

Edge Orbits and Cyclic and r -Pyramidal Decompositions of Complete Uniform Hypergraphs

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Abstract

Let \mathbb{Z}_n denote the group of integers modulo n and let $\mathcal{E}_n^{(k)}$ be the set of all k -element subsets of \mathbb{Z}_n where $1 \leq k < n$. If $E \in \mathcal{E}_n^{(k)}$, let $[E] = \{E + r : r \in \mathbb{Z}_n\}$. Then $[E]$ is the orbit of E where \mathbb{Z}_n acts on $\mathcal{E}_n^{(k)}$ via $(r, E) \mapsto E + r$. Furthermore, $\{[E] : E \in \mathcal{E}_n^{(k)}\}$ is a partition of $\mathcal{E}_n^{(k)}$ into \mathbb{Z}_n -orbits. In this article, we count the total number of \mathbb{Z}_n -orbits of $\mathcal{E}_n^{(k)}$, count the number of orbits of each size, determine the corresponding results when fixed points are introduced, and give an application to cyclic and r -pyramidal decompositions of complete uniform hypergraphs into isomorphic subgraphs.

1 Introduction

If (G, \oplus) is a group with identity 0, X is a set, and f is a function from $G \times X$ into X such that $f(0, x) = x$ for all $x \in X$ and $f(g_1 \oplus g_2, x) = f(g_1, f(g_2, x))$ for all $g_1, g_2 \in G$ and $x \in X$, then it is said that G acts on X . Furthermore, f partitions X into G -orbits, where two elements $x, y \in X$ are in the same orbit if and only if $x = f(g, y)$ for some $g \in G$. For $g \in G$, let $\text{Fix}(g) = \{x \in X : f(g, x) = x\}$. Thus $\text{Fix}(g)$ is the set of elements of X that are fixed by g . Burnside's lemma (see [10]) gives the number of G -orbits of X .

Lemma 1. (Burnside's Lemma) *Let a finite group G act on a set X . The number of orbits that G induces is given by:*

$$\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|.$$

Let \mathbb{Z}_n denote the group of integers modulo n and let $\mathcal{E}_n^{(k)}$ be the set of all k -element subsets of \mathbb{Z}_n where $1 < k < n$. If $E \in \mathcal{E}_n^{(k)}$ and $r \in \mathbb{Z}_n$, let $E+r$ be formed by replacing each element $x \in E$ with $x+r$; so $(r, E) \mapsto E+r$ maps $\mathbb{Z}_n \times \mathcal{E}_n^{(k)}$ into $\mathcal{E}_n^{(k)}$. It can be seen that the group \mathbb{Z}_n acts on the set $\mathcal{E}_n^{(k)}$ partitioning it into \mathbb{Z}_n -orbits, where $E_1, E_2 \in \mathcal{E}_n^{(k)}$ are in the same orbit if and only if $E_1+r = E_2$ for some $r \in \mathbb{Z}_n$. We define $[E]$ to be $\{E+r : r \in \mathbb{Z}_n\}$, which we refer to as the \mathbb{Z}_n -orbit of E . If $S \subseteq \mathcal{E}_n^{(k)}$ and $r \in \mathbb{Z}_n$, let $S+r = \{E+r : E \in S\}$. By *clicking* S , we shall mean replacing S with $S+1$.

1.1 Applications in Hypergraphs

The set $\mathcal{E}_n^{(k)}$ can be thought of as being the edge set of the complete k -uniform hypergraph $K_n^{(k)}$ with vertex set \mathbb{Z}_n . The \mathbb{Z}_n -orbit of an edge E can be viewed as the set resulting from successively clicking E .

Let K and G be k -uniform hypergraphs with G a subgraph of K . A G -decomposition of K is a set $\Gamma = \{G_1, G_2, \dots, G_t\}$ of subgraphs of K each of which is isomorphic to G and such that each edge of K appears in exactly one G_i . In this case, we may refer to the elements of Γ as G -blocks. A G -decomposition of K is also known as a (K, G) -design. A $(K_n^{(k)}, K_n^{(k)})$ -design is known as a *Steiner system* $S(k, m, n)$. A summary of results on Steiner systems $S(k, m, n)$ can be found in [6].

