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Subspace partitions of $\mathbb{F}_q^n$ containing direct sums II: General case

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ABSTRACT

A *subspace partition* of a finite vector space $\mathbb{F}_q^n$ of dimension n over the field $\mathbb{F}_q$ with q elements is a collection Π of subspaces of $\mathbb{F}_q^n$ consisting of subspaces with mutually zero intersection that partition the nonzero vectors in $\mathbb{F}_q^n$. Clearly, Π satisfies the equation $\sum_{W\in\Pi}(q^{\dim W}-1)=q^n-1$, which is called the *packing condition*. We say that Π *contains a direct sum* if there exist $W_1,\ldots,W_r\in\Pi$ with $W_1\oplus\cdots\oplus W_r=\mathbb{F}_q^n$. Partitions with direct sums are ubiquitous in practice and form an important subfamily of the lattice of all subspace partitions of $\mathbb{F}_q^n$, which is a combinatorial q-analogue of the lattice of all set partitions of the set with n elements. In this paper, we show that subspace partitions with direct sums (where the number h of distinct dimensions among subspaces of Π is arbitrary) exist when their type (the multiset of subspace dimensions found in Π) is in the convex hull of certain kinds of vertices in the lattice of solutions of the packing condition. This generalizes a result in a previous paper, where we had considered subspace partitions containing at most $h=2$ distinct subspace dimensions. We also construct an infinite family of *Frobenius subspace partitions* with h distinct dimensions that cannot contain a direct sum, due to the fact that no combination of dimensions adds up to n.

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1. Introduction

1.1. Subspace partitions and Gaussian partitions of finite vector spaces

For any positive integer n, let $\mathbf{n}$ denote a set with n elements, $\mathcal{P}(\mathbf{n})$ denote the collection of set partitions of $\mathbf{n}$, and $\mathcal{P}(n)$ denote the collection of integer partitions of n. The cardinalities of the last two sets, commonly known as the Bell number and the partition function respectively, will be denoted by $\mathrm{B}(n) = |\mathcal{P}(\mathbf{n})|$ and $\mathrm{p}(n) = |\mathcal{P}(n)|$.

For a prime power q, the symbol $\mathbb{F}_q^n$ will denote the vector space of dimension n over the Gaussian field $\mathbb{F}_q$. A *subspace partition* Π of $\mathbb{F}_q^n$, also known as a *vector space partition* in the literature, is a collection of subspaces of $\mathbb{F}_q^n$ such that any two subspaces in Π have zero intersection, and every nonzero vector in $\mathbb{F}_q^n$ appears in exactly one element of Π. We say that a subspace partition Π *contains a direct sum* if there exist r subspaces $W_i \in \Pi$ with $W_1 \oplus \cdots \oplus W_r = \mathbb{F}_q^n$ for some $r \geq 1$. Two special cases are the set of all one-dimensional subspaces of $\mathbb{F}_q^n$ and the singleton $\{\mathbb{F}_q^n\}$, denoted by $\mathbf{0}$ and $\mathbf{1}$ respectively. The set $\mathcal{P}(\mathbb{F}_q^n)$ of all subspace partitions of $\mathbb{F}_q^n$ is known to be a lattice with the underlying "refinement" partial order, the minimal object $\mathbf{0}$, and the maximal object $\mathbf{1}$ [1]. A similar statement applies to $\mathcal{P}(\mathbf{n})$. In this paper, we will argue in favor of considering the set $\mathcal{P}_D(\mathbb{F}_q^n)$ of subspace partitions of $\mathbb{F}_q^n$ that contain direct sums to be a valid q-analogue of $\mathcal{P}(\mathbf{n})$, and prove a theorem that gives us necessary conditions for a subspace partition be in $\mathcal{P}_D(\mathbb{F}_q^n)$. The special case of the theorem with at most two distinct dimensions among the subspaces of a partition was the subject of [4].

If a positive integer a divides n, then it is always possible to construct an *a-spread*, which is a subspace partition of $\mathbb{F}_q^n$ consisting entirely of subspaces of dimension a (see André [5] and Segre [18]). For an integer a such that $1 < a < n$, the term *partial a-spread* of $\mathbb{F}_q^n$ refers to a collection of a-dimensional subspaces of $\mathbb{F}_q^n$ with mutually zero intersections by convention. Now, any finite collection of subspaces that have pairwise zero intersections can be completed to a subspace partition by appending all one-dimensional subspaces of $\mathbb{F}_q^n$ that are not contained in any of these subspaces. Therefore, we can also envision a partial a-spread as an element of $\mathcal{P}(\mathbb{F}_q^n)$ exhibiting two distinct subspace dimensions, a and 1, and we will henceforth use the term in this manner. Furthermore, we will often use the shorthand notation "a-D" instead of "a-dimensional." In [4], we proved that both a-spreads and partial a-spreads contain direct sums unconditionally.

The *type* of a subspace partition Π, which we call a *Gaussian partition* of $\mathbb{F}_q^n$, is the multiset of dimensions of the subspaces in Π. Thus, if a subspace partition Π contains u_i subspaces of dimension a_i with $1 \leq i \leq h$, where the positive integers a_i are assumed to be distinct, then the corresponding Gaussian partition is denoted by $T(\Pi) = a_1^{u_1} \cdots a_h^{u_h}$. The vector $\mathbf{u} = (u_1, \ldots, u_h)$ is hence a solution of the *packing condition*

$$(q^{a_1}-1)\,u_1+\cdots+(q^{a_h}-1)\,u_h=q^n-1, \tag{1}$$

which indicates that all nonzero vectors in $\mathbb{F}_q^n$ are allocated to (or, partitioned by) the subspaces of Π. We will reserve the symbol $\mathcal{P}_G(\mathbb{F}_q^n)$ for the set of all Gaussian partitions of $\mathbb{F}_q^n$. The set $\mathcal{P}_G(\mathbb{F}_q^n)$ (resp., $\mathcal{P}(n)$) is a poset, whose partial ordering reflects that of $\mathcal{P}(\mathbb{F}_q^n)$ (resp., $\mathcal{P}(\mathbf{n})$). The number of Gaussian partitions of $\mathbb{F}_q^n$ will be denoted by $\mathrm{p}_q(n)$.

The set of subspace partitions of $\mathbb{F}_q^n$ that do not admit direct sums is nonempty for any h. Some, but not all, of these subspace partitions have a type T which does not contain an integer partition of n as a sub-multiset, hence excluding the possibility of a direct sum from the outset (see [4] for examples of both kinds). We will call the latter kind of subspace partition a *Frobenius partition*, after the Frobenius number: given any set of relatively prime positive integers $a_1,\ldots,a_d$, the *Frobenius number* $g(a_1,\ldots,a_d)$ is the largest positive integer that cannot be represented as a linear combination of a_i with nonnegative integer coefficients.

1.2. The argument for combinatorial q-analogues

One of the analogies between the set $\mathbf{n}$ and the vector space $\mathbb{F}_q^n$ is provided by the *Gaussian binomial coefficient*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q=\frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)},$$

which is a monic polynomial in $\mathbb{Z}[q]$ that counts the number of k-dimensional subspaces of $\mathbb{F}_q^n$. After clearing the denominator and substituting $q=1$, we obtain the regular binomial coefficient

$$\binom{n}{k},$$

which is the number of k-subsets of $\mathbf{n}$. Therefore, we recognize $\mathbb{F}_q^n$ as a *combinatorial* object that is a q-analogue of $\mathbf{n}$, just as the collection of subspaces of $\mathbb{F}_q^n$ is a *combinatorial* q-analogue of the power set of $\mathbf{n}$. Indeed, finite sets have been dubbed “vector spaces over the field $\mathbb{F}_1$” by Tits [20].

A substantial amount of research has been done on the topic of q-analogues of classical combinatorial objects and their enumeration. For instance, pioneering work on q-analogues of the Stirling numbers of both kinds was done by Gould [13] and Carlitz [10,11], which were further studied by Gessel [12], Wachs and White [22], Cai and Readdy [9], Milne [17], and others. In addition, q-analogues of partially ordered sets were studied by Simion [21] in general, while the particular case of the lattice of set partitions was investigated by Bennett, Dempsey, and Sagan [6]. Our work is primarily concerned with a new q-analogue of the lattice of set partitions of $\mathbf{n}$, which has not been studied in this sense by other authors. The q-Bell numbers based on this combinatorial object are also different from those in Milne [17]. Thus, in recent papers (see [1–3]), we followed the relationship between $\mathbf{n}$ and $\mathbb{F}_q^n$ to its natural conclusion. To support the analogies, we needed to define explicitly a combinatorial relationship between $\mathcal{P}(\mathbf{n})$ and $\mathcal{P}(\mathbb{F}_q^n)$. We established the following steps [1]:

- Identify $\mathbf{n}$ with a fixed basis of $\mathbb{F}_q^n$.
- Consider the action of the diagonal subgroup G of $\mathrm{GL}_q(n)$ with respect to this basis on $\mathcal{P}(\mathbb{F}_q^n)$.
- Show that an orbit of $\mathcal{P}(\mathbb{F}_q^n)$ under G is one of two types: either (a) it contains only one subspace partition Π, whose significant subspaces are spanned by parts of a fixed set partition of $\mathbf{n}$, with 1-D subspaces completing Π, or (b) its size is a multiple of $q-1$.

The inference is that there exists a lattice embedding of $\mathcal{P}(\mathbf{n})$ into $\mathcal{P}(\mathbb{F}_q^n)$, and that $\mathrm{B}_q(n) \equiv \mathrm{B}(n) \pmod{q-1}$. We were also able to show that the Möbius functions of the two lattices are related by the congruence $\mu_q(\mathbf{0},\mathbf{1}) \equiv \mu(0,1) \pmod{q-1}$. On the other hand, lacking a lattice structure or a group action on $\mathcal{P}_G(\mathbb{F}_q^n)$, we had to restrict ourselves to reasonably large and well-understood subfamilies of this set for a similar reduction in size to $\mathrm{p}(n)$ modulo $q-1$ [2,3]. Let us emphasize that even the maximal admissible types of partial spreads are still unknown except in a few special cases, and hence, counting *all* Gaussian partitions of $\mathbb{F}_q^n$ modulo $q-1$ presents an insurmountable difficulty at the current level of our knowledge. In short, the statement $\mathrm{p}_q(n) \equiv \mathrm{p}(n) \pmod{q-1}$ remains a conjecture.

