RECONFIGURATION OF SUBSPACE PARTITIONS

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Abstract. Let q be a fixed prime power and let $V(n,q)$ denote a vector space of dimension n over the Galois field with q elements. A *subspace partition* (also called "vector space partition") of $V(n,q)$ is a collection of subspaces of $V(n,q)$ with the property that every nonzero element of $V(n,q)$ appears in exactly one of these subspaces.

Given positive integers a, b, n such that $1 \le a < b < n$, we say a subspace partition of $V(n,q)$ has *type* $a^x b^y$ if it is composed of x subspaces of dimension a and y subspaces of dimension b. Let $c = \gcd(a,b)$. In this paper, we prove that if b divides n, then one can (algebraically) construct every possible subspace partition of $V(n,q)$ of type $a^x b^y$ whenever $y \ge (q^e - 1)/(q^b - 1)$, where $0 \le e < ab/c$ and $n \equiv e \pmod{ab/c}$. Our construction allows us to sequentially reconfigure batches of $(q^a - 1)/(q^c - 1)$ subspaces of dimension b into batches of $(q^b - 1)/(q^c - 1)$ subspaces of dimension a. In particular, this accounts for all numerically allowed subspace partition types $a^x b^y$ of $V(n,q)$ under some additional conditions, e.g., when $e = b$.

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1. Introduction

Let q be a fixed prime power throughout this article and $V(n,q)$ denote a vector space of dimension n over the Galois field $\mathbb{F}_q$ with q elements. A *subspace partition* (also called "vector space partition") of $V(n,q)$ is a collection of subspaces of $V(n,q)$ with the property that every nonzero element of $V(n,q)$ appears in exactly one of these subspaces (e.g., see [3, 8, 12]). Subspace partitions can be used to construct combinatorial designs, classical codes, and more recently subspace codes (e.g., [4, 7, 9, 10]). Let a, b, m, n be positive integers. We will use the common notation

$$[n]_q = \frac{q^n - 1}{q - 1}$$

for the Gaussian coefficient counting the 1-dimensional subspaces of $V(n,q)$, which form its finest subspace partition; note that $[1]_q = 1$. We have $(q^m - 1) \mid (q^n - 1)$, and hence, $[m]_q \mid [n]_q$, if and only if $m \mid n$ (the vertical bar denotes "divides in $\mathbb{Z}$.") It follows that for $c = \gcd(a,b)$, we have

$$\gcd(q^a - 1, q^b - 1) = q^c - 1 \Longrightarrow \gcd([a]_q, [b]_q) = [c]_q.$$

It was shown by André [1] and Segre [19] that $V(n,q)$ admits a subspace partition whose subspaces are of the same dimension a (i.e., a *spread*) if and only if a divides n. However, the problem of finding necessary and sufficient conditions for all possible combinations of a-dimensional and b-dimensional subspaces is still open in general.

For simplicity, we say a subspace partition has *type* $a^x b^y$ if it is composed of x subspaces of dimension a and y subspaces of dimension b. Note that the following necessary (but in general, not sufficient) condition on the counts of nonzero vectors must be satisfied for the existence of a subspace partition of type $a^x b^y$:

$$x(q^a - 1) + y(q^b - 1) = q^n - 1. \tag{1}$$

If $a = 1$ and $b > 1$, then the problem of finding necessary and sufficient conditions for the existence of a subspace partition of $V(n, q)$ of type $1^x b^y$ is equivalent to finding the maximum size of a *partial b-spread*, i.e., the maximum number of mutually skewed subspaces of dimension b in $V(n, q)$. See Beutelspacher [3], Drake-Freeman [6], and Hong-Patel [11] for earlier results, and for some recent progress in this direction, see Kurz [13, 14] and Năstase-Sissokho [15]. For $a = 2$ and $b = 3$, the necessary and sufficient conditions for the existence of a subspace partition of $V(n, q)$ of type $2^x 3^y$ are given in El-Zanati et al. [16].

If $a \mid n$ and $b \mid n$, then it is shown by Blinco et al. [4] that Eq. (1) is also a sufficient condition for a subspace partition of type $a^x b^y$ to exist. For $a = 2$ and $b > 3$, the problem of determining the partitions of $V(n, q)$ of type $2^x b^y$ was considered by Seelinger et al. in a series of two papers [17, 18]. In [17], they proved that the existence of subspace partitions of $V(n, q)$ of type $2^x b^y$ for a suitable range of solutions (x, y) implies the existence of subspace partitions of $V(n + b, q)$ of type $2^x b^y$ for almost all solutions (x, y). In their follow-up paper [18], they focused on the case $q = 2$ and proved the existence of partitions of $V(n, 2)$ of type $2^x b^y$ for almost all solutions (x, y) without any pre-condition.

In the current article, we consider the extension of this case where the smaller dimension a is arbitrary but $b \mid n$. Under this hypothesis, we show that for any nonnegative solution (x, y) of Eq. (1) where y is large enough, we can construct a subspace partition of $V(n, q)$ of type $a^x b^y$. More precisely, we prove the following theorem.