Let G be a subgraph of $K_n^{(k)}$, where $V(K_n^{(k)}) = \mathbb{Z}_n$, and let Γ be a G -decomposition of $K_n^{(k)}$. Then Γ is said to be *cyclic* if Γ is closed under clicking. Thus if $G_i \in \Gamma$, then $G_i+1 \in \Gamma$. If we partition $\mathcal{E}_n^{(k)}$ into m distinct \mathbb{Z}_n -orbits each of size n and if G a subgraph of $K_n^{(k)}$ consisting of one edge from each of the m distinct \mathbb{Z}_n -orbits, then $\Gamma = \{G+i : i \in \mathbb{Z}_n\}$ is a cyclic G -decomposition of $K_n^{(k)}$. For example, if G is the subgraph of $K_8^{(3)}$ with edge-set $S = \{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 1, 5\}, \{0, 1, 6\}, \{0, 2, 4\}, \{0, 2, 5\}\}$, then $\Gamma = \{G+i : i \in \mathbb{Z}_8\}$ constitutes a cyclic G -decomposition of $K_8^{(3)}$.

The requirement that the hypergraph G in the previous paragraph must contain exactly one edge from each of the different \mathbb{Z}_n -orbits of size n can be viewed as an extension of the notion of a ρ -labeling of G in the case $k=2$ as introduced by Rosa in [11]. Although stated differently in [11],

a subgraph G of $K_{2m+1}^{(2)}$ with m edges admits a ρ -labeling if each of the m edges of G belongs to a different orbit under the action of \mathbb{Z}_{2m+1} . In [9], Meszka and Rosa extend this definition to 3-uniform hypergraphs.

Most of the work on G -decompositions of $K_n^{(k)}$ focuses on the case $k = 2$ (see [1] for a summary of known results). In general, little is known when $k \geq 3$. Some of the focus has been on G -decompositions of $K_n^{(3)}$, where G is a graph with a relatively small number of edges (see for example, [4] and [5]). Perhaps the best known general result on decompositions of complete k -uniform hypergraphs is Baranyai's result [3] on the existence of 1-factorizations of $K_{mk}^{(k)}$ for all positive integers m . There are, however, several articles on decompositions of complete k -uniform hypergraphs (see [2] and [8]) and of k -uniform k -partite hypergraphs (see [7] and [12]) into variations on the concept of a Hamilton cycle.

In this article, we count the total number of \mathbb{Z}_n -orbits of $\mathcal{E}_n^{(k)}$, count the number of orbits of each size, determine the corresponding results when fixed points are introduced, and give an application to cyclic and r -pyramidal decompositions of complete uniform hypergraphs into isomorphic subgraphs.

2 Counting Orbits

In this paper lowercase letters represent integers. If a and b are integers, we define $[a, b]$ to be $\{r \in \mathbb{Z} : a \leq r \leq b\}$. In sums of the form $\sum_{e|m} f(e)$ we assume e is restricted to positive divisors. Let $d = \gcd(n, k)$, with $n = dn_0$ and $k = dk_0$, so $\gcd(n_0, k_0) = 1$. By ϕ and μ we mean the Euler ϕ -function and the Möbius function.