We were thus motivated to find a q-extension of $\mathcal{P}(\mathbf{n})$ to a natural class of the subspace partitions of $\mathbb{F}_q^n$, and consequently, of $\mathcal{P}(n)$ to a natural class of the Gaussian partitions of $\mathbb{F}_q^n$. Instead of an embedding, we explored the concept of several subspace partitions that could be seen as representing a given set partition. This led us to the last paper [4] in our program, which we will summarize next. Note that subspace partitions of $\mathbb{F}_q^n$ containing direct sums not only extend set partitions of various bases of $\mathbb{F}_q^n$, but provide generalizations of *flags* (nested sequences of subspaces with dimension increasing by 1) of $\mathbb{F}_q^n$ as well, which were studied more narrowly by Milne [16] and Bennett et al. [6] in connection with q-analogues of the Stirling numbers of the second kind. Most importantly, in [4], we deduced that since singleton G-orbits contain direct sums by definition, and every larger orbit has either only subspace partitions containing direct sums or no such partition, the first congruence holds for the cardinality of $\mathcal{P}_D(\mathbb{F}_q^n)$ as well. To summarize, we have so far proven:

Proposition 1 *([1,4]). Let $B(n)$, $B_q(n)$, and $B_{D,q}(n)$ be the cardinalities of $\mathcal{P}(\mathbf{n})$, $\mathcal{P}(\mathbb{F}_q^n)$, and $\mathcal{P}_D(\mathbb{F}_q^n)$ respectively. Then*

Table 1
q-analogies.

Classical object/symbol	$\mathbf{q}$-analogue/symbol
Set $\mathbf{n} = \{1, \ldots, n\}$	Vector space $\mathbb{F}_q^n$ with bases of n elements each
Subset of $\mathbf{n}$ with k elements	Subspace of $\mathbb{F}_q^n$ of dimension k
Number of k-subsets of $\mathbf{n}$: binomial coefficient $\binom{n}{k}$	Number of k-subspaces of $\mathbb{F}_q^n$: Gaussian binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$
Subset lattice (power set) of $\mathbf{n}$	Subspace lattice of $\mathbb{F}_q^n$
Set partition of $\mathbf{n}$	Subspace partition of $\mathbb{F}_q^n$
Lattice of set partitions of $\mathbf{n}$: $\mathcal{P}(\mathbf{n})$	Lattice of subspace partitions of $\mathbb{F}_q^n$: $\mathcal{P}(\mathbb{F}_q^n)$
Set partition of $\mathbf{n}$ with subsets $A_1, \ldots, A_m$	Subspace partition of $\mathbb{F}_q^n$ with a direct sum $W_1 \oplus \ldots \oplus W_m = \mathbb{F}_q^n$, where $\dim W_i = \|A_i\|$
Number of set partitions of $\mathbf{n}$: Bell number $B(n)$	Number of subspace partitions of $\mathbb{F}_q^n$: q-Bell number $B_q(n)$
	Number of subspace partitions of $\mathbb{F}_q^n$ containing direct sums: (D, q)-Bell number $B_{D,q}(n)$
Möbius number of $\mathcal{P}(\mathbf{n})$: $\mu(0, 1)$	Möbius number of $\mathcal{P}(\mathbb{F}_q^n)$: $\mu_q(\mathbf{0}, \mathbf{1})$
n	q-number of n: $[n]_q \stackrel{\text{def}}{=} (q^n - 1)/(q - 1)$
Integer partition of n	Gaussian partition of $\mathbb{F}_q^n$
Positive solution of $a_1x_1 + \cdots + a_hx_h = n$: element of $\mathcal{S}_1$ (Section 1.4)	Positive solution of $[a_1]_q u_1 + \cdots + [a_h]_q u_h = [n]_q$: element of $\mathcal{S}_q$ (Section 1.4)
Number of integer partitions of n: partition function $\mathrm{p}(n)$	Number of Gaussian partitions of $\mathbb{F}_q^n$: $\mathrm{p}_q(n)$ (We conjecture: $\mathrm{p}_q(n) \equiv \mathrm{p}(n) \pmod{q-1}$.)

$$B_q(n) \equiv B_{D,q}(n) \equiv B(n) \ \ (\mathit{mod}\ q - 1).$$

Moreover, if μ and μ_q denote the Möbius functions of the lattices $\mathcal{P}(\mathbf{n})$ and $\mathcal{P}(\mathbb{F}_q^n)$ respectively, then we have

$$\mu_q(\mathbf{0}, \mathbf{1}) \equiv \mu(0, 1) \ \ (\mathit{mod}\ q - 1).$$

Table 1 displays a summary of the combinatorial objects and their q-analogues of interest in this paper. The reader may refer to it for convenience.

1.3. Families of subspace partitions that contain direct sums

Since we will be generalizing the results in [4] in great detail, we shall only provide a brief version of them in this introductory section. The main concept is that for subspace partitions $\Pi \in \mathcal{P}(\mathbb{F}_q^n)$ of type $a^u b^v$, where the Diophantine equation $ax + by = n$ has nonnegative solutions (i.e., there exist integer partitions of n of the form $a^x b^y$ with nonnegative parts, so that it is conceivable that we might have a direct sum), we can be sure that Π contains a direct sum as long as v (equivalently, u) is contained in a union of two intervals, whose endpoints are given by polynomials in $\mathbb{Z}[q]$.

Theorem 1. *[4] Let q be a prime power, n, a, b be integers such that $n > a > b > 0$, and Π be a subspace partition of $\mathbb{F}_q^n$ of type $a^u b^v$. Define $\mathcal{S}$ to be the set of all solutions (x, y) of the Diophantine equation $ax + by = n$ with $x, y \geq 0$. Let $y_0 = \min_{(x,y)\in\mathcal{S}_1} y$, $y_M = \max_{(x,y)\in\mathcal{S}_1} y$, $x_0 = (n - ay_0)/b$, $x_M = (n - ay_M)/b$, and consider the intervals*

$$I_1 = \left[\frac{q^{by_0}-1}{q^b-1}, \frac{q^{by_M}-1}{q^b-1}\right] \quad and \quad I_2 = \left[\frac{q^n-q^{ax_0}}{q^b-1}, \frac{q^n-q^{ax_M}}{q^b-1}\right].$$

If $\mathcal{S} \neq \emptyset$ and $v \in I_1 \cup I_2$, then Π contains a direct sum. Conversely, if Π contains a direct sum, then $\mathcal{S} \neq \emptyset$.

When either a or b divides n, one of the two intervals is contained in the other. Otherwise, they are disjoint, with I_1 to the left of I_2. Now, Theorem 1 does not explicitly say that subspace partitions of the given types are the only ones with direct sums. However, there are many indications that the lower and upper limits of the intervals are important in their own right [4]. First, the lower limit for v, namely $(q^{by_0}-1)/(q^b-1)$, seems to be strict for *all* known subspace partitions of $\mathbb{F}_q^n$ of type $a^u b^v$ as far as we can tell. Second, we were able to construct a Frobenius partition that contains no direct sums, due to the fact that its Gaussian partition contains no integer partition of n (i.e., we have $\mathcal{S} = \emptyset$). Third, the upper bound is also significant, as we found an infinite class $\mathcal{C}$ of examples of subspace partitions of type $a^u b^v$ for which the solution set $\mathcal{S}$ is nonempty; however, v is strictly larger than $\max(I_1 \cup I_2)$ for each partition in $\mathcal{C}$, and no partition in $\mathcal{C}$ contains a direct sum.

The reader may consult [4] for the details of our claims in this section. Since subspace partitions with exactly one or two distinct subspace dimensions have been studied extensively, one might gain some insights from the proofs and examples therein.

1.4. Main theorem: generalization to any number of distinct subspace dimensions in a partition

Let h denote the number of distinct subspace dimensions $a_1, \ldots, a_h$ that appear in a subspace partition of $\mathbb{F}_q^n$, $\mathcal{S}_1$ denote the set of *positive* solutions of the Diophantine equation (see Remark 16 for an extension to nonnegative solutions)

$$a_1x_1 + \cdots + a_hx_h = n, \tag{2}$$

and $\mathcal{S}_q$ denote the set of *positive* solutions of the corresponding packing condition (1). Furthermore, we will use the notation $[r]_q$ for the Gaussian coefficient $(q^r-1)/(q-1)$ (see Section 3.1).

In order to state and prove the main theorem, we will introduce some underlying features that we were able to take for granted for $h = 2$. We shall devote the rest of this article to expanding upon the following concepts, then proving the main theorem, and finally, providing an infinite family of subspace partitions of Frobenius type for arbitrary h.

(i) The solutions of a homogeneous linear Diophantine equation in $h \geq 2$ variables form a Euclidean lattice L' of dimension $h-1$. As a result, the solutions of a nonhomogeneous linear Diophantine equation in at least two variables, if any, form an affine Euclidean lattice, L, which is a translation of the lattice L' of solutions of the associated homogeneous equation. It is understood that if $h = 1$, then $|L| = |L'| = 1$.

(ii) The following vector is in L', the lattice of solutions of the homogeneous Diophantine equation associated to Eq. (2):

$$\mathbf{v} = \begin{pmatrix} -a_2/d_2 \\ a_1/d_2 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3}$$

where $d_2 = \gcd(a_1, a_2)$ (see Section 2.1). This solution is the trivial extension for the case $h = 2$. Note that it is possible to permute the terms of Eq. (2), rename the coefficients, and utilize other similar vectors. This will give us the flexibility to potentially find more Gaussian partitions that belong to subspace partitions containing direct sums. Moreover, we will show that we can complete $\{\mathbf{v}\}$ to a basis of L', where each vector has a negative first component, followed by nonnegative components.

(iii) The solution set $\mathcal{S}_1$ of Eq. (2), if nonempty, can always be embedded into the solution set $\mathcal{S}_q$ via one of $h!$ different *coordinate maps* f_q. Without loss of generality, we may assume that the image of any solution $\mathbf{x} = (x_1, \ldots, x_h) \in \mathcal{S}_1$ is

$$\mathbf{X} = (X_1, \ldots, X_h) = f_q(\mathbf{x}) = \left(q^{a_2x_2+\cdots+a_hx_h}[x_1]_{q^{a_1}}, \ldots, q^{a_hx_h}[x_{h-1}]_{q^{a_{h-1}}}, [x_h]_{q^{a_h}}\right), \tag{4}$$

where $[x]_q$ is the Gaussian binomial coefficient $(q^x - 1)/(q - 1)$ (see Section 3.1). The complete set of coordinate maps is obtained by permuting the terms in Equation (2). For example, for $h = 2$, we have two intervals for the variable v in Theorem 1, associated with the 2! maps

$$f_q^{(1)}(x, y) = \left(q^{by}[x]_{q^a}, [y]_{q^b}\right) \quad \text{and} \quad f_q^{(2)}(x, y) = \left([x]_{q^a}, q^{ax}[y]_{q^b}\right), \tag{5}$$

corresponding to the permuted equations $ax + by = n$ and $by + ax = n$ respectively.

With this notation, we can now introduce Lemma 2 (a synthesis of Lemmas 19 and 21 that we proved in [4]), which forms the foundation of Theorem 1. It is in fact this result, rather than the condensed Theorem 1, that we are generalizing as Theorem 2 in this paper; in a one-dimensional lattice, it is very easy to describe unions of intervals, as we were able to do in Theorem 1. A description of the union of an unknown number of convex regions in higher dimensions would be inelegant and unnecessary in comparison.

