THE LATTICE OF FINITE SUBSPACE PARTITIONS

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Abstract. Let V denote $V(n, q)$, the vector space of dimension n over $\mathrm{GF}(q)$. A *subspace partition* of V is a collection Π of subspaces of V such that every nonzero vector in V is contained in exactly one subspace belonging to Π. The set $\mathcal{P}(V)$ of all subspace partitions of V is a lattice with minimum and maximum elements $\mathbf{0}$ and $\mathbf{1}$ respectively. In this paper, we show that the number of elements in $\mathcal{P}(V)$ is congruent to the number of all set partitions of $\{1, \ldots, n\}$ modulo $q-1$. Moreover, we show that the Möbius number $\mu_{n,q}(\mathbf{0}, \mathbf{1})$ of $\mathcal{P}(V)$ is congruent to $(-1)^{n-1}(n-1)!$ (the Möbius number of set partitions of $\{1, \ldots, n\}$) modulo $q-1$.

1. Introduction

Let V denote $V(n, q)$, the vector space of dimension n over $\mathrm{GF}(q)$, and $\mathbf{n}$ denote the set $\{1, \ldots, n\}$, throughout the rest of this article. A *subspace partition* of V is a collection Π of subspaces of V such that each nonzero vector in V is in exactly one subspace of Π. In this paper, we state that the set $\mathcal{P}(V)$ of subspace partitions of V is a lattice (Theorem 1), analogous to the lattice $\mathcal{P}(\mathbf{n})$ of set partitions of $\mathbf{n}$. We construct an order-preserving surjective map from $\mathcal{P}(V)$ onto $\mathcal{P}(\mathbf{n})$. Through this map, we are able to prove that $|\mathcal{P}(V(n, q))| \equiv |\mathcal{P}(\mathbf{n})|$ modulo $q-1$ (Theorem 2). We also show that the number of subspace partitions of $V(n, q)$ of any type, where there are more than n subspaces with dimension greater than one, is congruent to zero modulo $q-1$ (Theorem 3). Another important result is that the Möbius number of $\mathcal{P}(V(n, q))$ is congruent to the Möbius number of $\mathcal{P}(\mathbf{n})$ modulo $q-1$ (Theorem 4). Finally, we give a characterization of the Möbius number of $\mathcal{P}(V(4, q))$ (Theorem 5).

2. Subspace partitions of a finite vector space

A *subspace partition* of V is a collection Π of nonzero subspaces $U_1, \ldots, U_k$ such that $V = \bigcup_{i=1}^{k} U_i$ and $U_i \cap U_j = \{0\}$ for $i \neq j$. Let $n_1, \ldots, n_k$ and $d_1, \ldots, d_k$, with $d_1 < \cdots < d_k$, be positive integers. We say that Π is a partition of V of *type* $d_k^{n_k} \ldots d_1^{n_1}$ if it consists of n_i subspaces of dimension d_i for all i with $1 \leq i \leq k$; thus necessarily the condition $\sum_{i=1}^{k} n_i(q^{d_i} - 1) =$

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$q^n - 1$ holds. This condition is not sufficient; e.g., a partition of $V(5,2)$ of type $2^{10}1^1$ does not exist [6]. Despite extensive research (see [2, 3, 4, 6, 7] and the references therein) that settled some special cases, the general problem of finding necessary and sufficient conditions for V to admit a partition of type $d_k^{n_k} \ldots d_1^{n_1}$ is still open. Subspace partitions have applications in several areas, including coding theory, design theory, and finite geometry.

We denote the partition of V consisting entirely of one dimensional subspaces by $\mathbf{0}$ and the one consisting of V only by $\mathbf{1}$. We also use the shorthand notation s-D to denote "s-dimensional" from now on.

3. Lattice of subspace partitions

Bu states briefly in [3], *"If we define the meet between the two partitions of* V *by* $\{V_i\} \cap \{U_i\} = \{V_i \cap U_j\}$*, it is easy to see that the collection of all partitions of* V *form a finite lattice. It should be an aim for further research to characterize the structure of this lattice."* We have not found any other references than these two sentences about the lattice structure of subspace partitions in the literature. Below are the formal definitions of refinement, meet, and join.