Theorem 1. *Let a, b, n be fixed positive integers such that $b > a$, $n \geq ab/\gcd(a, b)$, and b divides n. Let d be the largest common multiple of a and b such that $d \leq n$. For convenience, set $c = \gcd(a, b)$, $e = n - d$, and $s = e/b$. Then*
(i) *There is a subspace partition of $V(n, q)$ of each type*

$$a^{i[b]_q/[c]_q}\, b^{[n]_q/[b]_q - i[a]_q/[c]_q}$$

for all i such that

$$0 \leq i \leq \frac{q^e [d]_q [c]_q}{[a]_q [b]_q}.$$

Thus, this accounts for all subspace partitions of $V(n, q)$ of type $a^x b^y$ with $y \geq [e]_q/[b]_q$.
(ii) *Moreover, if $[s]_{q^b} < [a/c]_{q^c}$, e.g., when $s = 1$ and $e = b$, then all possible subspace partition types $a^x b^y$ of V are covered by this list.*

Remark 1. (1) *If $c = 1$, then there is a subspace partition of $V(n, q)$ of each type*

$$a^{i[b]_q}\, b^{[n]_q/[b]_q - i[a]_q} \text{ for all } i \text{ such that } 0 \leq i \leq \tfrac{q^e [d]_q}{[a]_q [b]_q}.$$

(2) *If $y < [e]_q/[b]_q$, then our results are limited by fact that we do not have necessary and sufficient conditions for the existence of subspace partitions of $V(e, q)$ type $a^x b^y$ when $e < ab/\gcd(a, b)$.*

Example 2. *Let $a = 6$ and $b = 15$ so that $c = \gcd(15, 6) = 3$. Let $n = d + e$, with $d = 30t$ for some positive integer t, and $e = 15$. Since a and b both divide d, and $s = 1$, it follows from Theorem 1 that for any prime power q, there are subspace partitions of $V(30t + 15, q)$ containing $x_i = i[15]_q/[3]_q$ subspaces of dimension 6 and $y_i = [30t+15]_q/[15]_q - i[6]_q/[3]_q$ subspaces of dimension 15, where $0 \leq i \leq q^{15}[30t]_q[3]_q/[6]_q[15]_q$. Moreover, this accounts for all possible subspace partitions of $V(30t + 15, q)$ with subspaces of dimensions 6 and 15.*

Our approach combines the methods in [17, 18] with the properties of double cosets[1]. This is similar to the method of switching reguli in finite geometry. In particular, given a direct sum

[1]These double coset properties are implicit [4, Lemma 2.1], but we use them in a more extensive and explicit way here.

decomposition of $V(n,q) = U \oplus W$, where $U, W \neq \{\mathbf{0}\}$, b divides $\dim U$, and both a and b divide $\dim W$, we use an imposed field structure on W to find ways of partitioning the set $U + W$ into all possible combinations of a-dimensional and b-dimensional subspaces. If S, S' are subsets of a field $W = \mathbb{F}_{q^d}$ and ω is an element of W, then we will denote by $S+S'$ the subset $\{\sigma+\sigma' \colon \sigma \in S,\ \sigma' \in S'\}$, by SS' the subset $\{\sigma\sigma' \colon \sigma \in S,\ \sigma' \in S'\}$, by ωS the subset $\{\omega\sigma \colon \sigma \in S\}$, and by S^* the subset $\{\sigma \in S \colon \sigma \neq \mathbf{0}\}$ of W.

In order to properly manage the various combinations of a- and b-subspaces mentioned above, we use the set partition of the multiplicative group W^* into double cosets relative to its subgroups Γ^* and Δ^*, where Γ and Δ are the unique subfields of W of orders q^a and q^b, and hence, a-dimensional and b-dimensional subspaces of W over $\mathbb{F}_q$, respectively. The existence of such subfields of $\mathbb{F}_{q^d}$ is due to the following well-known characterization.

Lemma 3. *Let a, b, d be positive integers.*
(i) Let Γ be an a-dimensional subspace of $W = \mathbb{F}_q^d$. Then W can be simultaneously identified with the field $\mathbb{F}_{q^d}$ and Γ with the unique subfield of $\mathbb{F}_{q^d}$ of order q^a if and only if $a \mid d$. In this case, we write $W \simeq \mathbb{F}_{q^d}$ and $\Gamma \simeq \mathbb{F}_{q^a}$.
(ii) The intersection of the unique subfields of $\mathbb{F}_{q^d}$ of orders q^a and q^b (for a, b dividing d) is a subfield of all three, i.e., the unique one of order $q^{\gcd(a,b)}$.
(iii) If $a \mid d$, then $\mathbb{F}_{q^a}$ is an $\mathbb{F}_q$-subspace of $\mathbb{F}_{q^d}$ of dimension a.