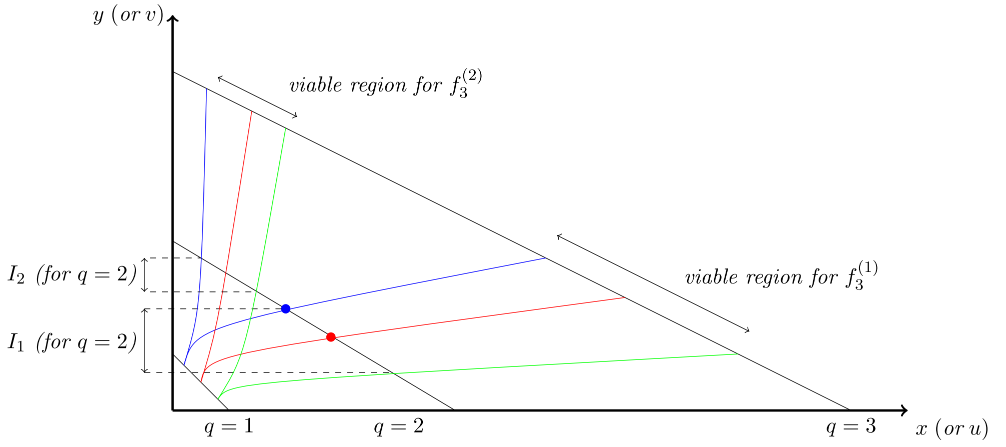

Fig. 1. Regions of solution lattice where direct sums are guaranteed. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Lemma 2 *([4]). Let a and b be distinct positive integers, and Π be a subspace partition of $\mathbb{F}_q^n$ with u subspaces of dimension a and v subspaces of dimension b so that (u, v) is a solution of the packing condition $[a]_q u + [b]_q v = [n]_q$. Suppose that (u, v) lies on the line joining the special solutions $f_q(x_j, y_j)$ and $f_q(x_{j+1}, y_{j+1})$, where $f_q = f_q^{(1)}$ (resp., $f_q = f_q^{(2)}$) in Eq. (5), and (x_j, y_j), (x_{j+1}, y_{j+1}) are two "consecutive" solutions of $ax+by = n$ with $x_j > x_{j+1}$ (resp., $y_j > y_{j+1}$). Then Π contains a direct sum with x_j subspaces of dimension a and y_j subspaces of dimension b.*

Example 4 below provides explicit constructions of some direct sums predicted by Lemma 2 in two subspace partitions of the same type. How Lemma 2 fits into the larger picture of Theorem 1 is explained in Example 7 and further illustrated in Fig. 1.

Note that there exists $\mathbf{v} = (v_1, v_2)$ such that $(x_j, y_j) + \mathbf{v} = (x_{j+1}, y_{j+1})$ and $v_1 < 0$, $v_2 > 0$ in the Lemma when $f_q = f_q^{(1)}$: it is the well-known vector $\mathbf{v} = (-b/d, a/d)$ from the theory of linear Diophantine equations in two variables, where $d = \gcd(a, b)$ (the signs are swapped for $f_q = f_q^{(2)}$). Our main theorem is thus a natural generalization of Lemma 2:

Theorem 2. *Let $a_1, \dots, a_h, n$ be fixed, distinct, positive integers, with $h \geq 1$, and Π be a subspace partition of $\mathbb{F}_q^n$ of type $a_1^{u_1} \cdots a_h^{u_h}$, so that $\mathbf{U} = (u_1, \dots, u_h) \in \mathcal{S}_q$, and let L' be the lattice of solutions of the homogeneous equation $a_1x_1 + \cdots + a_hx_h = 0$. Suppose that $\mathbf{U}$ is in the convex hull of $\{\mathbf{X}, \mathbf{Y}^{(1)}, \dots, \mathbf{Y}^{(\ell)}\} \subset \mathcal{S}_q$, where $\mathbf{X} = f_q(\mathbf{x})$ and $\mathbf{Y}^{(i)} = f_q\left(\mathbf{y}^{(i)}\right) = f_q\left(\mathbf{x} + \mathbf{v}^{(i)}\right)$ for some $\mathbf{x}, \mathbf{y}^{(i)} \in \mathcal{S}_1$, with ℓ ($\ell \leq h-1$) linearly independent vectors $\mathbf{v}^{(i)} \in L'$ satisfying $v_1^{(i)} < 0$ and $v_j^{(i)} \geq 0$ for $1 \leq i \leq \ell$ and $2 \leq j \leq h$. Then Π contains a direct sum consisting of x_k direct summands of dimension a_k for $1 \leq k \leq h$.*

Remark 3.

(i) The theorem easily expands to include *nonnegative* solutions $\mathbf{x}$, $\mathbf{y}^{(i)}$ in many cases. See Remark 16 after the proof.

(ii) There are in general many sets of ℓ linearly independent vectors $\mathbf{v}^{(i)}$ $(\ell \leq h-1)$ as described in the hypothesis; see the vector $\mathbf{v}$ shown in Eq. (3) and Proposition 5. Note that for each i, $1 \leq i \leq \ell$, it follows from the hypothesis that at least one entry $v_j^{(i)}$ of $\mathbf{v}^{(i)}$ needs to be strictly positive.

(iii) Strictly speaking, the choice of the coordinate map f_q given in Eq. (4) among all others is correlated with the choice of ordering of the a_j. If a permutation σ of the terms of Eq. (2) is employed instead, then we can replace a_j by $a_{\sigma(j)}$, x_j by $x_{\sigma(j)}$, u_j by $u_{\sigma(j)}$, and $v_j^{(i)}$ by $v_{\sigma(j)}^{(i)}$ for all j, $1 \leq j \leq h$. For example, the formula $f_q^{(2)}(x,y)$ for solutions (x,y) of $ax+by=n$ in Eq. (5) is essentially the same as the formula $f_q^{(1)}(y,x)$ for solutions (y,x) of $by+ax=n$; the two are related by the transposition $\sigma=(12)$.

(iv) Although we know that q must be a prime power, the arithmetic part of the proof does not depend on this fact, nor on the particular method of construction of the subspace partition. However, see Example 4 for other types of direct sums not predicted by this Theorem.

We should also point out that while Theorem 2 is a universal statement that applies to any prime power $q \geq 2$, it does not exhaust all possibilities for the existence of a direct sum. The existence of a direct sum may depend on the particular subspace partition, not just the type, and the multisets of dimensions of direct sums in a subspace partition may not be unique. For example, in Section 1.2, we mentioned very special subspace partitions that contain direct sums: those with some subspaces spanned by the parts of a set partition of a given basis of $\mathbb{F}_q^n$, together with the remaining 1-D subspaces.

Example 4. Consider the integer partition 2^3 of $n=6$ and a basis $\mathcal{B}=\{\mathbf{w}_1,\ldots,\mathbf{w}_6\}$ of $\mathbb{F}_q^6$. Let $\langle I\rangle$ denote the subspace of $\mathbb{F}_q^6$ spanned by a subset I of $\mathcal{B}$. Then the subspace partition Π_1 of $\mathbb{F}_q^6$ consisting of the 2-subspaces

$$\langle \mathbf{w}_1,\mathbf{w}_2\rangle,\ \langle \mathbf{w}_3,\mathbf{w}_4\rangle,\ \langle \mathbf{w}_5,\mathbf{w}_6\rangle$$

and the remaining 1-D subspaces contains a direct sum of dimensions $2+2+2$. However, the subspace partition Π_2 (of the same type as Π_1) consisting of the 2-subspaces

$$\langle \mathbf{w}_1,\mathbf{w}_2\rangle,\ \langle \mathbf{w}_3,\mathbf{w}_4\rangle,\ \langle \mathbf{w}_1+\mathbf{w}_3,\mathbf{w}_2+\mathbf{w}_4\rangle$$

together with the remaining 1-D subspaces cannot contain a direct sum of dimensions $2+2+2$, because the ordinary sum of the only three subspaces of dimension two does not contain $\mathbf{w}_5$ or $\mathbf{w}_6$. This does not clash with our assertion that we can always predict some direct sums of certain dimensions regardless of construction. In this case, the Gaussian partition $T = T(\Pi_1) = T(\Pi_2) = 2^3 1^{q^5+q^4+q^3+(q-3)q+(q-2)}$ falls between

$$f_q^{(2)}(1,4) = \left(1, q^5+q^4+q^3+q^2\right) \quad \text{and} \quad f_q^{(2)}(2,2) = \left(q^2+1, q^5+q^4\right),$$

where $(1,4)$ and $(2,2)$ are consecutive solutions of the integer partition equation $2x+y = 6$ with $4 > 2$ (see Eq. (5) and Lemma 2). If we considered only the type T, then we would expect a direct sum of dimensions $2+1+1+1+1$ by Lemma 2, which does exist in both Π_1 and Π_2: for example, we have

$$\langle \mathbf{w}_1, \mathbf{w}_2 \rangle \oplus \langle \mathbf{w}_1 + \mathbf{w}_3 \rangle \oplus \langle \mathbf{w}_2 + \mathbf{w}_4 \rangle \oplus \langle \mathbf{w}_4 + \mathbf{w}_5 \rangle \oplus \langle \mathbf{w}_4 + \mathbf{w}_6 \rangle = \mathbb{F}_q^6$$

in Π_1 and

$$\langle \mathbf{w}_1, \mathbf{w}_2 \rangle \oplus \langle \mathbf{w}_5 \rangle \oplus \langle \mathbf{w}_6 \rangle \oplus \langle \mathbf{w}_3 + \mathbf{w}_5 \rangle \oplus \langle \mathbf{w}_4 + \mathbf{w}_6 \rangle = \mathbb{F}_q^6$$

in Π_2.

2. The lattice of solutions of a diophantine equation

2.1. The Bond solutions

It is well known that if a Diophantine equation in $h \geq 2$ variables has solutions, then these solutions form an affine Euclidean lattice L of rank $h-1$. There are many algorithms that describe the general solution of a Diophantine equation (2), where $h, n, a_1, \ldots, a_h \in \mathbb{Z}$, $h \geq 2$, and the a_i are distinct and nonzero. The following formula is due to J. Bond [8]: Let $d_i = \gcd(a_1, \ldots, a_i)$, with $d_1 = a_1$, and $x_i^{(j)}$ be non-unique Bézout coefficients such that

$$a_1 x_1^{(i)} + \cdots + a_i x_i^{(i)} = d_i$$

for $1 \leq i \leq h$ (then $x_1^{(1)} = 1$). For Eq. (2) to have solutions, we must have $n = d_h t_h$ for some $t_h \in \mathbb{Z}$. The Bond solutions are of the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{h-2} \\ x_{h-1} \\ x_h \end{pmatrix} = \begin{pmatrix} x_1^{(h)} t_h \\ x_2^{(h)} t_h \\ x_3^{(h)} t_h \\ \vdots \\ x_{h-2}^{(h)} t_h \\ x_{h-1}^{(h)} t_h \\ x_h^{(h)} t_h \end{pmatrix} + i_{h-1} \begin{pmatrix} -x_1^{(h-1)} \frac{a_h}{d_h} \\ -x_2^{(h-1)} \frac{a_h}{d_h} \\ -x_3^{(h-1)} \frac{a_h}{d_h} \\ \vdots \\ -x_{h-2}^{(h-1)} \frac{a_h}{d_h} \\ -x_{h-1}^{(h-1)} \frac{a_h}{d_h} \\ \frac{d_{h-1}}{d_h} \end{pmatrix} + i_{h-2} \begin{pmatrix} -x_1^{(h-2)} \frac{a_{h-1}}{d_{h-1}} \\ -x_2^{(h-2)} \frac{a_{h-1}}{d_{h-1}} \\ -x_3^{(h-2)} \frac{a_{h-1}}{d_{h-1}} \\ \vdots \\ -x_{h-2}^{(h-2)} \frac{a_{h-1}}{d_{h-1}} \\ \frac{d_{h-2}}{d_{h-1}} \\ 0 \end{pmatrix} \tag{6}$$