Consider subspace partitions Π and Γ of V, with $\Pi = \{U_1, \ldots, U_k\}$ and $\Gamma = \{W_1, \ldots, W_t\}$. We say Π is a *refinement* of Γ in $\mathcal{P}(V)$, and write $\Pi \preceq \Gamma$, if each subspace W_i is the union of a number of the subspaces U_j. We also say Γ is *coarser than* Π, and write $\Gamma \succeq \Pi$. Clearly, $\mathbf{0}$ is the minimum element and $\mathbf{1}$ is the maximum element of the poset $\mathcal{P}(V)$. The *meet* of two arbitrary partitions Π and Γ, denoted by $\Pi \wedge \Gamma$, is the partition $\{U_i \cap W_j : 1 \le i \le k,\ 1 \le j \le t,\ U_i \cap W_j \ne \{0\}\}$ of V. Since $\mathcal{P}(V(n,q))$ is a poset with a meet operation and a maximum element, we have

Theorem 1. *$\mathcal{P}(V(n,q))$ is a lattice.*

The *join* of Π and Γ, denoted by $\Pi \vee \Gamma$, is defined by $\Pi \vee \Gamma = \bigwedge_{\substack{\Omega \succeq \Pi \\ \Omega \succeq \Gamma}} \Omega$.

4. Preliminaries

Recall that $V = V(n,q)$. In this section, E denotes a fixed ordered basis $\{e_1, \ldots, e_n\}$ of V, which we identify with $\mathbf{n}$ as needed. We denote the lattice of subspace partitions of V by $\mathcal{P}(V)$ and the lattice of set partitions of E (hence, of $\mathbf{n}$) by $\mathcal{P}(\mathbf{n})$.

4.1. **The good subspace partitions.** Let τ be a generic set partition of E, with $\tau = \{E_1, \ldots, E_t\}$. Define the injective map $\varphi : \mathcal{P}(\mathbf{n}) \to \mathcal{P}(V)$ by $\varphi(\tau) = \{\langle E_1 \rangle, \ldots, \langle E_t \rangle, U_1, \ldots, U_s\}$, where $\langle E_i \rangle$ denotes the subspace of V generated by the set E_i, and $U_1, \ldots, U_s$ are the 1-D subspaces not included in any $\langle E_i \rangle$. Let Π be a partition in $\mathcal{P}(V)$, and define the surjective map $\psi : \mathcal{P}(V) \to \mathcal{P}(\mathbf{n})$ by $\psi(\Pi) = \{W \cap E : W \in \Pi,\ W \cap E \ne \emptyset\}$.

Lemma 1. *For ψ and φ defined as above, the following hold:*
(i) *Both ψ and φ are monotone maps.*
(ii) $\psi\varphi(\tau) = \tau$.
(iii) $\varphi\psi(\Pi) \preceq \Pi$.

Lemma 1 shows that the map φ is in fact a "Galois function", the monotone counterpart of the well-known "Galois connection" (see Ore [8], Rota [9], and Aigner [1]). This makes the subset $\varphi(\mathcal{P}(\mathbf{n}))$ of $\mathcal{P}(V)$ a lattice isomorphic to $\mathcal{P}(\mathbf{n})$. We will denote this subset of $\mathcal{P}(V)$ by $\mathcal{A}$, and call its members *good* partitions. Accordingly, the remaining partitions in $\mathcal{P}(V)$ will be called *bad*. We organize our observations as follows:

Proposition 2. *The set $\mathcal{A}$ of good partitions of V consists of all $\Pi \in \mathcal{P}(V)$ with the property $\varphi\psi(\Pi) = \Pi$, and its elements are in 1-1 correspondence with the set partitions of E. The explicit characterization of $\mathcal{A}$ is as follows:*

$$\mathcal{A} = \{\Pi \in \mathcal{P}(V) : \ \textit{for all } W \in \Pi,\ W = \langle W \cap E\rangle \ \textit{if} \dim(W) \geq 2\}.$$

