THE COMPLETE CHARACTERIZATION OF THE MINIMUM SIZE SUPERTAIL IN A SUBSPACE PARTITION

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Abstract. Let q be a prime power and let n be a positive integer. Let $V = V(n, q)$ denote the vector space of dimension n over $\mathbb{F}_q$. A *subspace partition* $\mathcal{P}$ of V is a collection of subspaces of V with the property that each nonzero vector is in exactly one of the subspaces in $\mathcal{P}$. Suppose that $d_1, \ldots, d_k$ are the different dimensions, in increasing order, that occur in the subspace partition $\mathcal{P}$. For any integer s, with $2 \leq s \leq k$, the d_s*-supertail* $\mathcal{S}$ of $\mathcal{P}$ is the collection of all subspaces $X \in \mathcal{P}$ such that $\dim X < d_s$. It was shown that $|\mathcal{S}| \geq \sigma_q(d_s, d_{s-1})$, where $\sigma_q(d_s, d_{s-1})$ denotes the minimum number of subspaces over all subspace partitions of $V(d_s, q)$ in which the largest subspace has dimension d_{s-1}. Moreover, it was shown that if $d_s \geq 2d_{s-1}$ and equality holds in the previous bound on $|\mathcal{S}|$, then the union of the subspaces in $\mathcal{S}$ forms a subspace. This characterization was also conjectured to hold if $d_s < 2d_{s-1}$. This conjecture was recently proved in certain cases. In this paper, we use a much simpler approach to completely settle this conjecture.

1. Introduction

Let q be a prime power and let n be a positive integer. Let $V = V(n, q)$ denote the vector space of dimension n over $\mathbb{F}_q$. A subspace of dimension t is referred to as a t*-subspace*. A *subspace partition*, or *vector space partition*, $\mathcal{P}$ of V, is a collection of subspaces of V with the property that each nonzero vector is in exactly one of the subspaces in $\mathcal{P}$. A well-known example of a subspace partition is a *spread*, which is a subspace partition in which all subspaces have the same dimension. Pioneering work on spreads has been done by several researchers, e.g., André [1] and Segre [14]. Research work on subspace partitions has also been carried since the early 1900's, e.g., see Heden [7] for a survey. A special feature of subspace partitions is that they naturally occur in various fields such as finite geometry, coding theory, and design theory, e.g., see [1, 2, 9, 10, 14] and the references therein.

One main line of research in the area of subspace partitions is the *Classification Problem*, which we shall define after introducing some notation. Given a subspace partition $\mathcal{P}$, of V, the *type* of $\mathcal{P}$ is the multiset that consists of $\dim X$ for all subspaces $X \in \mathcal{P}$. The Classification Problem consists

2010 *Mathematics Subject Classification.* Primary 51E20; Secondary 51E23.
Key words and phrases. Subspace partition; vector space partition; supertail of a subspace partition.
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of finding necessary and/or sufficient conditions for a given multiset of integers that is realizable as the type of a subspace partition of V. Although there are many results related to the Classification Problem, e.g., see [3, 4, 5, **?**, 11], the main question is still wide open.

Before we describe the main contribution of this paper (Theorem 5), we introduce two necessary conditions and a few more definitions. Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ that contains m_{d_i} subspaces of dimension d_i for $1 \le i \le k$. In other words, the type of $\mathcal{P}$ is the multiset that consists of m_{d_i} copies of d_i for $1 \le i \le k$. We denote such a multiset by $d_1^{m_{d_1}} \ldots d_k^{m_{d_k}}$. The following necessary conditions are trivial to derive:

$$\sum_{i=1}^{k} m_{d_i}(q^{d_i}-1) = q^n - 1 \qquad \textit{(packing condition)} \tag{1}$$

$$\begin{cases} n \ge d_i + d_j & \text{if } m_{d_i}+m_{d_j} \ge 2 \text{ and } i \ne j; \\ n \ge 2d_i & \text{if } m_{d_i} \ge 2. \end{cases} \qquad \textit{(dimension condition)} \tag{2}$$

Let k, d_i, and m_{d_i} be as defined, and let s be an integer such that $2 \le s \le k$. We define the d_s-*supertail* of $\mathcal{P}$ to be the set of all subspaces $X \in \mathcal{P}$ such that $\dim X < d_s$. For any integers d and t such that $1 \le t \le d$, we also define $\sigma_q(d,t)$ to be the minimum number of subspaces over all subspace partitions of $V(d,q)$ in which the largest subspace has dimension t. It is easy to see that if $t \mid d$, then $\sigma_q(d,t) = (q^d-1)/(q^t-1)$, which is the number of subspaces in a t-spread of $V(d,q)$, i.e., a spread whose subspaces have dimension t. In fact, the exact value of $\sigma_q(d,t)$ is given by the following theorem (see André [1] and Beutelspacher [2] for $d \pmod t \equiv 0$, and see [**?**, 12] for $d \pmod t \not\equiv 0$).