$$+\cdots+i_2\begin{pmatrix} -x_1^{(2)}\frac{a_3}{d_3} \\ -x_2^{(2)}\frac{a_3}{d_3} \\ \frac{d_2}{d_3} \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}+i_1\begin{pmatrix} -\frac{a_2}{d_2} \\ \frac{d_1}{d_2} \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $i_1, \ldots, i_{h-1}$ are arbitrary integers (we have changed the signs of all vectors but the first one and absorbed the sign in the coefficients i_j so that the notation fits our conventions). We may write this in symbols as

$$\mathbf{x} = \mathbf{x}_0 + i_{h-1}\mathbf{v}^{(h-1)} + i_{h-2}\mathbf{v}^{(h-2)} + \cdots + i_2\mathbf{v}^{(2)} + i_1\mathbf{v}^{(1)},$$

where $\mathbf{x}_0$ is a particular solution of the nonhomogeneous equation, and $\{\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(h-1)}\}$ is a "primitive basis" of the solution lattice L' of the associated homogeneous equation. This is similar to the real solution set of a single linear equation in h real variables. Note that the last vector, $\mathbf{v}^{(1)}$, is the vector $\mathbf{v}$ we mention in Eq. (3), where $d_1 = a_1$.

2.2. A special basis of the solution lattice

Let us show that it is always possible to find a basis for the associated homogeneous lattice consisting of vectors with a negative component in the first position and nonnegative components in the remaining positions.

Proposition 5. *Let $a_1x_1 + \cdots + a_hx_h = 0$ be a Diophantine equation with distinct positive integer coefficients a_i, where $h \geq 2$. Then there exists a basis of the solution lattice L' consisting of $h-1$ vectors such that the first component is negative and the remaining components are nonnegative in each case.*

Proof. We shall send the basis $\{\mathbf{v}^{(i)} : 1 \leq i \leq h-1\}$ given by the Bond solutions (6) to another by a matrix that is invertible over $\mathbb{Z}$. Note that $\mathbf{v}^{(1)}$ has a negative first component and a positive second component, followed by zeros. Subsequently, all other $\mathbf{v}^{(i)}$ have arbitrary integers in the first i components, a positive integer in the $(i+1)$st component, and zeros elsewhere. We leave $\mathbf{v}^{(1)}$ as is, and add it sufficiently many times -if necessary- to $\mathbf{v}^{(2)}$ so that the first component becomes negative and the second becomes positive by the Archimedean property. This process does not change the positive third component and the zeros below. Hence, there exists a nonnegative integer N_1 such that $N_1\,\mathbf{v}^{(1)} + \mathbf{v}^{(2)} \in L'$ has the desired properties. Next, we add this new vector sufficiently many times to $\mathbf{v}^{(3)}$, if necessary, to obtain a negative first component and positive second and third components, and keep the positive fourth component as well as the zeros underneath intact. The newly constructed vector in L' is thus of the form

$N_2(N_1\,\mathbf{v}^{(1)}+\mathbf{v}^{(2)})+\mathbf{v}^{(3)}$ for some $N_1, N_2 \geq 0$, etc. The transformation is represented by the $(h-1)\times(h-1)$ matrix

$$\begin{pmatrix} 1 & N_1 & N_1N_2 & \cdots & N_1N_2\cdots N_{h-2} \\ 0 & 1 & N_2 & \cdots & N_2N_3\cdots N_{h-2} \\ 0 & 0 & 1 & \cdots & N_3N_4\cdots N_{h-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N_{h-2} \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

(in the given ordered basis), which is invertible over $\mathbb{Z}$. □

3. The geometry of the coordinate maps

3.1. Some properties of the q-numbers

For a nonnegative integer a, the Gaussian binomial coefficient

$$[a]_q = \begin{bmatrix} a \\ 1 \end{bmatrix}_q = \frac{q^a-1}{q-1} = q^{a-1}+\cdots+q+1,$$

sometimes called a *q-number*, is the number of 1-D subspaces of $\mathbb{F}_q^a$. As $q\to 1$, we have $[a]_q \to a$. Although it is more common to encounter quotients such as $(q^a-1)/(q-1)$ in the literature, the ease of working with q-numbers cannot be overstated. First, we note that the packing condition (1) can be written in the more compact form

$$[a_1]_q\,u_1+\cdots+[a_h]_q\,u_h = [n]_q \tag{7}$$

after dividing both sides by $q-1$, and letting $q\to 1$ gives us back the *integer partition equation* shown in (2):

$$a_1x_1+\cdots+a_hx_h = n$$

(the variables having been changed from u_i to x_i for consistency of notation). Next, we showed in [4] that the identities

$$\frac{[xy]_q}{[y]_q} = [x]_{q^y} \iff [xy]_q = [x]_{q^y}[y]_q = [x]_q[y]_{q^x} \tag{8}$$

can be used to obtain the special solutions f_q of the packing condition in Eq. (7). Moreover, since

$$\gcd\left(q^a-1, q^b-1\right) = q^{\gcd(a,b)}-1 \;\text{ and }\; \gcd\left([a]_q,[b]_q\right) = [\gcd(a,b)]_q \in \mathbb{Z}[q],$$

Bézout identities of the form $[a]_q u(q) + [b]_q v(q) = [\gcd(a,b)]_q$ are reduced to Bézout identities for a and b as $q \to 1$. All principles we have discussed here can be extended to three or more q-numbers. Many generic results about Gaussian partitions are traditionally expressed in terms of polynomials in q thanks to the availability of solutions of Eq. (7), such as the Bond solutions (6) that make use of Bézout coefficients, in $\mathbb{Z}[q]^h$.

3.2. The coordinate maps from $\mathcal{S}_1$ into $\mathcal{S}_q$

Let us consider the following foundational statement from the paper [4], with the order of variables reversed:

Proposition 6. *Let $a_1, \ldots, a_h, n$ be distinct positive integers, with $h \geq 2$, and $\gcd(a_1, \ldots, a_h)$ dividing n. If $\mathbf{x} = (x_1, \ldots, x_h)$ is a nonnegative solution of the Diophantine equation*

$$a_1 x_1 + \cdots + a_h x_h = n,$$

then

$$\mathbf{X} = (X_1, \ldots, X_h) = f_q(\mathbf{x}) = (q^{a_2 x_2 + \cdots + a_h x_h}[x_1]_{q^{a_1}}, \ldots, q^{a_h x_h}[x_{h-1}]_{q^{a_{h-1}}}, [x_h]_{q^{a_h}})$$

is a solution of the packing condition

$$[a_1]_q u_1 + \cdots + [a_h]_q u_h = [n]_q$$

for all integers $q \geq 2$, where the entries $X_i = X_i(q) \in \mathbb{Z}[q]$ are polynomials with nonnegative values for $q \geq 1$. In particular, $X_i(1) = x_i$ for all i, and the packing condition is reduced to the first Diophantine equation as $q \to 1$. When $a_1 < \cdots < a_h$, the solution $\mathbf{X}$ is a known Gaussian partition, corresponding to a constructible subspace partition of $\mathbb{F}_q^n$ that we called "basic" in [4].

Proof. Given the equation

$$a_1 x_1 + \cdots + a_{h-1} x_{h-1} + a_h x_h = n,$$

it is straightforward to prove that

$$[n]_q = [a_h\, x_h]_q + q^{a_h\, x_h}[a_{h-1}\, x_{h-1}]_q + \cdots + q^{a_2 x_2 + \cdots + a_h x_h}[a_1 x_1]_q$$

is a true identity. We work on the right-hand side and use Eq. (8) several times:

$$\begin{aligned} [n]_q &= [a_h]_q\, [x_h]_{q^{a_h}} + [a_{h-1}]_q\, q^{a_h\, x_h}\, [x_{h-1}]_{q^{a_{h-1}}} + \cdots + [a_1]_q\, q^{a_2 x_2 + \cdots + a_h x_h}\, [x_1]_{q^{a_1}} \\ &= [a_h]_q\, X_h(q) + [a_{h-1}]_q\, X_{h-1}(q) + \cdots + [a_1]_q\, X_1(q). \quad \square \end{aligned}$$

Example 7 *($h = 2$).* An example of a Diophantine equation with two variables where there are no zero coordinates among the solutions, and hence, there are two distinct intervals I_1 and I_2 as described in Theorem 1, is $2x+3y = 17$. Fig. 1 shows a not-to-scale drawing of the lines $2x+3y = 17$ ($q = 1$), $[2]_2u+[3]_2v = [17]_2$, and $[2]_3u+[3]_3v = [17]_3$, together with the extensions of the positive integer solutions

$$(x, y) = (1, 5),\ (4, 3),\ (7, 1)$$

of the first equation to the $2! = 2$ sets of solutions (among many) on the lines corresponding to $q = 2$ and $q = 3$ via the coordinate maps $f_q^{(1)}$ and $f_q^{(2)}$ (see Eq. (5)).

There are two sets of corresponding solutions of the packing condition for each of $q = 2$ and $q = 3$, depending on how we order the variables. The first set (closer to the x- or u-axis) correspond to

$$f_q^{(1)}(x, y) = (X, Y) = (q^{3y}[x]_{q^2}, [y]_{q^3}),$$

and we have three (basic) Gaussian partitions

$$\begin{aligned} f_q^{(1)}(1, 5) &= (q^{15}, q^{12} + q^9 + q^6 + q^3 + 1) \\ f_q^{(1)}(4, 3) &= (q^{15} + q^{13} + q^{11} + q^9, q^6 + q^3 + 1) \\ f_q^{(1)}(7, 1) &= (q^{15} + q^{13} + q^{11} + q^9 + q^7 + q^5 + q^3, 1). \end{aligned}$$

In the notation of Lemma 2, the interval I_1 is the region on the v-axis between $v = 1$ and $v = q^{12} + q^9 + q^6 + q^3 + 1$ (see Fig. 1, where both intervals are indicated for $q = 2$). The second set (closer to the y- or v-axis) is

$$f_q^{(2)}(x, y) = (X, Y) = ([x]_{q^2}, q^{2x}[y]_{q^3}),$$

with

$$\begin{aligned} f_q^{(2)}(1, 5) &= (1, q^{14} + q^{11} + q^8 + q^5 + q^2), \\ f_q^{(2)}(4, 3) &= (q^6 + q^4 + q^2 + 1, q^{14} + q^{11} + q^8), \\ f_q^{(2)}(7, 1) &= (q^{12} + q^{10} + q^8 + q^6 + q^4 + q^2 + 1, q^{14}). \end{aligned}$$

The interval I_2 of Lemma 2 is then the line segment between $v = q^{14}$ and $v = q^{14}+q^{11}+q^8+q^5+q^2$ on the v-axis. In general, the theory of subspace partitions does not guarantee that the last three solutions above are actual Gaussian partitions. In particular, since for any subspace partition Π the number of subspaces of smallest dimension is at least $q + 1$ (e.g., see [15, Theorem 1]), it follows that $f^{(2)}(1, 5)$ is not a Gaussian partition (but the remaining two are). The intervals I_1 and I_2 on the v-axis for $q = 2$ are those for which any subspace partition of $\mathbb{F}_2^{17}$ of type 2^u3^v, where $v \in I_1 \cup I_2$, must contain a direct sum. In fact, we can tell what the dimensions of the subspaces of the direct sum (that is, the one we predict in Theorem 1) should be, depending on where v falls in I_1 or I_2. For example, if (u,v) is on the line between $f_2^{(1)}(4,3)$ and $f_2^{(1)}(1,5)$ (the red and blue dots in Fig. 1 respectively) for a subspace partition Π of $\mathbb{F}_2^{17}$ of type 2^u3^v, where $x_j = 4 > 1 = x_{j+1}$, then Π has a direct sum with four subspaces of dimension 2 and three subspaces of dimension 3 (see Lemma 2).