4.2. **Canonical bases.** Let $\langle -, -\rangle_E$ denote the inner product on V determined by the orthonormal basis E. For any $v \in V$, define the *support* of v, written $\operatorname{supp}(v)$, as the set $\{e_i : \langle v, e_i\rangle_E \neq 0\}$, and define the support of a subspace $W \subseteq V$, written $\operatorname{supp}(W)$, as the set $\bigcup_{w\in W} \operatorname{supp}(w)$. We will make use of the *canonical bases* of subspaces of V. The following fact is standard.

Proposition 3. *If W is a k-D subspace of $V(n, q)$, and E is a fixed ordered basis (e.g., the standard basis) of $V(n, q)$, then there exists a unique $n \times n$ matrix over* $\mathrm{GF}(q)$ *of rank k in reduced row echelon form, whose nonzero rows form a basis of W (with respect to E). This basis is called the* canonical basis *of W.*

We omit the proof of the following simple lemma.

Lemma 4. *If W is a nonzero subspace of V, and β is the canonical basis of W, then the following properties hold:*
(i) *If $w, w' \in \beta$ and $w \neq w'$, then we cannot have* $\operatorname{supp}(w) \subseteq \operatorname{supp}(w')$.
(ii) *If $w \in W\backslash\{0\}$, with* $\dim(W) \geq 2$, *then* $\operatorname{supp}(w)$ *cannot be a proper subset of* $\operatorname{supp}(w')$ *for any $w' \in \beta$.*
(iii) *$W \cap E$ is a subset of β.*
(iv) *If $U \subseteq W$ and $U = \langle U \cap \beta\rangle$, then $U \cap \beta$ is the canonical basis for U.*

Using Lemma 4, we now describe a natural decomposition for nonzero subspaces W of V. If W has canonical basis β, then we define an equivalence relation $\sim$ on elements of β by the following rule: $w_i \sim w_j$ if and only if there exists a chain $v_0, v_1, \ldots, v_r$ of elements of β such that $v_0 = w_i$, $v_r = w_j$, and $\operatorname{supp}(v_{s-1}) \cap \operatorname{supp}(v_s) \neq \emptyset$ for all s with $1 \leq s \leq r$. The equivalence classes of $\sim$ partition $E \cap \operatorname{supp}(W)$.

Lemma 5. *If W is a subspace of V of dimension k with $k \geq 2$, and β is the canonical basis of W with respect to a fixed ordered basis E of V, then*

E is the disjoint union of the following three sets (any one of which could be empty):

$$E_1 = W \cap E = \beta \cap E, \quad E_2 = \bigcup_{w \in \beta \setminus E_1} \operatorname{supp}(w), \quad E_3 = E \setminus (E_1 \cup E_2),$$

where E_1 is the union of equivalence classes of $\sim$ that are singletons. Moreover, β can be written uniquely as a disjoint union of E_1 and larger equivalence classes $\beta_1, \ldots, \beta_t$, with $\bigcup_{w \in \beta_j} \operatorname{supp}(w) \stackrel{\text{def}}{=} E_2^{(j)}$, and consequently, E_2 can be written uniquely as a disjoint union of indecomposable subsets $E_2 = E_2^{(1)} \cup \cdots \cup E_2^{(t)}$. Therefore, we have the unique decomposition of W into a direct sum $W = \langle E_1 \rangle \oplus W^{(1)} \oplus \cdots \oplus W^{(t)}$, where each β_j is the canonical basis of $W^{(j)}$, and $\operatorname{supp}(W^{(j)}) = E_2^{(j)}$.