Theorem 1. *Let d, k, t, and r be integers such that $0 \le r < t$, $k \ge 1$, and $d = kt + r$. Then*

$$\sigma_q(d,t) = \begin{cases} (q^{kt}-1)/(q^t-1) & \text{for } r = 0, \\ q^t + 1 & \text{for } r \ge 1 \text{ and } 3 \le d < 2t, \\ (q^d - q^{t+r})/(q^t-1) + q^{\lceil \frac{t+r}{2} \rceil} + 1 & \text{for } r \ge 1 \text{ and } d \ge 2t. \end{cases}$$

Remark 2. *If $d = 2$, then either $(k,t,r) = (1,2,0)$, or $(k,t,r) = (2,1,0)$. Thus, this $d = 2$ possibility in Theorem 1 is implicitly covered by the "$r = 0$" case.*

The following theorem generalizes a theorem of Heden [6, Theorem 1], although Heden's theorem is stronger and more detailed for the particular case $s = 1$ for which it holds.

Theorem 3 ([8]). *Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of type $d_1^{m_{d_1}} \ldots d_k^{m_{d_k}}$ and let $2 \le s \le k$. If $\mathcal{S}$ is a d_s-supertail of $\mathcal{P}$, then*

$$|\mathcal{S}| \ge \sigma_q(d_s, d_{s-1}) \,. \tag{3}$$

If equality holds in (3), then $\mathcal{S}$ is called a *minimum size supertail*, and Theorem 3 has the following interesting corollary.

Corollary 4 ([8]). *If $|\mathcal{S}| = \sigma_q(d_s, d_{s-1})$ and $d_s \ge 2d_{s-1}$, then the union of the subspaces in $\mathcal{S}$ forms a subspace.*

Note that the union of the subspaces in a d_s-supertail does not have to be a subspace in general.

For $s = 2$, i.e., when $\mathcal{S}$ consists of subspaces of dimension d_1 only, Heden [6, Theorem 3] proved that the conclusion of Corollary 4 also holds when $d_s < 2d_{s-1}$. That result was recently extended in [13] for the following three additional cases:

(i) $s - 1 \leq 2$, that is $\mathcal{S}$ contains subspaces of at most 2 different dimensions;
(ii) $d_s = 2d_{s-1} - 1$; or
(iii) the subspaces in $\mathcal{P} \setminus \mathcal{S}$ have the same dimension.

In this paper, we completely settle the case when $d_s < 2d_{s-1}$, and give a complete characterization of the structure of a minimum size supertail. Our main theorem is as follows.

Theorem 5. *Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of type $d_1^{m_{d_1}} \ldots d_k^{m_{d_k}}$. Let $2 \leq s \leq k$, $d = d_s$, and $t = d_{s-1}$, and suppose $\mathcal{S}$ is a d-supertail of $\mathcal{P}$ such that $|\mathcal{S}| = \sigma_q(d,t) = q^t + 1$ and $d < 2t$. Then, the set of points of $V(n,q)$ covered by the subspaces in the supertail $\mathcal{S}$ forms a subspace W. Moreover, $\mathcal{S}$ is a subspace partition of W whose type is either t^{q^t+1}, or $t^1 a^{q^t}$, for some integer $a \geq 1$.*

By combining Corollary 4 and Theorem 5, we thus have the following theorem. This proves the conjecture stated in [8].

Theorem 6. *Let $\mathcal{P}$ be a subspace partition of $V(n,q)$ of type $d_1^{m_{d_1}} \ldots d_k^{m_{d_k}}$. If $\mathcal{S}$ is a d_s-supertail of $\mathcal{P}$ of size $|\mathcal{S}| = \sigma_q(d_s, d_{s-1})$, then the set of points of $V(n,q)$ covered by the subspaces in $\mathcal{S}$ forms a subspace.*

2. Proof of the main theorem

We now give some notation which is used in the rest of the paper.