In the next few sections, we will show that the coordinate maps f_q not only extend nonnegative integer solutions of $ax + by = n$ to nonnegative integer solutions of $[a]_q u + [b]_q v = [n]_q$ (where the positive integer q is not necessarily a prime power), but also map the part of the line $ax + by = n$ that lies in the first quadrant homeomorphically to the parts of all lines $[a]_q u + [b]_q v = [n]_q$, where $[a]_q, [b]_q, [n]_q$ are defined continuously, that lie in the first quadrant. In fact, this result will be proven more generally for hyperplanes defined by linear equations in h variables.

3.3. Continuous q-numbers

Expanding the familiar formula

$$[x]_q = \frac{q^x - 1}{q - 1}$$

for $x \in \{0, 1, 2, \dots\}$ and a prime power q, one can define

$$[x]_q = \frac{q^x - 1}{q - 1}$$

for $q, x \in \mathbb{R}$ and $q \geq 1$, because the limit

$$\lim_{q \to 1} \frac{q^x - 1}{q - 1} = \lim_{q \to 1} \frac{xq^{x-1}}{1} = x$$

exists.

Definition 8. Let $x, q \in \mathbb{R}$, with $x \in \mathbb{R}$ and $q \geq 1$. The *continuous q-number* (of x) is defined to be

$$[x]_q = \begin{cases} \frac{q^x-1}{q-1} & \text{if } q > 1, \text{ and} \\ x & \text{if } q = 1. \end{cases}$$

Lemma 9. *For $x, q \in \mathbb{R}$ and $q \geq 1$, the continuous q-number $[x]_q$ is a continuous function of x and q.*

Lemma 10. *For real numbers r, s, q, where $q \geq 1$, we have*

$$[rs]_q = [r]_{q^s}[s]_q = [r]_q[s]_{q^r}.$$

3.4. Coordinate maps as homeomorphisms between $H_1 \cap K$ and $H_q \cap K$

Let H_q be the hyperplane in $\mathbb{R}^h$ defined by the packing condition (7), where substituting $q = 1$ gives us the hyperplane H_1 in $\mathbb{R}^h$ defined by Eq. (2). Moreover, let K denote the positive orthant of $\mathbb{R}^h$. Regarding our notation introduced just before Theorem 2, note that if L_1 and L_q denote the solution lattices of Equations (2) and (7) respectively, then the positive solutions of interest are $\mathcal{S}_1 = L_1 \cap K \subset H_1 \cap K$ and $\mathcal{S}_q = L_q \cap K \subset H_q \cap K$.

Corollary 11 *(Generalization of Proposition 6). Let $a_1, \ldots, a_h, n$ be distinct positive real numbers, with $h \geq 2$. If $\mathbf{x} = (x_1, \ldots, x_h)$ is a real solution of the equation*

$$a_1x_1 + \cdots + a_hx_h = n,$$

then

$$\mathbf{X} = (X_1, \ldots, X_h) = f_q(\mathbf{x}) = (q^{a_2x_2+\cdots+a_hx_h}[x_1]_{q^{a_1}}, \ldots, q^{a_hx_h}[x_{h-1}]_{q^{a_{h-1}}}, [x_h]_{q^{a_h}})$$

is a solution of

$$[a_1]_q u_1 + \cdots + [a_h]_q u_h = [n]_q$$

for all real numbers $q \geq 1$.

Proof. Imitate the proof of Proposition 6 using Definition 8 and Lemma 10. □

Definition 12. The continuous functions $f_q : H_1 \to H_q$ defined in Corollary 11 for fixed $q \geq 1$ will be called *coordinate maps*. There are $h!$ of them, corresponding to the permutations of the terms of the Diophantine equation $a_1x_1 + \cdots + a_hx_h = n$.

Proposition 13. *Let f_q be any one of the $h!$ transformations that send integer solutions of $a_1x_1+\cdots+a_hx_h = n$ in $H_1 \cap K$ to integer solutions of $[a_1]_q u_1+\cdots+[a_h]_q u_h = [n]_q$ in $H_q \cap K$ for some integer $q \geq 2$. Then the continuous coordinate map f_q of Definition 8 is a homeomorphism between $H_1 \cap K$ and $H_q \cap K$ as a real function of the x_i and for a fixed real value of $q > 1$. All f_q are reduced to the identity map as $q \to 1$, which justifies the notation H_1 and $\mathcal{S}_1$.*

Proof. We can invert each coordinate map f_q that sends the hyperplane H_1 to a hyperplane H_q, where $q > 1$ is real, using induction on i, with i descending from h to 1. Let

$$f_q(x_1, \ldots, x_h) = (u_1, \ldots, u_h) = \left(q^{a_2x_2+\cdots+a_hx_h}[x_1]_{q^{a_1}}, \cdots, q^{a_hx_h}[x_{h-1}]_{q^{a_{h-1}}}, [x_h]_{q^{a_h}}\right)$$

for some $q > 1$. Then

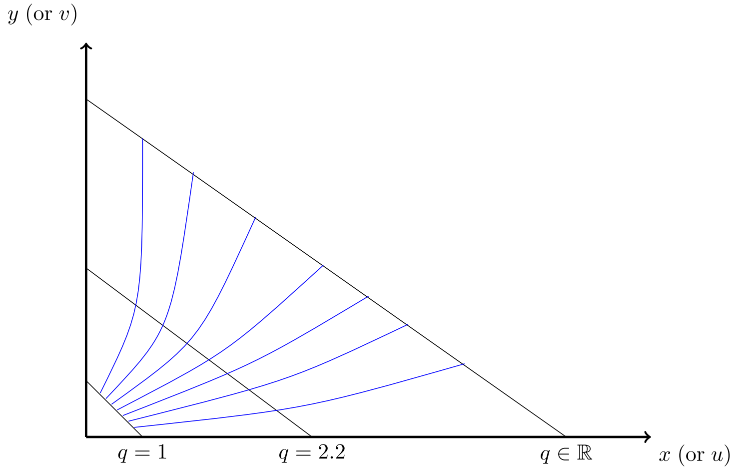

Fig. 2. Continuous deformation of hyperplanes.

$$u_h = \frac{q^{a_h x_h} - 1}{q^{a_h} - 1} \Rightarrow x_h = \frac{\log_q \left[u_h \left(q^{a_h} - 1\right) + 1\right]}{a_h}.$$

Assume that $(x_{k+1}, \ldots, x_h)$ has been solved for some $k \geq h - 1$. We solve for x_k:

$$u_k = q^{a_{k+1}x_{k+1} + \cdots + a_h x_h} \frac{q^{a_k x_k} - 1}{q^{a_k} - 1}$$
$$\Rightarrow x_k = \frac{\log_q \left[u_k \left(q^{a_k} - 1\right) q^{-(a_{k+1}x_{k+1} + \cdots + a_h x_h)} + 1\right]}{a_k}.$$

This shows that f_q^{-1} exists on the range of f_q. Also, given $(u_1, \ldots, u_h) \geq (0, \ldots, 0)$, we can see that $(x_1, \ldots, x_h)$ defined as above dominates the zero vector. Indeed, since $q > 1$, the a_i are positive integers, and all $u_h \geq 0$, we may conclude that each x_k is nonnegative. □

Once again, in the not-to-scale Fig. 2, we see how a coordinate map f_q continuously deforms the line $ax + by = n$ (at $q = 1$) into all lines $[a]_q u + [b]_q v = [n]_q$ for real $q > 1$ (in the first quadrant) and the points on the line $ax + by = n$ into points on all lines $[a]_q u + [b]_q v = [n]_q$ for real $q > 1$, though not along straight lines. A similar result is achieved for higher-dimensional hyperplanes. Although f_q is clearly not linear in $\mathbf{x}$ for fixed q, we can prove the following by showing that the derivative of $[x]_q$ is positive:

Lemma 14. *The continuous function $[x]_q$ is strictly increasing for $x > 0$ and fixed $q > 1$.*

Corollary 15. *Let $\mathbf{x}$ and $\mathbf{y}$ be points of real nonnegative components on the hyperplane H_1 given by $a_1x_1 + \cdots + a_hx_h = n$, with a_i, h, n as before, and $h \geq 2$. Assume that $x_1 > y_1$ and $x_j \leq y_j$ for $2 \leq j \leq h$. Then the order relationships of x_j and y_j are preserved for the nonnegative components $X_j = X_j(\mathbf{x})$ and $Y_j = Y_j(\mathbf{y})$ of $\mathbf{X} = f_q(\mathbf{x})$ and $\mathbf{Y} = f_q(\mathbf{y})$*

in H_q *for all* j, $1 \le j \le h$. *In particular, if we define* $\mathbf{V} = \mathbf{Y} - \mathbf{X}$, *then* $V_1 < 0$, $V_2 > 0$, *and* $V_j \ge 0$ *for* $3 \le j \le h$, *if any.*

Proof. Induction: the statement is true for $j = h$ by Lemma 14. Assume that we have proven the statement for $k+1 \le j \le h$ for some $k \ge 2$. Let us prove it for $j = k$: we have

$$V_k = Y_k(\mathbf{y}) - X_k(\mathbf{x}) = q^{a_h\, y_h + \cdots + a_{k+1}\, y_{k+1}}[y_k]_{q^{a_k}} - q^{a_h\, x_h + \cdots + a_{k+1}\, x_{k+1}}[x_k]_{q^{a_k}} \ge 0$$

since

$$q > 1, \quad a_h\, y_h + \cdots + a_{k+1}\, y_{k+1} \ge a_h\, x_h + \cdots + a_{k+1}\, x_{k+1}, \quad \text{and} \quad [y_k]_{q^{a_k}} \ge [x_k]_{q^{a_k}}$$

by Lemma 14. Finally, since all entries except the first one in $f_q(\mathbf{x})$ are less than or equal to their counterparts in $f_q(\mathbf{y})$, the first entries must be reversed in order, so that the real h-tuples $f_q(\mathbf{x})$ and $f_q(\mathbf{y})$ still satisfy the -continuous- packing condition (7). In fact, as at least one pair (x_j, y_j) must satisfy $x_j < y_j$ with $2 \le j \le h$, we must have

$$[y_2]_{q^{a_2}} > [x_2]_{q^{a_2}} \quad \text{or} \quad a_h\, y_h + \cdots + a_3\, y_3 > a_h\, x_h + \cdots + a_3\, x_3,$$

so that the second and first inequalities are strict: $V_2 > 0$, and hence, $V_1 < 0$. The last statement is true as both $\mathbf{X}$ and $\mathbf{Y}$ are solutions of Eq. (7). □

4. Proof of Theorem 2

Since we proved Theorem 2 for $h = 1, 2$ in [4], we may assume that $h \ge 3$. Let us now finish the proof of the theorem.