Lemma 2. *If $r > 0$ and $\gcd(r, n) = e$, then $\text{Fix}(r) = \text{Fix}(e)$.*

Proof. There exists a positive integer s such that $r = se$. First suppose $E \in \text{Fix}(e)$, so $E+e = E$. Then $E+r = E+se = E+e+e+\dots+e = E$, and so $E \in \text{Fix}(r)$. Conversely, suppose $E \in \text{Fix}(r)$. Since $\gcd(r, n) = e$, we can find integers $x > 0$ and y such that $xr + yn = e$. Then, performing addition in \mathbb{Z}_n , we have $E+e = E+xr+yn = E+xr+0 = E+r+r+\dots+r = E$, and so $E \in \text{Fix}(e)$. \square

Lemma 3. *Let n and k be positive integers and set $d = \gcd(n, k)$, $n = dn_0$, and $k = dk_0$. Suppose r is a positive integer such that $r \mid n$ and $n \mid kr$ (so $n_0 \mid r$). For any positive integer h , we have $h \mid r$ and $n \mid hk$ if and only if $h = en_0$, where $e \mid \frac{r}{n_0}$. Moreover, if $h \mid n$ then the number of integers r , $0 \leq r < n$, with $\gcd(n, r) = h$ is $\phi(n/h)$.*

Proof. First suppose $h \mid r$ and $n \mid hk$. Now $n_0 \mid hk_0$, so $n_0 \mid h$. Let $h = en_0$, and let $r = hh'$. Then $r/n_0 = hh'/n_0 = eh'$, so $e \mid r/n_0$.

Conversely, suppose $h = en_0$, where $e \mid r/n_0$. Then $h = en_0 \mid (r/n_0)n_0 = r$. Also $kh = ken_0 = (k/d)e(dn_0) = k_0en$, so $n \mid kh$.

To see the second statement note that $\gcd(r, n) = h$ if and only if $\gcd(r/h, n/h) = 1$, $0 \leq r/h < n/h$. \square

Theorem 4. *Let k and n be positive integers, $d = \gcd(n, k)$, and let \mathbb{Z}_n act on $\mathcal{E}_n^{(k)}$. Then the number t of \mathbb{Z}_n -orbits of $\mathcal{E}_n^{(k)}$ is given by*

$$t = \frac{1}{n} \sum_{e \mid d} \phi(e) \binom{n/e}{k/e}.$$

Proof. Suppose $\text{Fix}(q) \neq \emptyset$. Then by Lemma 2, $\text{Fix}(q) = \text{Fix}(h)$, where $h = \gcd(q, n)$. Suppose $E \in \text{Fix}(h)$. This implies that if $A \subseteq [0, h-1]$ and $A \subseteq E$, then $A + jh \subseteq E$ for $j \in [0, n/h-1]$. Hence E must have the form

$$A \cup (h + A) \cup (2h + A) \cup \cdots \cup ((n/h - 1)h + A).$$

Then $(n/h)|A| = |E| = k$, so $|A| = hk/n$ and $n \mid hk$. The number of ways of choosing $|A|$ elements from $\{0, 1, \dots, h-1\}$ is $\binom{h}{|A|} = \binom{h}{hk/n}$, and this is the same for all the $\phi(n/h)$ values of r such that $\gcd(r, n) = h$. Thus by Burnside's Lemma and the first part of Lemma 3 with $r = n$,

$$\begin{aligned} t &= \frac{1}{n} \sum_{\substack{h \mid n \\ n \mid hk}} \phi(n/h) \binom{h}{hk/n} = \frac{1}{n} \sum_{e \mid d} \phi\left(\frac{n}{en_0}\right) \binom{en_0}{en_0k/n} \\ &= \frac{1}{n} \sum_{e \mid d} \phi(d/e) \binom{n/(d/e)}{k/(d/e)}. \end{aligned}$$

Now if we notice that as e runs through the positive divisors of d , so does d/e , we get the formula of the theorem. \square

Example 1. When $n = 30$ and $k = 24$, we have $d = 6$, so

$$\begin{aligned} t &= \frac{1}{30} \sum_{e \mid 6} \phi(e) \binom{30/e}{24/e} \\ &= \frac{1}{30} \left(\phi(1) \binom{30}{24} + \phi(2) \binom{15}{12} + \phi(3) \binom{10}{8} + \phi(6) \binom{5}{4} \right) \\ &= \frac{1}{30} (1 \cdot 593,775 + 1 \cdot 455 + 2 \cdot 45 + 2 \cdot 5) = 19,811. \end{aligned}$$