4.3. **The group of invertible diagonal matrices.** Every subgroup of $GL(n,q)$, the group of automorphisms of $V(n,q)$, acts on $\mathcal{P}(V)$ via the rule $g(\{W_1, \ldots, W_k\}) = \{g(W_1), \ldots, g(W_k)\}$. The type of a partition is invariant under this action. Let G be the abelian subgroup of $GL(n,q)$ represented by the invertible diagonal matrices with respect to the fixed basis E. Clearly, the order of G is $(q-1)^n$, where $q-1$ is the order of the cyclic group $\mathrm{GF}(q)^*$. The subgroup $Z = \{cI \,:\, c \in \mathrm{GF}(q)^*\}$ of G is isomorphic to $\mathrm{GF}(q)^*$, and is contained in the stabilizer G_Π of any partition $\Pi \in \mathcal{P}(V)$ in G.

4.4. **The bad subspace partitions.** Recall that

$$\mathcal{A} = \{\Pi \in \mathcal{P}(V) : \text{ for all } W \in \Pi,\ W = \langle W \cap E \rangle \text{ if } \dim(W) \geq 2\}.$$

Define

$$\mathcal{B} = \{\Pi \in \mathcal{P}(V) \setminus \mathcal{A} : \text{ for all } W \in \Pi,\ \langle W \cap E \rangle \neq \emptyset \text{ if } \dim(W) \geq 2\},$$

and let

$$\mathcal{C} = \{\Pi \in \mathcal{P}(V) \setminus \mathcal{A} : \text{ there exists } W \in \Pi \text{ such that } \langle W \cap E \rangle = \emptyset \text{ and } \dim(W) \geq 2\}.$$

These three sets partition the set $\mathcal{P}(V)$.

Lemma 6. *If W is a subspace of V of dimension at least two such that $W \neq \langle W \cap E \rangle$, then the subgroup of G fixing W consists of elements that have blocks of equal (otherwise free) entries on the diagonal. Hence, this subgroup has order $(q-1)^m$, where m is the number of blocks, and $1 \leq m \leq n-1$.*

Proof. Suppose that W is as in Lemma 5, and let g be the diagonal matrix $\operatorname{diag}(g_{11}, \ldots, g_{nn}) \in G$. If g satisfies the condition

$$e_j \text{ and } e_k \text{ are in the same set } E_2^{(i)} \ \Rightarrow\ g_{jj} = g_{kk}, \tag{1}$$

then $g(W^{(i)}) = W^{(i)}$ for all i, which is equivalent to $g(W) = W$. Conversely, assume that $g(W) = W$. Without loss of generality, let $g_{11} = 1$. Fix any element (say e_1) of $E_2^{(i)}$, and let u be an element of β_i with $e_1 \in \operatorname{supp}(u)$.

By Lemma 4(ii), $g(u) - u \in W^{(i)}$ must be zero, because it has strictly smaller support than u. It follows that $g(u) = u$ and $g_{ss} = 1$ for all s with $e_s \in \mathrm{supp}(u)$. That is, the diagonal entries of g corresponding to the support of a canonical basis element are constant, and since it is possible to go from any element of $E_2^{(i)}$ to any other via linked supports of basis elements, we obtain Eq. (1). Thus, if there are k equivalence classes in β (including singletons) and l elements of E_3, then there are $(q-1)^{k+l}$ elements in the subgroup of G defined by Eq. (1). □

Proposition 7. *If $\Pi \in \mathcal{B}$, then the order of the orbit of Π under G is equal to $(q-1)^m$, with $1 \leq m \leq n-1$. Consequently, $(q-1)$ divides $|\mathcal{B}|$.*

Proof. Note that there may be some subspaces W' in Π that are generated by $W' \cap E$; these are fixed by any $g \in G$ and do not affect the order of the stabilizer G_Π of Π in G, nor the size of the orbit of Π. However, there must be at least one $W \in \Pi$ with $\dim(W) \geq 2$, $W \cap E \neq \emptyset$, and $W \neq \langle W \cap E \rangle$, because Π is in $\mathcal{B}$.

Any $g \in G_\Pi$ necessarily fixes such W, as the subspace $\langle W \cap E \rangle$ must go to itself under multiplication by a diagonal matrix. The group G_Π is then the intersection of all subgroups of G fixing any of the subspaces W described above. By Lemma 6, we know that each such subgroup consists of elements with blocks of equal (but otherwise free) elements on the diagonal.