Proof. Assume that $\mathbf{U} = (u_1, \ldots, u_h)$ is in the convex hull of

$$\mathbf{X} = (X_1, \ldots, X_h) = f_q(\mathbf{x}) = \left(q^{a_2 x_2 + \cdots + a_h x_h}[x_1]_{q^{a_1}}, \ldots, q^{a_h x_h}[x_{h-1}]_{q^{a_{h-1}}}, [x_h]_{q^{a_h}}\right)$$

together with ℓ additional solutions (with $1 \le \ell \le h-1$) $\mathbf{Y}^{(i)} = f_q\left(\mathbf{y}^{(i)}\right) = f_q\left(\mathbf{x} + \mathbf{v}^{(i)}\right)$ of the packing condition for linearly independent vectors $\mathbf{v}^{(i)}$ as described in the hypothesis. Then all vectors $\mathbf{V}^{(i)} = \mathbf{Y}^{(i)} - \mathbf{X}$ satisfy the properties in Corollary 15: for each i with $1 \le i \le \ell$, we have $V_1^{(i)} < 0$, $V_2^{(i)} > 0$, and $V_j^{(i)} \ge 0$ for $3 \le j \le h$. Let

$$\mathbf{U} = \lambda_0\, \mathbf{X} + \lambda_1 \mathbf{Y}^{(1)} + \cdots + \lambda_\ell \mathbf{Y}^{(\ell)} \quad (1 \le \ell \le h-1)$$

for unique $\lambda_i \in \mathbb{R}$ such that

$$0 \le \lambda_i \le 1 \quad \text{and} \quad \sum_{i=0}^{\ell} \lambda_i = 1.$$

Thus, since $\mathbf{Y}^{(i)} = \mathbf{X} + \mathbf{V}^{(i)}$ for $1 \leq i \leq \ell$, we must have

$$\mathbf{U} = \mathbf{X} + \lambda_1 \mathbf{V}^{(1)} + \cdots + \lambda_\ell \mathbf{V}^{(\ell)},$$

where $0 \leq \sum_{i=1}^{\ell} \lambda_i \leq 1$. Note that this condition does not exclude the case $\mathbf{U} = \mathbf{X}$. Also, for later use, let (μ) be the superscript of a vector $V^{(i)}$ for which the first component, $V_1^{(\mu)}$, is minimum among all first components. Then it must be true that

$$u_1 = X_1 + \sum_{i=1}^{\ell} \lambda_i V_1^{(i)} \geq X_1 + \left(\sum_{i=1}^{\ell} \lambda_i \right) V_1^{(\mu)} \geq X_1 + V_1^{(\mu)} = Y_1^{(\mu)} \tag{9}$$

(because $\sum_{i=1}^{\ell} \lambda_i \leq 1$ and $V_1^{(\mu)} < 0$).

We shall now start building the direct sum from the bottom up. Suppose that

$$\widehat{W}^{(a_h)} = W_1^{(a_h)} \oplus \cdots \oplus W_t^{(a_h)}$$

is a maximal direct sum that consists of t a_h-D subspaces of Π out of all u_h of them. As $u_h \geq 1$, we know that $t \geq 1$. Each of the remaining $u_h - t$ subspaces must have an intersection of at least dimension 1 with this direct sum by maximality. That is, there exist at least $u_h - t$ available 1-D subspaces in $\widehat{W}^{(a_h)}$, which houses $[a_h t]_q$ of them in all, but the $t[a_h]$ subspaces that are already in $W_1^{(a_h)}, \ldots, W_t^{(a_h)}$ cannot be used, because mutual intersections of the subspaces of Π are the zero subspace by definition. We write this condition counting the available 1-D subspaces for anticipated intersections with the direct sum as

$$\begin{aligned}
& [a_h t]_q - t[a_h]_q \geq u_h - t \\
\Rightarrow\ & [a_h t]_q + t \geq u_h + t[a_h]_q \geq u_h + t \\
\Rightarrow\ & [a_h t]_q \geq u_h \geq X_h = [x_h]_{q^{a_h}} \geq 0 \\
\Rightarrow\ & q^{a_h t - 1} + \cdots + q + 1 \geq q^{a_h(x_h - 1)} + \cdots + q^{a_h} + 1 \\
\Rightarrow\ & a_h t - 1 \geq a_h(x_h - 1) \\
\Rightarrow\ & a_h t > a_h t - 1 \geq a_h(x_h - 1) \\
\Rightarrow\ & t > x_h - 1 \\
\Rightarrow\ & t \geq x_h.
\end{aligned}$$

Without loss of generality, choose the first x_h direct summands, and complete this truncated direct sum to a maximal subspace by using direct summands from among the a_{h-1}-D subspaces of Π, etc.

By induction, assume that we already have a direct sum

$$[W_1^{(a_h)} \oplus \cdots \oplus W_{x_h}^{(a_h)}] \oplus \cdots \oplus [W_1^{(a_k)} \oplus \cdots \oplus W_{x_k}^{(a_k)}]$$

made up of direct summands from Π, where $3 \leq k \leq h$, and the superscripts indicate the dimensions of the subspaces. Let us add as many direct summands as possible from Π of dimension a_{k-1} and obtain a direct sum

$$[W_1^{(a_h)} \oplus \cdots \oplus W_{x_h}^{(a_h)}] \oplus \cdots \oplus [W_1^{(a_k)} \oplus \cdots \oplus W_{x_k}^{(a_k)}] \oplus [W_1^{(a_{k-1})} \oplus \cdots \oplus W_t^{(a_{k-1})}],$$

where $t \geq 0$. Again, by counting the available 1-D subspaces, we must similarly have

$$\begin{aligned}
&[a_h x_h + \cdots + a_k x_k + a_{k-1} t]_q - x_h[a_h]_q - \cdots - x_k[a_k]_q - t[a_{k-1}]_q \geq u_{k-1} - t \\
\Rightarrow &[a_h x_h + \cdots + a_k x_k + a_{k-1} t]_q + t \geq u_{k-1} + x_h[a_h]_q + \cdots + x_k[a_k]_q + t[a_{k-1}]_q \\
&\qquad = X_{k-1} + \sum_{i=1}^{\ell} \lambda_i V_{k-1}^{(i)} + x_h[a_h]_q + \cdots + x_k[a_k]_q + t[a_{k-1}]_q.
\end{aligned}$$

Since $V_{k-1}^{(i)} \geq 0$ for $i \in \{1, \ldots, \ell\}$, $x_j[a_j]_q \geq 0$ for $j \in \{1, \ldots, h\}$, and $t[a_{k-1}]_q \geq t$ (by Corollary 15 and the hypothesis), the above inequality becomes

$$\begin{aligned}
&[a_h x_h + \cdots + a_k x_k + a_{k-1} t]_q + t \geq X_{k-1} + t \\
\Rightarrow &[a_h x_h + \cdots + a_k x_k + a_{k-1} t]_q \geq q^{a_k x_k + \cdots + a_h x_h}[x_{k-1}]_{q^{a_{k-1}}} \\
\Rightarrow &q^{(a_h x_h + \cdots + a_k x_k + a_{k-1} t) - 1} + \cdots + 1 \geq q^{a_h x_h + \cdots + a_k x_k + a_{k-1}(x_{k-1} - 1)} + \cdots + q^{a_h x_h + \cdots + a_k x_k} \\
\Rightarrow &a_h x_h + \cdots + a_k x_k + a_{k-1} t - 1 \geq a_h x_h + \cdots + a_k x_k + a_{k-1}(x_{k-1} - 1) \\
\Rightarrow &a_{k-1} t - 1 \geq a_{k-1}(x_{k-1} - 1) \\
\Rightarrow &a_{k-1} t > a_{k-1} t - 1 \geq a_{k-1}(x_{k-1} - 1) \\
\Rightarrow &t > x_{k-1} - 1 \\
\Rightarrow &t \geq x_{k-1}.
\end{aligned}$$

This way, we have shown the existence of a direct sum

$$[W_1^{(a_h)} \oplus \cdots \oplus W_{x_h}^{(a_h)}] \oplus \cdots \oplus [W_1^{(a_2)} \oplus \cdots \oplus W_{x_2}^{(a_2)}] \tag{10}$$

built from subspaces of Π of dimensions $a_h, \ldots, a_2$ respectively by induction.

One may wonder why we did not push the induction up to $k = 1$. However, the lower bound condition $V_j^{(i)} \geq 0$, where $1 \leq i \leq \ell$, that we have used for $2 \leq j \leq h$ does not hold for $j = 1$, since $V_1^{(i)} < 0$. Thus, we use a more careful analysis to show that the above direct sum can be further extended with x_1 subspaces of dimension a_1 as required.