3 Difference Vectors

If $E \in \mathcal{E}_n^{(k)}$, we can write E uniquely as $\{a_1, a_2, \dots, a_k\}$ where $0 \leq a_1 < a_2 < \dots < a_k < n$. By the *difference vector* of E , we mean the k -tuple $\Delta E = (a_2 - a_1, a_3 - a_2, \dots, a_k - a_{k-1}, n + a_1 - a_k)$. Note that the components of ΔE are positive and sum to n . Also, distinct elements of a \mathbb{Z}_n -orbit may yield distinct difference vectors. For example, if $E = \{0, 3, 4\} \in \mathcal{E}_5^{(3)}$, then $\Delta E = (3, 1, 1)$, while $\Delta(E + 2) = \Delta\{0, 1, 2\} = (1, 1, 3)$. However, it is easy to see that edges are in the same \mathbb{Z}_n -orbit if and only if they have difference vectors that are cyclic permutations of each other. By the *reduced* difference vector $\Delta' E$ of an edge E , we mean the difference vector among the cyclic permutations of ΔE that is lexicographically first. In our example, the cyclic permutations of ΔE are $(3, 1, 1)$, $(1, 3, 1)$, and $(1, 1, 3)$, and thus $\Delta' E = (1, 1, 3) = \Delta'(E + 2)$.

Now let $\mathcal{D}_n^{(k)}$ be all ordered k -tuples of positive integers with sum n . Using standard counting techniques, it can be proved that the number of sequences of u positive integers with sum v is $\binom{v-1}{u-1}$. In particular, $|\mathcal{D}_n^{(k)}| = \binom{n-1}{k-1}$. If $D = (b_1, b_2, \dots, b_k)$ is a difference vector in $\mathcal{D}_n^{(k)}$ and $r \in \mathbb{Z}_k$, we define $D \oplus r$ to be $(b_{1+r}, b_{2+r}, \dots, b_{k+r})$, where the subscripts are taken modulo k . Then the group \mathbb{Z}_k acts on $\mathcal{D}_n^{(k)}$, and we will denote the \mathbb{Z}_k -orbit containing $D \in \mathcal{D}_n^{(k)}$ by $[D]$. There is a one-to-one correspondence between the \mathbb{Z}_k -orbits of $\mathcal{D}_n^{(k)}$ with respect to \oplus and the \mathbb{Z}_n -orbits of $\mathcal{E}_n^{(k)}$ with respect to $+$. In particular, if $E \in \mathcal{E}_n^{(k)}$, then the \mathbb{Z}_n -orbit $[E]$ contained in $\mathcal{E}_n^{(k)}$ corresponds to the \mathbb{Z}_k -orbit $[\Delta E]$ contained in $\mathcal{D}_n^{(k)}$.

We can count the \mathbb{Z}_k -orbits of $\mathcal{D}_n^{(k)}$ using Burnside's Lemma. This yields the formula

$$t = \frac{1}{k} \sum_{e|d} \phi(e) \binom{n/e - 1}{k/e - 1},$$

where as before $d = \gcd(n, k)$. This formula gives the same result as that of Theorem 4.

Example 2. If $n = 30$ and $k = 24$ so $d = 6$, we have

$$\begin{aligned} t &= \frac{1}{24} \sum_{e|6} \phi(e) \binom{30/e - 1}{24/e - 1} \\ &= \frac{1}{24} \left(\phi(1) \binom{29}{23} + \phi(2) \binom{14}{11} + \phi(3) \binom{9}{7} + \phi(6) \binom{4}{3} \right) \\ &= \frac{1}{24} (1 \cdot 475,020 + 1 \cdot 364 + 2 \cdot 36 + 2 \cdot 4) = 19,811. \end{aligned}$$

4 Orbits of a Given Size

If $\gcd(n, k) = 1$ the sum in Theorem 4 has a single term, and the number of \mathbb{Z}_n -orbits of $\mathcal{E}_n^{(k)}$ is $\frac{1}{n} \binom{n}{k}$. Since $|\mathcal{E}_n^{(k)}| = \binom{n}{k}$, every orbit in this case has size n . If $\gcd(n, k) > 1$ however, there will be orbits of various sizes. In this section we count the number of orbits of each size.