To avoid the extra calculations due to a possible non-unity greatest common divisor of subspace dimensions, we state a general conversion principle:

Lemma 4. *Let a, $a_1, \ldots, a_r$, c, n be positive integers and $x_1, \ldots, x_r$ be nonnegative integers. Then the following are true:*
(i) If $c \mid n$, then $V(n/c, q^c) \simeq \mathbb{F}_{q^{c(n/c)}}$ can be identified with $V(n,q) \simeq \mathbb{F}_{q^n}$, where $V(n,q)$ has the same underlying set as $V(n/c, q^c)$, the same vector addition, and the same multiplication restricted to scalars from the field $\mathbb{F}_q \subseteq \mathbb{F}_{q^c}$.
(ii) If $c \mid a$ and $a \mid n$, then every (a/c)-spread of $V(n/c, q^c)$ has cardinality

$$\frac{[n/c]_{q^c}}{[a/c]_{q^c}} = \frac{[n]_q}{[a]_q}.$$

In particular, the number of subspaces of $V(n/c, q^c)$ of dimension $1 = c/c$ is

$$[n/c]_{q^c} = \frac{[n]_q}{[c]_q}.$$

(iii) If $c \mid a_1, \ldots, c \mid a_r$, $c \mid n$, and $V(n/c, q^c)$ has a subspace partition of type

$$(a_1/c)^{x_1} \cdots (a_r/c)^{x_r},$$

then $V(n,q)$ has a subspace partition of type

$$a_1^{x_1} \cdots a_r^{x_r}.$$

We may therefore, without loss of generality, assume that the subspace dimensions a and b in our partitions are relatively prime where convenient. All vector spaces are understood to be over the generic field $\mathbb{F}_q$ (unless otherwise stated), which will in turn be identified with the unique 1-dimensional subfield if the vector space has been given a field structure.

2. Cosets and Double Cosets: Reconfiguration of Partitions of W

The next lemma is essentially due to Beutelspacher [3], and independently, to Bu [5]. It gives a construction of an a-spread of $V(q,d)$ for a divisor a of d via cosets of $\mathbb{F}_{q^a}^*$ in $\mathbb{F}_{q^d}^*$.

Lemma 5. *Let a and d be positive integers such that $a \mid d$. Let $W = \mathbb{F}_q^d$ with a field structure, so that $W \simeq \mathbb{F}_{q^d}$, and Γ be the unique a-dimensional subspace of W that corresponds to the subfield of $\mathbb{F}_{q^d}$ of order q^a.*
(i) If $\omega \in W^*$, then $\omega\Gamma$ is a linear a-dimensional subspace of W.
(ii) If $\omega_1\Gamma^*, \ldots, \omega_r\Gamma^*$ are the distinct cosets of Γ^* in the multiplicative subgroup W^* of W, where $r = |W^*|/|\Gamma^*| = [d]_q/[a]_q$, then the a-subspaces $\omega_1\Gamma, \ldots, \omega_r\Gamma$ form a subspace partition of W.

Note that the second part of Lemma 5 follows directly from Lemma 4 and the fact that all 1-dimensional subspaces of $V(d/a, q^a)$ form a spread; the $[d]_q/[a]_q$ elements ω_k are simply representatives of such subspaces.

We will now move to the next level and study the subspace partitions with two distinct dimensions a and b. Let us review the following properties of double cosets of two special subgroups of the multiplicative group of a finite field.

Lemma 6. *Let a, b, c, d be positive integers such that $c = \gcd(a, b)$, $a \mid d$, and $b \mid d$. Moreover, let $W = \mathbb{F}_{q^d}$ and Γ, Δ be the unique subfields of W of orders q^a and q^b respectively. Then $\Gamma \cap \Delta$ is the unique subfield of W of order q^c, containing the scalar field $\mathbb{F}_q$. The following properties hold:*
(i) There are

$$m = \frac{|W^*|\,|\Gamma^* \cap \Delta^*|}{|\Gamma^*|\,|\Delta^*|} = \frac{[d]_q[c]_q}{[a]_q[b]_q}$$

distinct double (Γ^*, Δ^*)-cosets in W^*, whose representatives will be denoted by $\omega_1, \ldots, \omega_m$:

$$\Gamma^*\omega_1\Delta^*, \ldots, \Gamma^*\omega_m\Delta^*.$$

Every double coset $\Gamma^*\omega_k\Delta^*$ contains

$$\frac{|\Gamma^*|\,|\Delta^*|}{|\Gamma^* \cap \Delta^*|} = \frac{(q^a-1)(q^b-1)}{(q^c-1)}$$

elements of W^*.
(ii) For each k with $1 \le k \le m$, the double coset $\Gamma^*\omega_k\Delta^*$ is the disjoint union of $[a]_q/[c]_q$ left cosets of Δ^*. The elements $\gamma_i \in \Gamma^*$ below can be chosen freely as coset representatives of $\mathbb{F}_{q^c} = \Gamma^* \cap \Delta^*$ in Γ^*:

$$\Gamma^*\omega_k\Delta^* = \bigsqcup_{i=1}^{[a]_q/[c]_q} (\gamma_i\omega_k)\,\Delta^*.$$

(iii) For each k with $1 \le k \le m$, the double coset $\Gamma^*\omega_k\Delta^*$ is the disjoint union of $[b]_q/[c]_q$ right cosets of Γ^*. The elements $\delta_j \in \Delta^*$ below (same for all k) can be chosen freely as coset representatives of $\mathbb{F}_{q^c} = \Gamma^* \cap \Delta^*$ in Δ^*:

$$\Gamma^*\omega_k\Delta^* = \bigsqcup_{j=1}^{[b]_q/[c]_q} \Gamma^*\,(\omega_k\delta_j)\,.$$

(iv) With notation as above, the $[d]_q/[c]_q$ elements $\gamma_i\,\omega_k\,\delta_j$ are distinct for $1 \le i \le [a]_q/[c]_q$, $1 \le j \le [b]_q/[c]_q$, and $1 \le k \le m$ and form representatives of a c-spread of W of the form $\{\gamma_i\omega_k\delta_j\mathbb{F}_{q^c}\}$.