We now complete the direct sum in (10) to a maximal one using a_1-D subspaces of Π,

$$[W_1^{(a_h)} \oplus \cdots \oplus W_{x_h}^{(a_h)}] \oplus \cdots \oplus [W_1^{(a_2)} \oplus \cdots \oplus W_{x_2}^{(a_2)}] \oplus [W_1^{(a_1)} \oplus \cdots \oplus W_s^{(a_1)}],$$

where $0 \leq s \leq x_1$, because $\sum a_i x_i = n$. The final calculation is as follows:

$$
\begin{aligned}
&[a_hx_h + \cdots + a_2x_2 + a_1s]_q - x_h[a_h]_q - \cdots - x_2[a_2]_q - s[a_1]_q \geq u_1 - s \\
\Rightarrow\ &[a_hx_h + \cdots + a_2x_2 + a_1s]_q + s \geq u_1 + x_h[a_h]_q + \cdots + x_2[a_2]_q + s[a_1]_q \\
\Rightarrow\ &[a_hx_h + \cdots + a_2x_2 + a_1s]_q + s \geq u_1 + s \quad (\text{since } x_j[a_j]_q \geq 0 \text{ and } [a_1]_q \geq 1) \\
\Rightarrow\ &[a_hx_h + \cdots + a_2x_2 + a_1s]_q \geq u_1 \geq Y_1^{(\mu)} \quad (\text{by Eq. (9)}) \\
\Rightarrow\ &[a_hx_h + \cdots + a_2x_2 + a_1s]_q \geq q^{a_hy_h+\cdots+a_2y_2}\,[y_1]_{q^{a_1}} \quad (\text{set } \mathbf{y}^{(\mu)} = \mathbf{y} \text{ for simplicity}) \\
&\qquad = q^{a_hy_h+\cdots+a_2y_2}\left((q^{a_1})^{(y_1-1)} + \cdots + q^{a_1} + 1\right) \\
&\qquad \geq q^{a_hy_h+\cdots+a_2y_2+a_1y_1-a_1} \\
\Rightarrow\ &\frac{q^{a_hx_h+\cdots+a_2x_2+a_1s} - 1}{q-1} \geq q^{a_hy_h+\cdots+a_2y_2+a_1y_1-a_1} \quad (\text{definition of } q\text{-number}) \\
\Rightarrow\ &q^{a_hx_h+\cdots+a_2x_2+a_1s} \geq (q-1)\,q^{a_hy_h+\cdots+a_2y_2+a_1y_1-a_1} + 1 \geq q^{a_hy_h+\cdots+a_2y_2+a_1y_1-a_1} + 1 \\
\Rightarrow\ &a_hx_h + \cdots + a_2x_2 + a_1s > a_hy_h + \cdots + a_2y_2 + a_1y_1 - a_1 \\
\Rightarrow\ &n - a_1x_1 + a_1s > n - a_1 \quad (\text{since } \textstyle\sum_{i=1}^n a_ix_i = \sum_{i=1}^n a_iy_i = n) \\
\Rightarrow\ &-a_1x_1 + a_1s > -a_1 \\
\Rightarrow\ &-x_1 + s > -1 \\
\Rightarrow\ &s > x_1 - 1 \\
\Rightarrow\ &s \geq x_1. \quad \square
\end{aligned}
$$

Remark 16. We proved Theorem 1 for nonnegative rather than positive solutions $\mathbf{x}$, $\mathbf{y}^{(i)}$ of $ax + by = n$. That proof was simplified by the fact that all solutions were linearly ordered at regular intervals. For the current theorem, it is again possible to choose $\mathbf{x}$ to be nonnegative without adding any new hypotheses: if any x_i happens to be zero, then we simply skip that part of the construction and add no subspaces of dimension a_i to the direct sum. This way, the expressions $[x_i]_{q^{a_i}}$ will be expanded as a nonzero sum of powers of q only when $x_i > 0$. As for the $\mathbf{y}^{(i)}$, our new proof above translates verbatim to the nonnegative case provided that $y_1^{(\mu)}$, the first component of the solution $\mathbf{y}^{(\mu)}$, is positive: we need to ascertain that $\left[y_1^{(\mu)}\right]_{q^{a_1}} \neq 0$ at only one point in the proof.

5. Universal solutions of the packing condition

In this section only, let $\mathcal{S}_1$ and $\mathcal{S}_q$ denote the *nonnegative* (not just positive) solutions of Equations (2) and (7) respectively. General statements about Gaussian partitions of $\mathbb{F}_q^n$ in the literature are almost always described in terms of *polynomial* expressions in $\mathbb{Z}[q]$, even if q is intended to be a fixed prime power.

Our choice of the special solutions $f_q(\mathbf{x}) \in \mathcal{S}_q$ of the packing condition introduced in Proposition 6 as polynomials in $\mathbb{Z}[q]$ is therefore traditional but not at all random. These polynomial solutions are among the most generic ones in a certain sense that we will describe below, which makes the statement of Theorem 2 more meaningful in the context of *all* prime powers q.

Definition 17. For the formal variable q, we define a set

$$\mathcal{U}[q] = \{g(q) = a_k q^k + \cdots a_1 q^1 + a_0 \in \mathbb{Z}[q] : k \geq 0,\ a_i \in \{0,1\} \text{ for } 0 \leq i \leq k\}.$$

We shall call the elements of $\mathcal{U}[q]$ *universal polynomials* in $\mathbb{Z}[q]$, and any polynomial solution

$$\mathbf{U}(q) = (g_1(q), \ldots, g_h(q)) \in (\mathcal{U}[q])^h$$

of the packing condition (7) will be called a *universal solution.*

Remark 18. The importance of universal solutions is that each of them represents a nonnegative solution of the packing condition for all prime powers (indeed, for all integers) $q_0 \geq 2$, since we can think of it as a set of integer components written in base q_0: the coefficients 0 and 1 of a universal polynomial are acceptable digits for any base. In particular, these polynomial solutions do not give us repeated powers of q (hence, coefficients ≥ 2) when substituted into the packing condition, as the right-hand side of the equation is also a universal polynomial, $[n]_q$.

Let us also emphasize the connection of universal polynomials to our q-analogy theme:

Proposition 19. *Let $a_1, \ldots, a_h, n$ be fixed distinct positive integers. The coefficients $[a_i]_q$ and the constant term $[n]_q$ of the packing condition (7) are universal polynomials, and the $h!$ special solutions $f_q(\mathbf{x})$ of this equation defined in Proposition 6 for each $x \in \mathcal{S}_1$ are universal solutions. Moreover,*

- *The substitution $q = 1$ into Eq. (7) with solution $f_q(\mathbf{x})$ restores the integer identity $a_1x_1 + \cdots + a_hx_h = n$, where $\mathbf{x} = (x_1, \ldots, x_h)$.*
- *More generally, for any universal solution $\mathbf{U}(q) = (g_1(q), \ldots, g_h(q))$ of Eq. (7), the substitution $q = 1$ into $\mathbf{U}(q)$ and $[a_i]_q$ gives rise to an identity $a_1x_1 + \cdots + a_hx_h = n$ in $\mathbb{Z}$, where $\mathbf{x} = \mathbf{U}(1) = (x_1, \ldots, x_h) \in S_1$.*

Here is a complete characterization of all universal solutions of the packing condition, which clearly play a special role among all polynomial solutions:

Proposition 20. *Let $a_1, \ldots, a_h, n$ be fixed distinct positive integers, with a nonempty solution set $\mathcal{S}_1$ of Eq. (2). Every solution $\mathbf{x} = (x_1, \ldots, x_h) \in \mathcal{S}_1$ and every permutation (with repetition) $\sigma(\mathbf{x})$ of the coefficient multiset that has x_i copies of a_i, where $1 \leq i \leq h$, corresponds to a unique universal solution of the packing condition (7). This correspondence exhausts all universal solutions of (7); hence, the number of such solutions is given by*

$$\sum_{\mathbf{x}\in\mathcal{S}_1} \frac{(x_1+\cdots+x_h)!}{x_1!\cdots x_h!}.$$

The $h!$ special solutions f_q in Proposition 6 correspond to those $\mathbf{x}$ and $\sigma(\mathbf{x})$ where all copies of a_i are permuted as a block by $\sigma(\mathbf{x})$ for all i, $1 \le i \le h$; equivalently, to all permutations σ of the coefficients a_i with multiplicity one, independent of $\mathbf{x}$.

Proof. By Definition 17 and Proposition 19, any universal solution

$$\mathbf{U}(q) = (g_1(q), \dots, g_h(q))$$

of the -polynomial- Eq. (7) must have components of the form

$$g_i(q) = \sum_{j=1}^{x_i} q^{c_j} \quad (1 \le i \le h),$$

where (i) $\mathbf{x} = (x_1, \dots, x_h) \in \mathcal{S}_1$, and (ii) either $x_i = 0$ and the empty sum $g_i(q)$ is zero, or, $x_i > 0$ and all the x_i exponents c_j, with $c_j > 0$, are distinct. Any power q^{c_j} in $g_i(q)$ multiplied by

$$[a_i]_q = q^{a_i-1} + q^{a_i-2} + \cdots + q + 1$$

produces a block-sum of a_i consecutive powers of q, with exponents increasing from c_j to $a_i - 1 + c_j$ in increments of 1. Each such block can occur exactly once in $[n]_q$; also, their sum over all coefficients $[a_i]_q$ and all terms of $g_i(q)$ must account for the n distinct powers of q in $[n]_q$. Therefore, we should be able to arrange the products $q^{c_j}[a_i]_q$ (hence, the x_i copies of $[a_i]_q$ for each i) end-to-end in some permutation of the multiset $\{a_1^{x_1}, \dots, a_h^{x_h}\}$ and add them in such a way that the expanded products give us $q^{n-1}+\cdots+q+1$, in this order. The distinct powers q^{c_j} in every $g_i(q)$ (times $[a_i]_q$) must then appear in strictly descending order from left to right in this arrangement. Thus, we have shown that every universal solution yields a solution $\mathbf{x} \in \mathcal{S}_1$ and a permutation $\sigma(\mathbf{x})$ of $\{a_1^{x_1}, \dots, a_h^{x_h}\}$.

Conversely, given $\mathbf{x} \in \mathcal{S}_1$ and a multiset permutation $\sigma(\mathbf{x})$, we can construct x_i distinct powers of q (if $x_i > 0$) for each i, which collectively add up to $g_i(q)$, in the following manner. Let a_k be the rightmost coefficient and a_l the next one to its left in the permutation, where it is possible that $k = l$. We multiply $[a_k]_q$ by $1 = q^0$ and $[a_l]_q$ by q^{a_k} to obtain the two rightmost block-sums:

$$q^{a_k}[a_l]_q + q^0[a_k]_q = \left(q^{a_k+a_l-1} + q^{a_k+a_l-2} + \cdots + q^{a_k}\right) + \left(q^{a_{k-1}} + q^{a_{k-2}} + \cdots + q + 1\right).$$

Then the coefficient of the next block must be $q^{a_k+a_l}$, and so on. That is, each time we add a new block $[a_t]_q$, we multiply it by q to an exponent that is the sum of all a_s's that are to the right of a_t, with multiplicities. Combining the powers of q in front of all x_i copies of $[a_i]_q$, we obtain the ith component $g_i(q)$ of the corresponding universal solution

$\mathbf{U}(q)$. Note that whichever $[a_t]_q$ is the leftmost object in this permutation, the exponent of q for $[a_t]_q$ must be the sum of all the remaining a_s's (with multiplicities according to $\mathbf{x}$), and the last power in the expanded sum will be

$$q^{\sum_{i=1}^{h} x_i a_i - a_t} q^{a_t - 1} = q^{\sum_{i=1}^{h} x_i a_i - 1} = q^{n-1},$$

as expected.