Notation 7. (1) *Let $\mathcal{P}$ denote a subspace partition of $V(n,q)$ of type $d_1^{m_{d_1}} \ldots d_k^{m_{d_k}}$, and let $\mathcal{S}$ denote a d_s-supertail of $\mathcal{P}$ of minimum size $|\mathcal{S}| = \sigma_q(d_s, d_{s-1})$.*
(2) *Set $t = d_{s-1}$ and $d = d_s$ with $t < d < 2t$. Thus, $|\mathcal{S}| = \sigma_q(d,t) = q^t + 1$.*
(3) *For any integer $i \geq 0$, let $\Theta_i = (q^i - 1)/(q-1)$. Thus, if $i \geq 1$, then Θ_i is the the number of points, i.e., 1-subspaces, in an i-subspace.*
(4) *Let $\mathcal{H}$ denote the set of all hyperplanes of $V(n,q)$.*
(5) *For $H \in \mathcal{H}$, and any integer $i \geq 1$, let $b_{H,i}$ denote the number of i-subspaces $X \in \mathcal{S}$ such that $X \subseteq H$.*
(6) *For $H \in \mathcal{H}$, let $\beta_H = \sum_{i=a}^{t} b_{H,i} q^i$, where $a \leq \dim X \leq t$ for any $X \in \mathcal{S}$.*

We will use the following elementary result that needs no further reference.

Proposition 8. *The number of hyperplanes $H \in \mathcal{H}$ that contain a given k-subspace of $V(n,q)$ is Θ_{n-k}. In particular, $\mathcal{H}$ contains Θ_n hyperplanes.*

We will also need the following lemma from [13, Lemma 16].

Lemma 9. *Let $\mathcal{P}$, $\mathcal{S}$, t, and d be as defined in Notation 7. If $H \in \mathcal{H}$, then*

$$\beta_H \geq q^t \quad \text{and} \quad \sum_{i=1}^{t} m_i \Theta_i = \frac{cq^d - 1}{q - 1}$$

for some integer $c \geq 1$.

Remark 10. *The bound $\beta_H \geq q^t$ in Lemma 9 plays an important role in the proof of our main theorem. The proof of this bound relies on results proved in [8] by Heden, Lehmann, and the authors of this paper.*

We will use the following easy proposition in the proof of Theorem 5. However, for the sake of completeness, we include a proof of it.

Proposition 11. *Let a and t be positive integers such that $t \geq a$. If $\mathcal{S}$ is a subspace partition of $V(t+a,q)$ that contains q^t+1 subspaces, of which one is a subspace X of dimension t, then all subspaces in $\mathcal{S} \setminus \{X\}$ have dimension a.*

Proof. For any $Y \in \mathcal{S} \setminus \{X\}$, the dimension condition (2) implies that $\dim Y \leq a$. Thus, the proposition follows from the packing condition (1) and the following identity

$$q^{t+a} - 1 = (q^t - 1) + \sum_{i=1}^{q^t} (q^a - 1).$$

□

Finally, we now prove the following technical lemma.

Lemma 12. *Let $\mathcal{P}$, $\mathcal{S}$, and t, be as defined in Notation 7. Let $W = \bigcup_{X \in \mathcal{S}} X$ and let $\delta = \delta(\mathcal{S})$ denote the number of points, i.e., 1-subspaces, of W. For $H \in \mathcal{H}$, let $\delta_H = \delta_H(\mathcal{S})$ be the number of points in $W \cap H$. Then*
(i) $|\mathcal{S}| - 1 = q^t \leq \beta_H \leq cq^d + q^t = \delta(q-1) + |\mathcal{S}|$
(ii) $\sum_H \delta_H = \delta\Theta_{n-1}$
(iii) $\sum_H \delta_H(\delta_H - 1) = \delta(\delta-1)\Theta_{n-2}$.
(iv) $\beta_H = q\delta_H - \delta + |\mathcal{S}|$.
(v) $\sum_H \beta_H = |\mathcal{S}|\Theta_n - \delta$.
(vi) $\sum_H \beta_H^2 = \Theta_n \left(|\mathcal{S}|^2 + \delta(q-1)\right) - \delta^2(q-1) - \delta(2|\mathcal{S}| - 1)$.
(vii) $\sum_H \left(\beta_H - (|\mathcal{S}| - 1)\right)\left(\beta_H - (\delta(q-1) + |\mathcal{S}|)\right) = 0$.

Proof. Applying Lemma 9 and using $|\mathcal{S}| = q^t + 1$, gives

$$\beta_H \geq q^t = |\mathcal{S}| - 1.$$

Using the definitions of δ and Θ_i, we obtain

$$\beta_H = \sum_{i=a}^{t} b_{H,i} q^i \leq \sum_{i=1}^{t} m_i q^i = \delta(q-1) + |\mathcal{S}|,$$

which concludes the proof of (i).