Proof. (i) Since W^* is abelian, all subgroups of W^* are normal, and the distinction between left and right cosets is for notational purposes only. In particular, the set $\Gamma^*\Delta^*$ is a subgroup of W^*. From the formula for the cardinality of the product of two subgroups of a finite group, we obtain

$$|\Gamma^*\Delta^*| = \frac{|\Gamma^*|\,|\Delta^*|}{|\Gamma^* \cap \Delta^*|} = \frac{(q^a-1)(q^b-1)}{(q^c-1)}.$$

Now, each double coset $\Gamma^*\omega_k\Delta^*$ is a left coset $\omega_k(\Gamma^*\Delta^*)$ of the subgroup $\Gamma^*\Delta^*$ and has the same cardinality as $\Gamma^*\Delta^*$, computed above. Therefore, the q^d-1 elements of W^* are partitioned into

$$m = \frac{|W^*|}{|\Gamma^*\Delta^*|} = \frac{(q^d-1)(q^c-1)}{(q^a-1)(q^b-1)} = \frac{[d]_q[c]_q}{[a]_q[b]_q}$$

double cosets.
(ii) We write

$$\Gamma^*\omega_k\Delta^* = \bigcup_{\gamma\in\Gamma^*}(\gamma\omega_k)\Delta^*,$$

where any two left cosets of Δ^* in W^* are either equal or disjoint. To compute the number of disjoint cosets in this union, we divide the total number of elements of $\Gamma^*\omega_k\Delta^*$ by $|\Delta^*|$ (see part (i)):

$$\frac{(q^a-1)(q^b-1)}{(q^c-1)(q^b-1)} = \frac{[a]_q}{[c]_q}.$$

Therefore, we can choose elements $\gamma_1,\dots,\gamma_{[a]_q/[c]_q}\in\Gamma^*$ such that $\gamma_1\omega_k,\dots,\gamma_{[a]_q/[c]_q}\omega_k$ are distinct coset representatives of Δ^* in W^*. Moreover, we have

$$\gamma_i\omega_k\Delta^* = \gamma_{i'}\omega_k\Delta^* \iff \gamma_i\Delta^* = \gamma_{i'}\Delta^* \iff \gamma_i\gamma_{i'}^{-1}\in\Gamma^*\cap\Delta^* = \mathbb{F}_{q^c}.$$

As a result, it suffices to fix a complete set $\gamma_1,\dots,\gamma_{[a]_q/[c]_q}$ of coset representatives of $\mathbb{F}_{q^c}$ in Γ^* regardless of the value of k.
(iii) Similar to the proof of part (ii).
(iv) If $(i,j,k)\neq(i',j',k')$, then at least one of the following must be true for the elements $\gamma_i\omega_k\delta_j$ and $\gamma_{i'}\omega_{k'}\delta_{j'}$ of W^*: they are (a) in distinct double (Γ^*,Δ^*)-cosets; or (b) in distinct left cosets of Δ^*; or (c) in distinct right cosets of Γ^*. Hence, such elements cannot be equal. In addition, we observe that there are

$$\frac{[a]_q}{[c]_q}\,\frac{[d]_q[c]_q}{[a]_q[b]_q}\,\frac{[b]_q}{[c]_q} = \frac{[d]_q}{[c]_q} = [W^*\colon \mathbb{F}_{q^c}^*]$$

of them, and

$$\begin{aligned}
&\gamma_i\omega_k\delta_j\mathbb{F}_{q^c} = \gamma_{i'}\omega_{k'}\delta_{j'}\mathbb{F}_{q^c}\\
\implies\ &\gamma_i\omega_k\delta_j = \gamma_{i'}\omega_{k'}\underbrace{\delta_{j'}\lambda}_{\in\Delta^*}\in\Gamma^*\omega_k\Delta^*\cap\Gamma^*\omega_{k'}\Delta^*,\ \lambda\in\mathbb{F}_{q^c}\\
\implies\ &k=k' \text{ by part } (i), \text{ and } \gamma_i\gamma_{i'}^{-1} = \delta_j^{-1}\delta_{j'}\lambda\in\Gamma^*\cap\Delta^* = \mathbb{F}_{q^c}\\
\implies\ &i=i' \text{ and } j=j' \text{ by parts } (ii) \text{ and } (iii). \quad \square
\end{aligned}$$

We may now assert that there are subspace partitions of W of all types a^xb^y that are allowed by the condition in Eq. (1), that is, $x(q^a-1)+y(q^b-1)=q^d-1$. Therefore, our statement recovers a result of Blinco et al. [4]

Lemma 7. *Let a, b, c, d be positive integers such that $c=\gcd(a,b)$, $a\mid d$, and $b\mid d$. Then the collection of $m=[d]_q[c]_q/([a]_q[b]_q)$ subsets $\Gamma\omega_k\Delta$ of $V(d,q)$ have pairwise zero intersection. Each set $\Gamma\omega_k\Delta$ in this collection is simultaneously the union of subspaces $(\gamma_i\omega_k)\Delta$ that form a partial b-spread of $V(d,q)$ of cardinality $[a]_q/[c]_q$ and the union of subspaces $\Gamma(\omega_k\delta_j)$ that form a partial a-spread of cardinality $[b]_q/[c]_q$. As a result, there exist subspace partitions of $V(d,q)$ of all types*

$$a^{i[b]_q/[c]_q}\,b^{[d]_q/[b]_q-i[a]_q/[c]_q},$$

with $0\le i\le m$.