We consider the \mathbb{Z}_n -orbits of $\mathcal{E}_n^{(k)}$. Let $N(h)$ be the number \mathbb{Z}_n -orbits in $\mathcal{E}_n^{(k)}$ with exactly h elements. By the *order* of $E \in \mathcal{E}_n^{(k)}$ we mean the least positive integer h such that $E + h = E$. Since $E + h = E + \gcd(h, n)$, this means $h \mid n$. Clearly the order of E is $|[E]|$. Recall that $\text{Fix}(h) = \{E \in \mathcal{E}_n^{(k)} : E + h = E\}$.

Lemma 5. *Let s be a positive divisor of n . Then $E \in \text{Fix}(s)$ if and only if E has order h , where $h \mid s$ and $n \mid hk$.*

Proof. Let $E \in \text{Fix}(s)$ have order h , and let $s = q'h + r'$, where $0 \leq r' < h$. Then $E = E + s = (E + h + h + \cdots + h) + r' = E + r'$, and so by the definition of order we must have $r' = 0$. Thus $h \mid s$. Since $\text{Fix}(h) \neq \emptyset$ we have $n \mid hk$ as in the proof of Theorem 4.

Now assume the order of E is h , where $h \mid s$ and $n \mid hk$. Say $s = qh$. Then $E + s = E + h + h + \cdots + h = E$, so $E \in \text{Fix}(s)$. \square

Since $\text{Fix}(n) = \mathcal{E}_n^{(k)}$, by taking $s = n$ in the previous lemma we see that $m(h) > 0$ if and only if $h \mid n$ and $n \mid hk$, and by Lemma 3 with $r = n$ this happens if and only if $h = en_0$, where $e \mid n/n_0 = d$. This is the first conclusion of the following theorem.

Theorem 6. *Let n and k be positive integers, and consider $\mathcal{E}_n^{(k)}$. Set $d = \gcd(n, k)$, $n = dn_0$, $k = dk_0$. The values of s such that $N(s) > 0$ are exactly the integers $s = en_0$, where e runs through the positive divisors of d . In this case*

$$N(s) = \frac{1}{s} \sum_{f|s_0} \mu(s_0/f) \binom{n_0 f}{k_0 f},$$

where $s_0 = s/n_0$.

Proof. Let s be a positive divisor of n such that $\text{Fix}(s) \neq \emptyset$. Then as in the proof of Theorem 4 we have $n \mid sk$. Since if E has order h , then $|[E]| = h$, the total number of edges of order h is $hN(h)$. By Lemma 5 we have

$$|\text{Fix}(s)| = \sum_{\substack{h|s \\ n|hk}} hN(h) = \sum_{e|s_0} en_0 N(en_0),$$

where in the second sum we have applied the first part of Lemma 3 and set $s_0 = s/n_0$. In the proof of Theorem 4 we found that $|\text{Fix}(s)| = \binom{s}{sk/n}$, so we have

$$\sum_{e|s_0} en_0 N(en_0) = \binom{s}{sk/n} = \binom{n_0 s_0}{k_0 s_0}.$$

The last equation has the form $G(s_0) = \sum_{f|s_0} g(f)$, where $G(s_0) = \binom{n_0 s_0}{k_0 s_0}$, and $g(f) = fn_0 N(fn_0)$. By the Möbius inversion formula we have

$$g(s_0) = \sum_{f|s_0} \mu(s_0/f) G(f) \quad \text{or} \quad s_0 n_0 N(s_0 n_0) = s N(s) = \sum_{f|s_0} \mu(s_0/f) \binom{n_0 f}{k_0 f},$$

so

$$N(s) = \frac{1}{s} \sum_{f|s_0} \mu(s_0/f) \binom{n_0 f}{k_0 f}. \quad \square$$