To prove (ii), we count in two ways the number N_1 of pairs (u, H) such that u is a point in both $\mathcal{S}$ and the hyperplane $H \in \mathcal{H}$, i.e., $u \in W \cap H$. We first determine N_1 by selecting u from $W \cap H$, and then summing over all $H \in \mathcal{H}$, to obtain $N_1 = \sum_H \delta_H$. Second, we determine N_1 by selecting u from $W = \bigcup_{X \in \mathcal{S}} X$ in δ ways, and multiplying it by the number Θ_{n-1} of $H \in \mathcal{H}$ that contain u. This yields $N_1 = \delta\Theta_{n-1}$, and the proof of (ii) is complete.

To prove (iii), we count in two ways the number N_2 of triples (u_1, u_2, H) such that u_1 and u_2 are distinct points that are contained in both W and the hyperplane $H \in \mathcal{H}$. First, we determine N_2 by fixing $H \in \mathcal{H}$ and by selecting from $W \cap H$ an ordered pair of distinct points (u_1, u_2). Since the number of points in H is denoted by δ_H, we can sum over all $H \in \mathcal{H}$ to obtain $N_2 = \sum_H \delta_H(\delta_H - 1)$. Second, we determine N_2 by selecting from W an ordered pair of distinct points (u_1, u_2) in $\delta(\delta - 1)$ ways and multiplying by the number Θ_{n-2} of $H \in \mathcal{H}$ that contain those two points. This yields $N_2 = \delta(\delta-1)\Theta_{n-2}$, and the statement in (iii) follows.

Next, we prove (iv). Since for $H \in \mathcal{H}$, δ_H is the number of points from $W \cap H$, it follows from the definitions of m_i, $b_{H,i}$, and Θ_i in Notation 7 that

$$\begin{aligned}
\delta_H &= \sum_{i=a}^{t} b_{H,i}\Theta_i + \sum_{i=a}^{t}(m_i - b_{H,i})\Theta_{i-1} \\
&= \sum_{i=a}^{t} b_{H,i}(\Theta_i - \Theta_{i-1}) + \sum_{i=a}^{t} m_i\Theta_{i-1} \\
&= \sum_{i=a}^{t} b_{H,i}q^{i-1} + \sum_{i=a}^{t} m_i\Theta_{i-1} \\
&= q^{-1}\sum_{i=a}^{t} b_{H,i}q^{i} + q^{-1}\sum_{i=a}^{t} m_i(\Theta_i - 1) \\
&= q^{-1}\beta_H + q^{-1}(\delta - |\mathcal{S}|).
\end{aligned}$$

Thus,

$$\beta_H = q\delta_H - \delta + |\mathcal{S}|.$$

To show (v), we use (iv) to obtain

$$\begin{aligned}
\sum_{H\in\mathcal{H}} \beta_H &= \sum_{H\in\mathcal{H}} (q\delta_H - \delta + |\mathcal{S}|) \\
&= q\sum_{H\in\mathcal{H}} \delta_H - (\delta - |\mathcal{S}|)\Theta_n \qquad \text{(by Proposition 8)} \\
&= q\delta\Theta_{n-1} - (\delta - |\mathcal{S}|)\Theta_n \qquad \text{(by (ii))} \\
&= |\mathcal{S}|\Theta_n - \delta.
\end{aligned}$$

Next, we prove (vi).

$$\begin{aligned}
\sum_H \beta_H^2 &= \sum_H (q\delta_H - \delta + |\mathcal{S}|)^2 \qquad \text{(by (iv))} \\
&= q^2\sum_H \delta_H^2 - 2q(\delta - |\mathcal{S}|)\sum_H \delta_H + \sum_H(\delta - |\mathcal{S}|)^2 \\
&= q^2\left(\delta(\delta-1)\Theta_{n-2} + \sum_H \delta_H\right) - 2q\,(\delta - |\mathcal{S}|)\sum_H \delta_H + (\delta - |\mathcal{S}|)^2\Theta_n \qquad \text{(by (iii) and Proposition 8)} \\
&= q^2\delta(\delta-1)\Theta_{n-2} + \big(q^2 - 2q(\delta - |\mathcal{S}|)\big)\,\delta\Theta_{n-1} + (\delta - |\mathcal{S}|)^2\Theta_n \qquad \text{(by (ii))} \\
&= \delta(\delta-1)(\Theta_n - \Theta_2) + \delta\,(q - 2\delta + 2|\mathcal{S}|)\,(\Theta_n - \Theta_1) + (\delta - |\mathcal{S}|)^2\Theta_n \\
&= \Theta_n\left(|\mathcal{S}|^2 + \delta(q-1)\right) - \delta(\delta-1)\Theta_2 - \delta(q - 2\delta + 2|\mathcal{S}|)\Theta_1 \\
&= \Theta_n\left(|\mathcal{S}|^2 + \delta(q-1)\right) - \delta^2(q-1) - \delta(2|\mathcal{S}| - 1).
\end{aligned}$$