Proof. See Lemma 6 and Lemma 5. Clearly, all nonnegative solutions (x,y) of the Diophantine equation $[a]_qx+[b]_qy=[d]_q$ are represented by the listed subspace partitions of types a^xb^y. $\square$

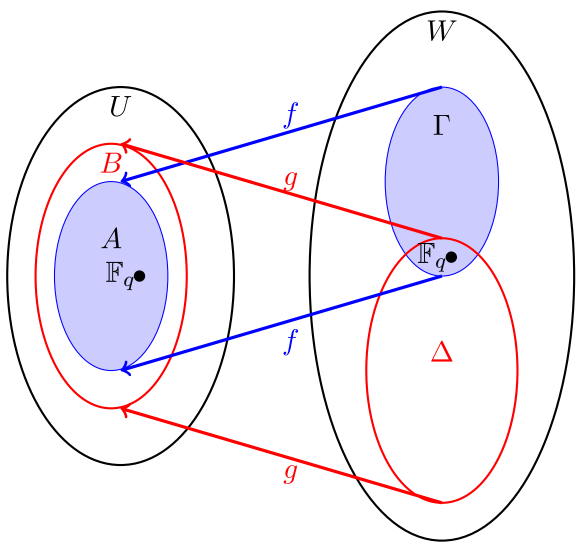
3. Reconfiguring the Transversal Subspaces of $U \oplus W$

In this section, we assume that a and b are relatively prime. We arrange the existing notation under this convention in Table 1. We now describe the construction depicted in Figure 1.

Table 1. Notation for W when $\gcd(a,b) = 1$

Object	**Description**	**Size/range**
W	$\mathbb{F}_q$-space with field structure	$\dim W = d > 0$
Γ	subfield of W	$\dim \Gamma = a, \quad a \mid d$
Δ	subfield of W	$\dim \Delta = b, \quad b \mid d$
γ_i	$\mathbb{F}_q^*$-coset rep in Γ^*, $\gamma_1 = 1$	$1 \leq i \leq [a]_q$
δ_j	$\mathbb{F}_q^*$-coset rep in Δ^*, $\delta_1 = 1$	$1 \leq j \leq [b]_q$
ω_k	double (Γ^*, Δ^*)-coset rep in W^*, $\omega_1 = 1$	$1 \leq k \leq m = [d]_q/([a]_q[b]_q)$
$\gamma_i\omega_k\delta_j$	$\mathbb{F}_q^*$-coset rep in W^*	$[d]_q$
$\omega_k\delta_j$	Γ^*-coset rep in W^*	$[d]_q/[a]_q$
$\gamma_i\omega_k$	Δ^*-coset rep in W^*	$[d]_q/[b]_q$

Figure 1. Configuration of the Construction



Construction 8. (1) *For positive integers a, b, d, e such that $a < b$, $\gcd(a,b) = 1$, $b \mid e$, $a \mid d$, and $b \mid d$, consider two vector spaces U and W over $\mathbb{F}_q$ of dimensions e and d respectively. We shall describe constructions of subspace partitions of $V = U \oplus W$ consisting of a- and b-subspaces only. We have $n \stackrel{\text{def}}{=} \dim V = e + d$, is divisible by b but not necessarily by a.*

(2) *We impose a field structure on W and let Γ, Δ be the unique subfields of orders q^a and q^b respectively, intersecting at the subfield isomorphic to the scalar field $\mathbb{F}_q$. Choose elements $\gamma_i \in \Gamma^*$, $\delta_j \in \Delta^*$, and $\omega_k \in W^*$ as in Lemma 6, with $1 \leq i \leq [a]_q$, $1 \leq j \leq [b]_q$, and $1 \leq k \leq m = [d]_q/([a]_q[b]_q)$. Without loss of generality, let $\gamma_1 = \delta_1 = \omega_1 = 1 \in W$.*

(3) *We also endow U with a field structure. Let B be its unique subfield of order q^b and $\upsilon_1, \ldots, \upsilon_{[e]_q/[b]_q}$ be a set of B^*-coset representatives in U^*. By Lemma 5, the b-subspaces*

$\upsilon_1 B, \dots, \upsilon_{[e]_q/[b]_q} B$ form a b-spread of U. Without loss of generality, let $\upsilon_1 = 1 \in U$. We also fix an a-dimensional subspace A of B such that $\mathbb{F}_q$ is contained in A.