Example 3. If $n = 30$ and $k = 24$ so that $d = 6$, we have $n_0 = 5$ and $k_0 = 4$. The divisors of d are 1, 2, 3, and 6, the corresponding values of s are 5, 10, 15, and 30, and of s_0 are 1, 2, 3, and 6. Then

$$\begin{aligned} N(5) &= \frac{1}{5} \sum_{f|1} \mu(1/f) \binom{5f}{4f} = \frac{1}{5} (1) \binom{5}{4} = 1, \\ N(10) &= \frac{1}{10} \sum_{f|2} \mu(2/f) \binom{5f}{4f} = \frac{1}{10} \left((-1) \binom{5}{4} + (1) \binom{10}{8} \right) = 4, \\ N(15) &= \frac{1}{15} \sum_{f|3} \mu(3/f) \binom{5f}{4f} = \frac{1}{15} \left((-1) \binom{5}{4} + (1) \binom{15}{12} \right) = 30, \\ N(30) &= \frac{1}{30} \sum_{f|6} \mu(6/f) \binom{5f}{4f} \\ &= \frac{1}{30} \left((1) \binom{5}{4} + (-1) \binom{10}{8} + (-1) \binom{15}{12} + (1) \binom{30}{24} \right) \\ &= 19,776. \end{aligned}$$

Note that $1 + 4 + 30 + 19,776 = 19,811$, which agrees with the results in Examples 1 and 2.

If $E \in \mathcal{E}_n^{(k)}$ and $D \in \mathcal{D}_n^{(k)}$ are in corresponding orbits, then $||D|| = (k/n)||E||$. Thus if we define $M(h)$ to be the number of orbits of $\mathcal{D}_n^{(k)}$ of size h , then $M(h) = N(nh/k)$. For example $\mathcal{D}_{30}^{(24)}$ has 1 orbit of size 4, 4 orbits of size 8, 30 orbits of size 12, and 19,776 orbits of size 24.

5 Orbits under r -Pyramidal Actions

Let n , k , and r be integers with $0 \leq r < k < n$ and let $I_r = \{\infty_1, \infty_2, \dots, \infty_r\}$. Also, let $\mathcal{E}_{n,r}^{(k)}$ be the set of all k -element subsets of $\mathbb{Z}_{n-r} \cup I_r$. Then the set $\mathcal{E}_{n,r}^{(k)}$ can be thought of as being the edge set of the complete k -uniform hypergraph $K_n^{(k)}$ with vertex set $\mathbb{Z}_{n-r} \cup I_r$. If $\infty_i \in I_r$ and $s \in \mathbb{Z}_{n-r}$, we define $\infty_i + s$ to be ∞_i . Furthermore if $E \in \mathcal{E}_{n,r}^{(k)}$, we let $E + s = \{x + s : x \in E\}$.

It is easy to see that the group \mathbb{Z}_{n-r} acts on $\mathcal{E}_{n,r}^{(k)}$ via $(s, E) \rightarrow E + s$, and so $\mathcal{E}_{n,r}^{(k)}$ is partitioned into \mathbb{Z}_{n-r} -orbits under this r -pyramidal action. As before if $s \in \mathbb{Z}_{n-r}$, we let $\text{Fix}(s) = \{E \in \mathcal{E}_{n,r}^{(k)} : E + s = E\}$. Clearly if E and E' are in the same orbit, then $E \cap I_r = E' \cap I_r$. Suppose $E \in \mathcal{E}_{n,r}^{(k)}$ and $E \cap I_r = J$, where $|J| = j$. Let $E^* = E \setminus I_r$, so $|E^*| = k - j$. Then $E + s = E$ exactly when $E^* + s = E^*$.