This is the same construction described in the proof of Proposition 6, where all x_i copies of the a_i's had been lumped together. □

Example 21. Consider the Diophantine equation

$$6x + 5y = 28,$$

with unique nonnegative solution $(3, 2)$. The corresponding packing condition

$$[6]_q u + [5]_q v = [28]_q$$

has exactly one universal solution for every permutation (with repetitions) of the sequence 6, 6, 6, 5, 5 and no others (note that 6 is repeated 3 times and 5 is repeated twice, which is dictated by the solution). The number of permutations in this example is $5!/(3!\,2!) = 10$. Two of these universal solutions are the f_q's that we already have identified; they correspond to the permutations 6, 6, 6, 5, 5 and 5, 5, 6, 6, 6. As another example, let us consider the arrangement 6, 5, 6, 5, 6. We construct

$$\begin{aligned}
&1 \cdot (q^5 + q^4 + q^3 + q^2 + q + 1) \quad \text{(multiply next by } q^6\text{)} \\
\Rightarrow\ &\text{(add this)}\ \ q^6 \cdot (q^4 + q^3 + q^2 + q + 1) \quad \text{(next: } q^{11}\text{)} \\
\Rightarrow\ &\text{(add this)}\ \ q^{11} \cdot (q^5 + q^4 + q^3 + q^2 + q + 1) \quad \text{(next: } q^{17}\text{)} \\
\Rightarrow\ &\text{(add this)}\ \ q^{17} \cdot (q^4 + q^3 + q^2 + q + 1) \quad \text{(next: } q^{22}\text{)} \\
\Rightarrow\ &\text{(add this)}\ \ q^{22} \cdot (q^5 + q^4 + q^3 + q^2 + q + 1).
\end{aligned}$$

The corresponding universal solution is

$$(u, v) = \left(q^{22} + q^{11} + 1,\ q^{17} + q^6\right).$$

In this manner, we obtain all 10 universal solutions $(u(q), v(q))$ of $[6]_q\, u + [5]_q\, v = [28]_q$, and display the matching solutions for $q = 2$. The arrangements in boldface in Table 2 point to the two solutions that correspond to the two ordinary permutations of 5 and 6, which we have called f_q.

We indicated in Remark 18 that not all $q = 2$ solutions of the packing condition can be obtained via universal solutions. Although each such numerical solution (u, v) can be converted to a pair of universal polynomials $(u(q), v(q))$ simply by writing u and v in

Table 2
Universal solutions of $[6]_q\, u + [5]_q\, v = [28]_q$.

Arrangement	$u(q)$	$v(q)$	$u(2)$	$v(2)$
55666	$q^{12} + q^6 + 1$	$q^{23} + q^{18}$	4 161	8 650 752
56566	$q^{17} + q^6 + 1$	$q^{23} + q^{12}$	131 137	8 392 704
56656	$q^{17} + q^{11} + 1$	$q^{23} + q^6$	133 121	8 388 672
56665	$q^{17} + q^{11} + q^5$	$q^{23} + 1$	133 152	8 388 609
65566	$q^{22} + q^6 + 1$	$q^{17} + q^{12}$	4 194 369	135168
65656	$q^{22} + q^{11} + 1$	$q^{17} + q^6$	4 196 353	131 136
65665	$q^{22} + q^{11} + q^5$	$q^{17} + 1$	4 196 384	131 073
66556	$q^{22} + q^{16} + 1$	$q^{11} + q^6$	4 259 841	2 112
66565	$q^{22} + q^{16} + q^5$	$q^{11} + 1$	4 259 872	2 049
66655	$q^{22} + q^{16} + q^{10}$	$q^5 + 1$	4 260 864	33

Table 3
Number of universal solutions of $[2]_q u + [3]_q v = [17]_q$.

$(1, 5)$	$6!/(1!\,5!) = 6$
$(4, 3)$	$7!/(4!\,3!) = 35$
$(7, 1)$	$8!/(7!\,1!) = 8$
Total	49

base 2 and changing all 2's to q's, upon substitution, we obtain a polynomial with some coefficients greater than 1 on the left-hand side in many cases, which does not match the universal polynomial $[n]_q$ on the right-hand side.

Example 22. Let us further examine Example 7, where we had the Diophantine equation

$$2x + 3y = 17,$$

with three positive solutions $(1, 5)$, $(4, 3)$, and $(7, 1)$. The number of universal solutions of the polynomial packing condition is broken down by solutions $\mathbf{x} \in \mathcal{S}_1$ in Table 3.

We display some of these solutions for the packing condition $[2]_q\, u + [3]_q\, v = [17]_q$ in Table 4.

Note that only the universal solution constructions that correspond to the two ordinary permutations of the dimensions, 2 and 3, can be defined in a way that is independent of $\mathbf{x}$, and be used in Theorem 2. In general, the number of universal solutions per $\mathbf{x} \in \mathcal{S}_1$ varies, and there is no common description of all solution types across the board that would be useful as a coordinate map.

6. Subspace partitions with no direct sums: Frobenius subspace partitions

In order to construct an infinite family of Frobenius subspace partitions for arbitrary h, we need to introduce several results from other papers. We start with the following proposition by Selmer [19] which gives a lower bound for the Frobenius number of certain sequences of integers.

Table 4
Universal solutions of $[2]_q u + [3]_q v = [17]_q$.

(x, y)	Arrangement	$u(q)$	$v(q)$	$u(2)$	$v(2)$
$(1, 5)$	**333332**	1	$q^{14}+q^{11}+q^8+q^5+q^2$	1	18724
$(1, 5)$	333323	q^3	$q^{14}+q^{11}+q^8+q^5+1$	8	18721
$(1, 5)$	333233	q^6	$q^{14}+q^{11}+q^8+q^3+1$	64	18697
$(1, 5)$	332333	q^9	$q^{14}+q^{11}+q^6+q^3+1$	512	18505
$(1, 5)$	323333	q^{12}	$q^{14}+q^9+q^6+q^3+1$	4096	16969
$(1, 5)$	**233333**	q^{15}	$q^{12}+q^9+q^6+q^3+1$	32768	4681
$(4, 3)$	**3332222**	$q^6+q^4+q^2+1$	$q^{14}+q^{11}+q^8$	85	18688
$(4, 3)$	3323222	$q^9+q^4+q^2+1$	$q^{14}+q^{11}+q^6$	533	18496
$(4, 3)$	3322322	$q^9+q^7+q^2+1$	$q^{14}+q^{11}+q^4$	645	18448
...	...	...	...	...	...
$(4, 3)$	**2222333**	$q^{15}+q^{13}+q^{11}+q^9$	q^6+q^3+1	43520	73
$(7, 1)$	**32222222**	$q^{12}+q^{10}+q^8+q^6+q^4+q^2+1$	q^{14}	5461	16384
$(7, 1)$	23222222	$q^{15}+q^{10}+q^8+q^6+q^4+q^2+1$	q^{12}	34133	4096
$(7, 1)$	22322222	$q^{15}+q^{13}+q^8+q^6+q^4+q^2+1$	q^{10}	41301	1024
$(7, 1)$	22232222	$q^{15}+q^{13}+q^{11}+q^6+q^4+q^2+1$	q^8	43093	256
$(7, 1)$	22223222	$q^{15}+q^{13}+q^{11}+q^9+q^4+q^2+1$	q^6	43541	64
$(7, 1)$	22222322	$q^{15}+q^{13}+q^{11}+q^9+q^7+q^2+1$	q^4	43653	16
$(7, 1)$	22222232	$q^{15}+q^{13}+q^{11}+q^9+q^7+q^5+1$	q^2	43681	4
$(7, 1)$	**22222223**	$q^{15}+q^{13}+q^{11}+q^9+q^7+q^5+q^3$	1	43688	1

Proposition 23 *(Selmer [19]). If a, d, h, k are positive integers such that $h \geq 2$ and $\gcd(a, d) = 1$, then*

$$g\left(a, ka+d, ..., ka+(h-1)d\right) = ka\left\lfloor \frac{a-2}{h-1} \right\rfloor + (k-1)a + d(a-1).$$

In particular, if $k = d = 1$, then

$$g(a, a+1, ..., a+h-1) = a\left\lfloor \frac{a-2}{h-1} \right\rfloor + a - 1.$$

We will need the following straightforward corollary.

Corollary 24 *(Selmer [19]). If a and h are integers such that $h \geq 2$ and $a\left\lfloor \frac{a-2}{h-1} \right\rfloor + a - 1 \geq 4a + 2h - 1$, then*

$$g(a, a+1, ..., a+h-1) \geq 4a + 2h - 1.$$

We will also use the following result of Heden [14].

Proposition 25 *(Heden [14, Theorem 1]). If $T = \{a_1, \ldots, a_h\}$ is a set of positive integers such that $2 \leq a_1 < \ldots < a_h$, then there exists a subspace partition Π of $\mathbb{F}_q^{2a_h}$ such that $\{\dim W : W \in \Pi\} = T$.*

Finally, we state the following results of Beutelspacher [7].

Lemma 26 *(Beutelspacher [7, Example 1 and Lemma 2]).*
(i) For any positive integers d and s, there exists a subspace partition Π of $\mathbb{F}_q^{d+s}$ such that $\{\dim W : W \in \Pi\} = \{d, s\}$.
(ii) For any positive integers s and a such that $s \geq 2a+1$, there exists a subspace partition Π of $\mathbb{F}_q^s$ such that $\{\dim W : W \in \Pi\} = \{a, a+1\}$.

We now state the main proposition of this section, which shows the existence of Frobenius subspace partitions.

Proposition 27. *For any integer h with $h \geq 2$, let a be an integer such that $a \left\lfloor \frac{a-2}{h-1} \right\rfloor + a - 1 \geq 4a + 2h - 1$. If $n = g(a, a+1, \ldots, a+h-1)$, then there exists a subspace partition Π of $\mathbb{F}_q^n$ such that $\{\dim W : W \in \Pi\} = \{a, a+1, \ldots, a+h-1\}$, and Π does not contain a direct sum.*

Proof. Let $a_i = a + i - 1$ for $1 \leq i \leq h$. It follows from Corollary 24 that

$$n = g(a_1, \ldots, a_h) \geq 4a + 2h - 1 = 2a_h + a_1 + a_2.$$

Let $d = 2a_h$ and $s = n - d$, so that $s \geq a_1 + a_2$. Then it follows from Lemma 26(i) that there exists a subspace partition Γ of $\mathbb{F}_q^n$ such that $\{\dim W : W \in \Gamma\} = \{d, s\}$. Let X be a subspace in Γ. If $\dim X = d = 2a_h$, then apply Proposition 25 to obtain a subspace partition Π_X of $X \cong \mathbb{F}_q^{2a_h}$ such that $\{\dim W : W \in \Gamma_X\} = \{a_1, \ldots, a_h\}$. If $\dim X = s$, where $s \geq a_1 + a_2$ and $a_2 = a_1 + 1$, then apply Lemma 26(ii) to obtain a subspace partition Γ_X of $X \cong \mathbb{F}_q^s$ such that $\{\dim W : W \in \Gamma_X\} = \{a_1, a_2\}$. Thus, $\Pi = \bigcup_{X \in \Gamma} \Gamma_X$ is a subspace partition of $\mathbb{F}_q^n$ such that $\{\dim W : W \in \Pi\} = \{a_1, \ldots, a_h\}$. Finally, we claim that Π does not contain a direct sum. Otherwise, there exists $W_1, \ldots, W_t \in \Pi$ such that $W_1 \oplus \ldots \oplus W_t = \mathbb{F}_q^n$. Thus, $\dim W_1 + \ldots + \dim W_t = n$. Since $\dim W_i \in \{a_1, \ldots, a_h\}$, there exist nonnegative integers x_i, $1 \leq i \leq h$, such that $x_1 a_1 + \ldots + x_h a_h = n$. However, this contradicts the definition of the Frobenius number $n = g(a_1, \ldots, a_h)$. □

Declaration of competing interest

There is no competing interest.

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