Finally, we prove (vii).

$$\sum_H \left(\beta_H - (|\mathcal{S}| - 1)\right)\left(\beta_H - (\delta(q-1) + |\mathcal{S}|)\right)$$

$$
\begin{aligned}
&= \sum_H \beta_H^2 - (\delta(q-1) + 2|\mathcal{S}| - 1)\sum_H \beta_H + \sum_H (|\mathcal{S}|-1)(\delta(q-1)+|\mathcal{S}|) \\
&= \sum_H \beta_H^2 - (\delta\,(q-1) + 2|\mathcal{S}| - 1)\,(|\mathcal{S}|\Theta_n - \delta) + \Theta_n\,(|\mathcal{S}|-1)\,(\delta(q-1)+|\mathcal{S}|) \qquad \text{(by (v) and Proposition 8)} \\
&= \sum_H \beta_H^2 - \Theta_n\left(|\mathcal{S}|^2 + \delta(q-1)\right) + \delta^2(q-1) + \delta(2|\mathcal{S}|-1) \\
&= 0 \qquad \text{(by (vi))}.
\end{aligned}
$$

□

Proof of Theorem 5. Let β_H be as defined in Notation 7. Then it follows from part (i) and part (vii) of Lemma 12 that for any hyperplane $H \in \mathcal{H}$, we have

$$\beta_H = |\mathcal{S}| - 1 = q^t \ \text{ or } \ \beta_H = \delta(q-1) + |\mathcal{S}| = cq^d + q^t. \tag{4}$$

Thus, if x denotes the number of hyperplanes H such that $\beta_H = q^t$, and if y denotes the number of hyperplanes H such that $\beta_H = cq^d + q^t$, then

$$
\begin{cases}
x + y = \Theta_n \\
xq^t + y(cq^d + q^t) = \sum_H \beta_H = |\mathcal{S}|\Theta_n - \delta = (q^t+1)\Theta_n - \dfrac{cq^d - 1}{q-1}
\end{cases}
\quad \text{(by Lemma 9 and Lemma 12 (v))}.
$$

Solving the above system, yields

$$x = \frac{q^{n-d}(cq^d - 1)}{c(q-1)} \ \text{ and } \ y = \frac{q^{n-d} - c}{c(q-1)}.$$

Since $\gcd(c, cq^d - 1) = 1$, $\gcd(q-1, q^{n-d}) = 1$, and x is an integer, it follows that $c \mid q^{n-d}$. Thus, $c = q^j$, for some positive integer j, which implies that

$$x = \frac{q^{n-(d+j)}(q^{d+j} - 1)}{q-1} \ \text{ and } \ y = \frac{q^{n-(d+j)} - 1}{q-1}. \tag{5}$$

Since for any hyperplane $H \in \mathcal{H}$ that contains $W = \bigcup_{X \in \mathcal{S}} X$, we have by the choices of β_H given in (4), that

$$\beta_H = \sum_{i=a}^{t} b_{H,i} q^i = \sum_{i=a}^{t} m_i q^i > q^t,$$

and thus, from (5), the number of hyperplanes containing W is exactly $y = (q^{n-(d+j)} - 1)/(q-1)$. Let $\langle W \rangle$ denote the space spanned by $W = \bigcup_{X \in \mathcal{S}} X$. For any hyperplane $H \in \mathcal{H}$, we have

$$W \subseteq H \iff \langle W \rangle \subseteq H.$$

Thus, the number of hyperplanes containing $\langle W \rangle$ is also y; which, by Proposition 8, implies that $\dim\langle W \rangle = d + j$. Moreover, since the number of points in W is

$$\delta = \frac{cq^d - 1}{q-1} = \frac{q^{d+j} - 1}{q-1},$$

which is equal to the number of points in $\langle W \rangle$, it follows that $W = \langle W \rangle$ is a subspace of dimension $d + j$.

We now prove the last part of the theorem. Let $a = d + j - t$. Then $a > 0$ and W is a subspace of dimension $t + a$ that admits a subspace partition $\mathcal{S}$ which satisfies the hypothesis of Proposition 11. Thus, $\mathcal{S}$ has type t^{q^t+1} if $a = t$, and type $t^1 a^{q^t}$ if $a \neq t$. □

Acknowledgement: We thank an anonymous referee for carefully reading the paper and for suggesting various changes which helped improve its presentation.

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