The intersection of the subgroups of G described above may make some blocks merge, but the structure of the elements of the group G_Π will essentially be the same. Since there is at least one block of length greater than or equal to two (due to the existence of some $E_2^{(i)}$; see Lemma 5), and $Z \subseteq G_\Pi$ for all $\Pi \in \mathcal{P}(V)$, we have $|G_\Pi| = (q-1)^r$, with $1 \leq r \leq n-1$. Thus, the same restrictions on the power of $q-1$ hold for the order of the orbit G/G_Π of Π. As $\mathcal{B}$ is a disjoint union of such orbits, $q-1$ divides $|\mathcal{B}|$. □

Finally, we turn our attention to the bad partitions in $\mathcal{C}$, namely, in $\mathcal{P}(V) \setminus (\mathcal{A} \cup \mathcal{B})$. For each i with $1 \leq i \leq n$, define $\mathcal{C}^{(i)}$ by

$$\mathcal{C}^{(i)} = \left\{ \Pi \in \mathcal{C} : \min \left[\bigcup_{\substack{W \in \Pi \\ \dim W \geq 2, W \cap E = \emptyset}} \{j : e_j \in \mathrm{supp}(W)\} \right] = i \right\}.$$

Note that $\mathcal{C}^{(n-1)} = \mathcal{C}^{(n)} = \emptyset$ and the sets $\mathcal{C}^{(i)}$, $1 \leq i \leq n-2$, form a set partition of $\mathcal{C}$. For all i with $1 \leq i \leq n-2$, we also define $G^{(i)}$ to be $\{\mathrm{diag}(g_{11}, \ldots, g_{nn}) \in G : g_{jj} = 1 \text{ for } j \neq i\}$, which is a subgroup of G of order $q-1$. Observe that $\mathcal{C}^{(i)}$ is closed under the action of $G^{(i)}$.

Proposition 8. *If $\Pi \in \mathcal{C}^{(i)}$, then the order of the orbit of Π under the action of $G^{(i)}$ is $q-1$. Consequently, $q-1$ divides $|\mathcal{C}|$.*

Proof. Let $\Pi \in \mathcal{C}^{(i)}$ and $g \in G^{(i)}$. There exists $W \in \Pi$ with $\dim(W) \geq 2$ and $W \cap E = \emptyset$, as well as canonical basis vectors u and $v \in W$ such that $e_i \in \mathrm{supp}(u)$ and $e_i \notin \mathrm{supp}(v)$ (Lemma 4). Since the action of $G^{(i)}$ affects only the coefficient of e_i, we have $g(v) = v$. Now any $g \in G^{(i)}$ that fixes Π must fix the subspace W.

Let us reconsider the first element $u \in W$. We have $u = e_i + z$ for some $z \in V \backslash \{0\}$ such that $e_i \notin \text{supp}(z)$. Let g be any element of the stabilizer $G^{(i)}_{\Pi}$, say $g = \text{diag}(1, \ldots, g_{ii}, \ldots, 1)$, necessarily fixing W. Now, since $g(u) - u = (g_{ii} - 1)e_i \in W$, and $W \cap E = \emptyset$, we must have $g_{ii} = 1$. That is, the only element in the stabilizer of Π is the identity matrix, I. Hence, the order of the orbit of Π is $|G^{(i)}|/|G^{(i)}_{\Pi}| = (q-1)/|\{I\}| = q - 1$. Thus $q - 1$ divides $|\mathcal{C}^{(i)}|$. As $\mathcal{C}$ is a disjoint union of the sets $\mathcal{C}^{(i)}$, $q - 1$ also divides $|\mathcal{C}|$. □

5. Final count: Theorems 2-5

We retain the notation of Section 4. The *Gaussian coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}_q$, defined by $\frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}$, counts the number of k-D subspaces of $V(n, q)$.