(4) *We now identify various subspaces and subfields. Let $g\colon \Delta \to B$ be a field isomorphism, which necessarily satisfies $g(\delta_1) = g(1) = 1$; set $\beta_j = g(\delta_j)$ for $1 \le j \le [b]_q$. Without loss of generality, assume that $\beta_1, \dots, \beta_{[a]_q} \in A$, and that the first a of them, $\beta_1, \dots, \beta_a$, form a basis of A. We define an $\mathbb{F}_q$-linear isomorphism $f\colon \Gamma \to A$ by setting $f(\gamma_i) \stackrel{\text{def}}{=} \beta_i = g(\delta_i)$ for $1 \le i \le a$. Finally, let $\alpha_i \stackrel{\text{def}}{=} \beta_i$ for $1 \le i \le [a]_q$.*

We display the new information from Construction 8 in Table 2.

TABLE 2. Notation for $V = U \oplus W$ when $\gcd(a, b) = 1$

Object	**Description**	**Size/range**
U	$\mathbb{F}_q$-space with field structure	$\dim U = e > 0$
W	$\mathbb{F}_q$-space with field structure	$\dim W = d > 0$
V	$U \oplus W$, $\mathbb{F}_q$-space	$\dim V = n = e + d$
B	subfield of U	$\dim B = b,\ \ b \mid e,\ b \mid d$
A	subspace of B containing $\mathbb{F}_q$	$\dim A = a < b,\ a \mid d$
f	linear isomorphism	$f\colon \Gamma \to A$
g	field isomorphism	$g\colon \Delta \to B$
υ_ℓ	B^*-coset reps in U^*, $\upsilon_1 = 1$	$1 \le \ell \le [e]_q/[b]_q$
β_j	$g(\delta_j)$, $\mathbb{F}_q^*$-coset reps in B^*, $\beta_1 = 1$	$1 \le j \le [b]_q$
α_i	$\beta_i = f(\gamma_i) = g(\delta_i)$, reps of 1-D subspaces in A, $\alpha_1 = 1$	$1 \le i \le [a]_q$
$\upsilon_\ell \beta_j$	$\mathbb{F}_q^*$-coset reps in U^*	$[e]_q$

We will now construct a partial a-spread of $V = U \oplus W$ whose subspaces have the property that all nonzero vectors have nonzero projections onto U and W: we will call any such subspace *transversal*. Hence, these subspaces have pairwise trivial intersection with those in each of the subspace partitions of W that we have described in Lemma 7.

Lemma 9. *Let the notation be as in Tables 1 and 2, with $\gcd(a, b) = 1$. For each triple (υ, j, k) with $\upsilon \in U^*$, $1 \le j \le [b]_q$, and $1 \le k \le [d]_q/([a]_q[b]_q)$, we define a map*

$$\phi_j^{(k,\upsilon)} \stackrel{\text{def}}{=} \upsilon \cdot f + \omega_k \delta_j \cdot id_\Gamma \colon \Gamma \to U \oplus W.$$

Then $\phi_j^{(k,\upsilon)}$ is $\mathbb{F}_q$-linear and injective. Hence, the subspaces

$$A_j^{(k,\upsilon)} \stackrel{\text{def}}{=} Im\,\phi_j^{(k,\upsilon)} = \{\upsilon f(\gamma) + \gamma\omega_k\delta_j \colon \gamma \in \Gamma\}$$

of $U \oplus W$ are a-dimensional over $\mathbb{F}_q$. Moreover, we have $A_j^{(k,\upsilon)} = A_{j'}^{(k',\upsilon')}$ if $(\upsilon, j, k) = (\upsilon', j', k')$ and $A_j^{(k,\upsilon)} \cap A_{j'}^{(k',\upsilon')} = \{\mathbf{0}\}$ otherwise, giving rise to a partial a-spread of $U \oplus W$ with $(q^e - 1)[d]_q/[a]_q$ transversal subspaces.

Proof. Linearity is clear. If $\upsilon f(\gamma) + \gamma\omega_k\delta_j = \mathbf{0}$, then we must have $f(\gamma) = \mathbf{0}$ since $\upsilon \ne \mathbf{0}$, and $\gamma = \mathbf{0}$ since f is injective. This shows that $\phi_j^{(k,\upsilon)}$ is injective.

Assume that for some υ, $\upsilon' \in U^*$, j, $j' \in \{1, \dots, [b]_q\}$, k, $k' \in \{1, \dots, [d]_q/([a]_q[b]_q)\}$, and γ, $\gamma' \in \Gamma^*$, we have

$$\upsilon f(\gamma) + \gamma\omega_k\delta_j = \upsilon' f(\gamma') + \gamma'\omega_{k'}\delta_{j'}.$$

By the construction of coset representatives of Γ^* in W^* (see Table 1), we must have $j = j'$ and $k = k'$. Thus, from the above, we first obtain $\gamma = \gamma'$ and $\upsilon f(\gamma) = \upsilon' f(\gamma')$, from which we may conclude $\upsilon = \upsilon'$. □

Remark 10. *This construction is optimal in the sense that if a were to divide e as well, then together with a-spreads of U and W, we would obtain a total of*

$$(q^e - 1)\frac{[d]_q}{[a]_q} + \frac{[e]_q}{[a]_q} + \frac{[d]_q}{[a]_q} = \frac{q^e[d]_q + [e]_q}{[a]_q} = \frac{[n]_q}{[a]_q}$$

subspaces, forming an a-spread of $V = U \oplus W$.

