,59,61\},\{30,36,40,51,69\},\{35,42,54,62,84\},\{39,48,71,74,76\}\}.\end{aligned}$$

$\mathcal{B} = \{\{0,1,2,25,26,50,198\},\{3,49,78,140,168,202,230\}\}$.
$C = \{7,14,19,27,34,44,47,57,65,73,77\}$.

Example 14. *Partition of type* $T(4,0,0)$.

$$\begin{aligned}\mathcal{A} &= \{\{1,8,20,28,50\},\{3,39,46,58,66\},\{4,21,38,55,72\},\{5,12,24,32,54\},\\ &\quad \{6,23,40,57,74\},\{7,15,37,73,80\},\{9,25,56,69,70\},\{10,27,44,61,78\},\\ &\quad \{13,30,47,64,81\},\{14,31,48,65,82\},\{22,29,41,49,71\}\}.\end{aligned}$$

$\mathcal{B} = \{\{17,18,19,42,43,67,215\},\{34,35,36,59,60,84,232\},\{51,52,53,76,77,101,249\}\}$.
$C = \{0,2,11,26,33,63,68,75,83\}$.

Example 15. *Partition of type* $T(5,0,0)$.

$$\begin{aligned}\mathcal{A} &= \{\{1,32,45,46,70\},\{2,25,28,30,78\},\{4,35,48,49,73\},\{6,15,38,41,43\},\\ &\quad \{9,12,14,62,71\},\{11,42,55,56,80\},\{13,16,18,66,75\},\{17,34,51,68,85\},\\ &\quad \{23,36,37,61,77\},\{33,40,52,60,82\}\}.\end{aligned}$$

$$\begin{aligned}\mathcal{B} &= \{\{22,53,67,79,105,194,251\},\{3,26,65,93,132,199,242\},\{7,59,64,90,104,148,197\},\\ &\quad \{10,39,83,191,244,246,254\},\{31,54,58,135,154,214,227\}\}.\end{aligned}$$

Example 16. *Partition of type* $T(6,0,0)$.

$$\begin{aligned}\mathcal{A} &= \{\{7,29,65,72,84\},\{4,27,30,32,80\},\{5,23,69,75,79\},\{14,20,24,35,53\},\\ &\quad \{22,31,54,57,59\},\{39,48,71,74,76\},\{46,55,78,81,83\}\}.\end{aligned}$$

$$\begin{aligned}\mathcal{B} &= \{\{0,1,2,25,26,50,198\},\{8,9,10,33,34,58,206\},\{11,13,16,47,61,149,236\},\\ &\quad \{17,18,19,42,43,67,215\},\{3,40,70,77,106,182,219\},\{6,37,38,41,51,62,137\}\}.\end{aligned}$$

$C = \{15,44,56,60,63,68,73,82\}$.

Example 17. *Partition of type* $T(7,0,0)$.

$$\begin{aligned}\mathcal{A} &= \{\{7,29,65,72,84\},\{6,23,40,57,74\},\{3,49,55,59,70\},\{4,27,30,32,80\},\\ &\quad \{20,56,63,75,83\},\{44,53,76,79,81\}\}.\end{aligned}$$

$$\begin{aligned}\mathcal{B} &= \{\{0,1,2,25,26,50,198\},\{8,9,10,33,34,58,206\},\{11,13,16,47,61,149,236\},\\ &\quad \{17,18,19,42,43,67,215\},\{22,39,78,90,145,153,205\},\{48,54,82,123,184,239,247\}\\ &\quad \{15,52,62,73,116,122,194\}\}.\end{aligned}$$

$C = \{12,21,41,46,51,71\}$.

Example 18. *Partition of type* $T(8,0,0)$.

$$\mathcal{A} = \{\{12,29,46,63,80\},\{14,31,48,65,82\},\{23,39,70,83,84\},\{40,49,72,75,77\}\}.$$

$$\begin{aligned}\mathcal{B} &= \{\{0,1,2,25,50,26,198\},\{3,5,6,53,226,30,107\},\{8,9,10,33,58,34,206\},\\ &\quad \{11,13,16,61,149,236,47\},\{17,18,19,42,67,43,215\},\{21,24,68,244,122,239,163\},\\ &\quad \{27,41,73,251,164,76,224\},\{38,44,51,229,137,156,202\}\}.\end{aligned}$$

$C = \{4,7,15,20,35,55,57,60,62\}$.