Theorem 2. *The number of subspace partitions of $V(n, q)$ is congruent to the number of set partitions of $\mathbf{n}$ modulo $q - 1$.*

Proof. By Proposition 2, the subspace partitions in $\mathcal{A}$ are in one-to-one correspondence with the set partitions of E. Moreover, the total size of $\mathcal{P}(V) \setminus \mathcal{A}$ is divisible by $(q - 1)$ (Propositions 7 and 8). □

Example 9. *The number of subspace partitions of $V(3, q)$ is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q + 2 = q^2 + q + 3 \equiv 5 \bmod (q - 1)$, the total number of set partitions of $\{1, 2, 3\}$.*

Theorem 3. *If $n_1, \ldots, n_k$ and $d_1, \ldots, d_k$ are positive integers such that $1 < d_1 < \cdots < d_k$ and $\sum_{i=1}^{k} n_i d_i > n$, then the number of subspace partitions of $V(n, q)$ of type $d_k^{n_k} \cdots d_1^{n_1} 1^{n_0}$ (where n_0 may be zero) is congruent to zero modulo $q - 1$.*

Proof. Let Π be a partition of V of type $d_k^{n_k} \cdots d_1^{n_1} 1^{n_0}$. Note that the type of a partition and the supports of its subspaces are constant throughout any orbit under the action of any subgroup of the group G introduced in Subsection 4.3. By Propositions 2, 7, and 8, it suffices to show that $\Pi \notin \mathcal{A}$. Assume, on the contrary, that $\Pi \in \mathcal{A}$. It follows that each subspace of dimension greater than one in Π is generated by a subset of $E = \{e_1, \ldots, e_n\}$; let $E' \subseteq E$ be the union of such subsets. Thus, the canonical bases of the subspaces of dimension greater than one form a partition of E', and we have

$$n = |E| \geq |E'| = \sum_{i=1}^{k} n_i d_i > n,$$

a contradiction. □

Let $\mathcal{P}$ be a finite poset with a minimum element $\mathbf{0}$ and a maximum element $\mathbf{1}$. For x and $y \in \mathcal{P}$, with $x \leq y$, the *Möbius function* is defined by $\mu(x, x) = 1$, and for $x < y$, by $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$. We will call the number $\mu(\mathbf{0}, \mathbf{1})$ the *Möbius number* of $\mathcal{P}$ (see Godsil [5] for slightly different terminology). One significance of this number is that the reduced order complex of $\mathcal{P}(V)$, with the vertex set consisting of the elements of $\mathcal{P}(V) \setminus \{\mathbf{0}, \mathbf{1}\}$ and the edge set consisting of the nonempty chains in $\mathcal{P}(V) \setminus \{\mathbf{0}, \mathbf{1}\}$, has Euler characteristic equal to $\mu(\mathbf{0}, \mathbf{1})$ (Rota [9]). We have the following result about the Möbius number of $\mathcal{P}(V)$:

Theorem 4. *The Möbius number $\mu_{n,q}(\mathbf{0}, \mathbf{1})$ of the poset $\mathcal{P}(V)$ of subspace partitions of $V(n, q)$ is congruent to $(-1)^{n-1}(n-1)!$, the Möbius number $\mu_n(0, 1)$ of the poset $\mathcal{P}(\mathbf{n})$ of set partitions of $\mathbf{n}$, modulo $q - 1$.*

Proof. Let T be a partition of the integer n, with $T = (n-1)^{m_{n-1}} \cdots 2^{m_2} 1^{m_1}$, let τ vary over all set partitions of $\mathbf{n}$ of type T (we write $t(\tau) = T$), and let Π vary over all corresponding subspace partitions of $V(n, q)$ of type $[T] = (n-1)^{m_{n-1}} \cdots 2^{m_2} 1^{x}$ (we write $t(\Pi) = [T]$). We want to prove by induction that

$$\sum_{\substack{\Pi \\ t(\Pi)=[T]}} \mu_{n,q}(\mathbf{0}, \Pi) \equiv \sum_{\substack{\tau \\ t(\tau)=T}} \mu_n(0, \tau) \pmod{q-1}. \tag{2}$$