Here is a similar construction of transversal b-subspaces of $U \oplus W$.

Lemma 11. *Let the notation be as in Tables 1 and 2, with* $\gcd(a, b) = 1$. *For each triple (ν, i, k) with $\nu \in U^*$, $1 \le i \le [a]_q$, and $1 \le k \le [d]_q/([a]_q[b]_q)$, we define a map*

$$\psi_i^{(k,\nu)} \overset{\text{def}}{=} \nu \cdot g + \gamma_i \omega_k \cdot id_\Delta \colon \Delta \to U \oplus W.$$

The maps $\psi_i^{(k,\nu)}$ are $\mathbb{F}_q$-linear and injective. The subspaces

$$B_i^{(k,\nu)} \overset{\text{def}}{=} Im\, \psi_i^{(k,\nu)} = \{\nu g(\delta) + \gamma_i \omega_k \delta \colon \delta \in \Delta\}$$

of $U \oplus W$ are b-dimensional over $\mathbb{F}_q$. Moreover, we have $B_i^{(k,\nu)} = B_{i'}^{(k',\nu')}$ if $(\nu, i, k) = (\nu', i', k')$ and $B_i^{(k,\nu)} \cap B_{i'}^{(k',\nu')} = \{\mathbf{0}\}$ otherwise, giving rise to a partial b-spread of $U \oplus W$ with $(q^e - 1)[d]_q/[b]_q$ transversal subspaces.

Proof. The proof is similar to that of Lemma 9, with Γ^*-coset representatives in W^* replaced by Δ^*-coset representatives. □

Remark 12. *Since b divides both d and e, together with b-spreads of U and W, we obtain a total of*

$$(q^e - 1)\frac{[d]_q}{[b]_q} + \frac{[e]_q}{[b]_q} + \frac{[d]_q}{[b]_q} = \frac{q^e[d]_q + [e]_q}{[b]_q} = \frac{[n]_q}{[b]_q}$$

subspaces, forming a b-spread of $V = U \oplus W$.

Remark 13. *Further partitioning the elements of U^* into B^*-cosets $\nu_\ell B^*$ (see Table 2), we refine our construction in Lemma 9 to the a-subspaces*

$$A_j^{(k,\ell,\beta)} \overset{\text{def}}{=} \{\upsilon_\ell \beta \beta_j f(\gamma) + \gamma \omega_k \delta_j \colon \gamma \in \Gamma\}$$

for (β, j, k, ℓ) with $\beta \in B^*$, $1 \le j \le [b]_q$, $1 \le k \le [d]_q/([a]_q[b]_q)$, and $1 \le \ell \le [e]_q/[b]_q$. This is still in full generality as every B^*-coset remains invariant under multiplication by the fixed element β_j of B^* (resp., by $\alpha_i = \beta_i$ below.) Similarly, we define b-subspaces that are more nuanced with respect to Lemma 11 via

$$B_i^{(k,\ell,\beta)} \overset{\text{def}}{=} \{\upsilon_\ell \beta \alpha_i g(\delta) + \gamma_i \omega_k \delta \colon \delta \in \Delta\}$$

for (β, i, k, ℓ) with $\beta \in B^*$, $1 \le i \le [a]_q$, $1 \le k \le [d]_q/([a]_q[b]_q)$, and $1 \le \ell \le [e]_q/[b]_q$.

We are ready to describe the main reconfiguration result for two distinct subspace dimensions a and b.

Proposition 14. *Let the notation be as in Table 1, Table 2, and Remark 13, where* $\gcd(a, b) = 1$. *Then for all fixed $(\beta,\, k,\, \ell)$ with $\beta \in B^*$, $1 \le k \le [d]_q/([a]_q[b]_q)$, and $1 \le \ell \le [e]_q/[b]_q$, we have*

$$\bigsqcup_{j=1}^{[b]_q} \left(A_j^{(k,\ell,\beta)}\right)^* = \bigsqcup_{i=1}^{[a]_q} \left(B_i^{(k,\ell,\beta)}\right)^*.$$

There are

$$(q^e - 1)\,\frac{[d]_q}{[a]_q[b]_q}$$

such possible reconfigurations as β, k, and ℓ vary.

Proof. For later use, first recall from Construction 8, part (4), that $\alpha_i \stackrel{\text{def}}{=} \beta_i$ for $1 \le i \le [a]_q$.