Example 19. *Partition of type* $T(9,0,0)$.

$$\mathcal{A} = \{\{7,29,65,72,84\},\{22,31,54,57,59\},\{44,53,76,79,81\}\}.$$

$$\begin{aligned}\mathcal{B} = \; & \{\{0,1,2,25,26,50,198\},\{8,9,10,33,34,58,206\},\{11,13,16,47,61,149,236\},\\ & \{17,18,19,42,43,67,215\},\{21,23,24,48,71,125,244\}\{3,49,78,140,168,202,230\},\\ & \{12,30,77,131,165,174,252\},\{5,56,69,75,148,155,243\},\{6,37,38,41,51,62,137\}\\ & \{14,15,20,39,112,153,205\}\}.\end{aligned}$$

Remark 20. *Heden* [11] *gave lower bounds for the number of subspaces of least dimension in a given partition of* $V(n,q)$. *The* $4^6 3^{21} 2^6$*-partition of* $V(8,2)$ *in Example 17 shows that the bound in Theorem* 1(*iii*) *from* [11] *is tight.*

4. Concluding Remarks

In this section, we give a few remarks regarding the computational aspects involved in the constructions of the $4^a 3^b 2^c$-partitions given in Section 3. In particular, the GAP [8] system along with the GRAPE [15] package made these computations much easier by allowing us to find large cliques in certain graphs.

Let $\mathcal{X}$ be the family of all 5-element subsets $A \subseteq \mathbb{Z}_{255}$ obtained using Construction 1 in Section 2.2. Recall that A yields a 4-dimensional subspace of $V(8,2)$. Let $\mathcal{Y}$ be the family of all 7-element subsets $B \subseteq \mathbb{Z}_{255}$ obtained using Construction 2 in Section 2.2. Recall that B yields 3 disjoint 3-dimensional subspaces of $V(8,2)$.

For any sub-families $\mathcal{X}' \subseteq \mathcal{X}$ and $\mathcal{Y}' \subseteq \mathcal{Y}$, consider the graph $G(\mathcal{X}',\mathcal{Y}')$ with vertex set

$$V(\mathcal{X}',\mathcal{Y}') = \big\{S \pmod{85} : \; S \in \mathcal{X}' \cup \mathcal{Y}'\big\}$$

and edge set

$$E(\mathcal{X}',\mathcal{Y}') = \big\{\{S_1,S_2\} : S_1,S_2 \in V(\mathcal{X}',\mathcal{Y}') \text{ and } S_1 \cap S_2 = \emptyset\big\}.$$

Let H be a complete subgraph of $G(\mathcal{X}',\mathcal{Y}')$ and define

$$\mathcal{A} = \big\{S \in H : \; |S| = 5\big\},$$
$$\mathcal{B} = \big\{S \in H : \; |S| = 7\big\},$$

and

$$C = \mathbb{Z}_{85} - \left(\bigcup_{S \in \mathcal{A}\cup\mathcal{B}} S\right).$$

Then the set families $\mathcal{A}$, $\mathcal{B}$, and C yield $4^{a-i}3^{b-j}2^{5i+7j+c}$-partitions of V, where $0 \le i \le a = |\mathcal{A}|$, $0 \le j \le b = 3|\mathcal{B}|$, and $c = |C|$. Note that we can always set $\mathcal{X}' = \mathcal{X}$ and $\mathcal{Y}' = \mathcal{Y}$, but if we are trying to construct families $\mathcal{A}$, $\mathcal{B}$, and C as in the Examples 11–19, we can select $\mathcal{X}'$ and $\mathcal{Y}'$ appropriately to reduce the search space for finding a suitable clique in $G(\mathcal{X}, \mathcal{Y})$. For instance, we can choose some $A \in \mathcal{X}$, set $\mathcal{X}_A = \{S \in \mathcal{X} : S \cap A = \emptyset\}$, and let $\mathcal{Y}' = \emptyset$. Then any clique of size 16 in $G(\mathcal{X}_A, \emptyset)$ together with the set A yield a set family $\mathcal{A}$ of size 17 that can be used in Example 11. All the partitions in Examples 11–19 were obtained in this way by using the GAP [8] system and the GRAPE [15] package to search for a suitable clique H.

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