Note that if Π and Π' are in the same orbit, then $\mu_{n,q}(\mathbf{0}, \Pi) = \mu_{n,q}(\mathbf{0}, \Pi')$, because the type of a partition completely determines this number. Hence, when a subspace partition Π is in $\mathcal{B}$ or $\mathcal{C}$, the Möbius function value $\mu_{n,q}(\mathbf{0}, \Pi)$ will occur in multiples of $q - 1$ by Propositions 7 and 8. Therefore, we may ignore partitions of types described in Theorem 3 above, modulo $q-1$. This observation together with Eq. (2) will immediately lead us to the desired result, since we will have

$$\begin{aligned}
\mu_{n,q}(\mathbf{0}, \mathbf{1}) &= -\sum_{\substack{\Pi \\ \Pi \neq \mathbf{1}}} \mu_{n,q}(\mathbf{0}, \Pi) \\
&\equiv -\sum_{[T] \neq n^1} \sum_{\substack{\Pi \\ t(\Pi)=[T]}} \mu_{n,q}(\mathbf{0}, \Pi) \pmod{q-1} \\
&\equiv -\sum_{T \neq n^1} \sum_{\substack{\tau \\ t(\tau)=T}} \mu_n(0, \tau) \pmod{q-1} \quad \text{(Eq. (2))} \\
&\equiv -\sum_{\tau \neq 1} \mu_n(0, \tau) \equiv \mu_n(0, 1) \pmod{q-1}
\end{aligned} \tag{3}$$

for each n. Hence, we only need to prove Eq. (2).

If $n = 2$, then the only possible nice partition types are $T = 1^2$ and $[T] = 1^{q+1}$, and Eq. (2) can only be in the form

$$\mu_{2,q}(\mathbf{0}, \mathbf{0}) = 1 = \mu_2(0, 0) \equiv \mu_2(0, 0) \pmod{q-1}.$$

Next, assume that Eq. (2) holds for all dimensions $< n$, where $n \geq 3$, and that T and $[T]$ are as described as above. We have

$$\begin{aligned}
\sum_{\substack{\Pi \\ t(\Pi)=[T]}} \mu_{n,q}(\mathbf{0},\Pi) &= \sum_{\substack{\Pi \\ t(\Pi)=[T]}} (\mu_{n-1,q}(\mathbf{0},\mathbf{1}))^{m_{n-1}} \cdots (\mu_{2,q}(\mathbf{0},\mathbf{1}))^{m_2} \\
&\equiv \sum_{\substack{\tau \\ t(\tau)=T}} (\mu_{n-1}(0,1))^{m_{n-1}} \cdots (\mu_2(0,1))^{m_2} \pmod{q-1} \\
&\qquad \text{(by induction step and Eq. (3))} \\
&\equiv \sum_{\substack{\tau \\ t(\tau)=T}} \mu_n(0,\tau) \pmod{q-1}.
\end{aligned}$$

$\square$

The Möbius numbers of $\mathcal{P}(V(n,q))$ for $n \leq 3$ are trivial to compute. We give below a formula for the Möbius number of $\mathcal{P}(V(4,q))$, characterized in terms of the numbers of certain partial spreads. Let t be a positive integer, with $t \leq n$. A *partial t-spread* (of size k) of $V(n,q)$ is a subspace partition Π of $V(n,q)$ of type $t^k 1^x$, where $k = |\Pi|$ and $x = \left((q^n-1) - k(q^t-1)\right)/(q-1)$.

Theorem 5. *The Möbius number of $\mathcal{P}(V(4,q))$ is given by*

$$\mu_{4,q}(\mathbf{0},\mathbf{1}) = -\sum_{k=0}^{q^2+1} (-1)^k c_k^4 - \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q (q^2+q),$$

where $c_0^4 = 1$, and c_k^4 is the number of partial 2-spreads of $V(4,q)$ of size k.

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