Let

$$x = \upsilon_\ell \beta\beta_j f(\gamma) + \gamma\omega_k\delta_j \in \left(A_j^{(k,\ell,\beta)}\right)^* .$$

The element $\gamma \in \Gamma^*$ has a unique representation $\gamma = \lambda\gamma_i$ for some $\lambda \in \mathbb{F}_q^*$ and $1 \le i \le [a]_q$. Hence, we have

$$\lambda^{-1}x = \upsilon_\ell\beta\beta_j f(\gamma_i) + \gamma_i\omega_k\delta_j = \upsilon_\ell\beta\alpha_i g(\delta_j) + \gamma_i\omega_k\delta_j \in \left(B_i^{(k,\ell,\beta)}\right)^* .$$

Conversely, if

$$y = \upsilon_\ell\beta\alpha_i g(\delta) + \gamma_i\omega_k\delta \in \left(B_i^{(k,\ell,\beta)}\right)^* ,$$

then the element $\delta \in \Delta^*$ has a unique representation as $\delta = \lambda\delta_j$ for some $\lambda \in \mathbb{F}_q^*$ and $1 \le j \le [b]_q$. We have

$$\lambda^{-1}y = \upsilon_\ell\beta\alpha_i g(\delta_j) + \gamma_i\omega_k\delta_j = \upsilon_\ell\beta\beta_j f(\gamma_i) + \gamma_i\omega_k\delta_j \in \left(A_j^{(k,\ell,\beta)}\right)^* . \qquad \square$$

Let us summarize our findings in this section.

Proposition 15. *Let a, b, d, e, n be positive integers such that $a < b$, $\gcd(a,b) = 1$, $a \mid d$, $b \mid d$, $b \mid e$, and $n = e + d$ (see Tables 1 and 2 for further notation.) If U is an $\mathbb{F}_q$-space of dimension e and W is an $\mathbb{F}_q$-space of dimension d, then there exist $(q^e - 1)[d]_q/([a]_q[b]_q)$ disjoint subsets of the set*

$$(U \oplus W)^* \setminus (U^* \sqcup W^*),$$

each of which can be set-partitioned both into the nonzero vectors of $[b]_q$ subspaces of dimension a and into those of $[a]_q$ subspaces of dimension b.

Remark 16. *In order to update the last result to the case $c = \gcd(a,b)$, we divide the dimensions of all subspaces under consideration by c and consider them subspaces over the field $\mathbb{F}_{q^c}$ according to Lemma 4. Over $\mathbb{F}_{q^c}$, we have*

$$|U^*| = (q^c)^{e/c} - 1 = q^e - 1 \quad \textit{(number of nonzero elements of } U(e/c, q^c)\textit{) and}$$

$$\frac{[d/c]_q}{[a/c]_q[b/c]_q} = \frac{[d]_q/[c]_q}{[a]_q[b]_q/[c]_q^2} = \frac{[d]_q[c]_q}{[a]_q[b]_q} \quad \textit{(number of double cosets in } W(d/c, q^c)\textit{.)}$$

Then there are

$$(q^e - 1)\,\frac{[d]_q[c]_q}{[a]_q[b]_q}$$

disjoint subsets in Proposition 15, each of which can give us a partial spread of type $a^{[b]_q/[c]_q}$ or a partial spread of type $b^{[a]_q/[c]_q}$ over $\mathbb{F}_q$.

4. Proof of the Main theorem

In this section, we prove our main theorem.

Proof of Theorem 1. The various partitions of the types listed can be directly constructed using Lemma 7 and Remark 16 following Proposition 15. Nonnegative solutions of the Diophantine equation Eq. (1) exist, as $[b]_q \mid [n]_q$, and are given by

$$(x_i, y_i) = \left(0, \frac{[n]_q}{[b]_q}\right) + i\left(\frac{[b]_q}{[c]_q}, -\frac{[a]_q}{[c]_q}\right), \quad 0 \le i \le \left\lfloor \frac{[n]_q[c]_q}{[a]_q[b]_q} \right\rfloor . \tag{2}$$

By Lemma 7, we are able to reconfigure between zero and $m = [d]_q[c]_q/([a]_q[b]_q)$ batches of a-subspaces of W (where each batch contains $[b]_q/[c]_q$ of them) from b-dimensional subspaces as expected. Next, Remark 16 shows us that there are up to $(q^e - 1)m$ additional batches (again, of $[b]_q/[c]_q$ subspaces of dimension a each) available by reconfiguration from some of the transversal b-subspaces. The total number of times we can convert b's into a's (the upper limit of i in the statement of this theorem) is then $M = q^e m = q^e[d]_q[c]_q/([a]_q[b]_q)$.

For the second part of the theorem, we obtain $y_{M+1} < 0$ in Eq. (2) if and only if the condition $[s]_{q^b} < [a/c]_{q^s}$ is satisfied. Thus,

$$\begin{aligned} y_{M+1} &= \frac{[sb+d]_q}{[b]_q} - \left(\frac{q^{sb}[d]_q[c]_q}{[a]_q[b]_q} + 1\right)\frac{[a]_q}{[c]_q} \\ &= \frac{[sb+d]_q - q^{sb}[d]_q}{[b]_q} - \frac{[a]_q}{[c]_q} \\ &= \frac{[sb]_q}{[b]_q} - \frac{[a]_q}{[c]_q} \\ &= [s]_{q^b} - [a/c]_{q^c}. \end{aligned}$$

If a divides b, then $V(n,q)$ clearly admits a subspace partition of type $a^{[n]_q/[a]_q}$, i.e., an a-spread. Thus, when $e = b$, we have $s = 1$ and either a divides b or $[s]_{q^b} < [a/c]_{q^c}$. In both cases all possible subspace partition types $a^x b^y$ of V are covered. □

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