Let $\mathcal{E}_J = \{E \in \mathcal{E}_{n,r}^{(k)} : E \cap I_r = J\}$. Then \mathbb{Z}_{n-r} acts on \mathcal{E}_J , and by Theorem 4 the number of \mathbb{Z}_{n-r} -orbits in \mathcal{E}_J is

$$\frac{1}{n-r} \sum_{e|\gcd(n-r, k-j)} \phi(e) \binom{(n-r)/e}{(k-j)/e}.$$

Let $\mathcal{E}_j = \{E \in \mathcal{E}_{n,r}^{(k)} : |E \cap I_r| = j\}$. Since there are $\binom{r}{j}$ subsets of I_r with j elements, the number of \mathbb{Z}_{n-r} -orbits in \mathcal{E}_j is

$$\binom{r}{j} \frac{1}{n-r} \sum_{e|\gcd(n-r, k-j)} \phi(e) \binom{(n-r)/e}{(k-j)/e}.$$

Noting that j can vary from 0 to r gives the following.

Theorem 7. *Let n , k , and r be integers with $0 \leq r < k < n$ and consider the complete k -uniform hypergraph $K_n^{(k)}$ with vertex set $\mathbb{Z}_{n-r} \cup I_r$. Let \mathbb{Z}_{n-r} act on the set of edges $\mathcal{E}_{n,r}^{(k)}$ as described above. Then the number of \mathbb{Z}_{n-r} -orbits is*

$$\frac{1}{n-r} \sum_{j=0}^r \left(\binom{r}{j} \sum_{e|\gcd(n-r, k-j)} \phi(e) \binom{(n-r)/e}{(k-j)/e} \right).$$

Example 4. If $n = 30$, $k = 24$, and $r = 3$, the number of \mathbb{Z}_{27} -orbits of

$\mathcal{E}_{30,3}^{(24)}$ is

$$\begin{aligned}
& \frac{1}{27} \sum_{j=0}^3 \left(\binom{3}{j} \sum_{e|\gcd(27,24-j)} \phi(e) \binom{27/e}{(24-j)/e} \right) \\
&= \frac{1}{27} \left(\binom{3}{0} \sum_{e|3} \phi(e) \binom{27/e}{24/e} + \binom{3}{1} \sum_{e|1} \phi(e) \binom{27/e}{23/e} \right. \\
&\quad \left. + \binom{3}{2} \sum_{e|1} \phi(e) \binom{27/e}{22/e} + \binom{3}{3} \sum_{e|3} \phi(e) \binom{27/e}{21/e} \right) \\
&= \frac{1}{27} \left(\left(\binom{27}{24} + 2 \binom{9}{8} \right) + 3 \binom{27}{23} + 3 \binom{27}{22} + \left(\binom{27}{21} + 2 \binom{9}{7} \right) \right) \\
&= \frac{1}{27} (2943 + 52,650 + 242,190 + 296,082) = 21,995.
\end{aligned}$$

Now we consider the number of orbits of a given size. With general n , k , and r as before the example consider a fixed set $J \subseteq I_r$ with $|J| = j$. The action of \mathbb{Z}_{n-r} on $\mathcal{E}_J = \{E \in \mathcal{E}_{n,r}^{(k)} : E \cap I_r = J\}$ amounts to the action of \mathbb{Z}_{n-r} on $\mathcal{E}_{n-r}^{(k-j)}$. Set $n' = n - r$, $k' = k - j$, $d' = \gcd(n', k')$, $n' = d'n'_0$, $k' = d'k'_0$. By Theorem 6 the orbit sizes are exactly the integers $s = en'_0$, where e runs through the positive divisors of d' . Then for such an integer e the number of \mathbb{Z}_{n-r} -orbits of \mathcal{E}_J of size en'_0 is

$$\frac{1}{s} \sum_{f|s_0} \mu(s_0/f) \binom{n'_0 f}{k'_0 f},$$

where $s_0 = s/n'_0$. Letting J run through the $\binom{r}{j}$ subsets of I_r of size j gives the following.

Theorem 8. *Let n , k , r , and j be integers with $n > k$, $0 \leq r < k$, and $0 \leq j \leq r$, and consider the complete k -uniform hypergraph $K_n^{(k)}$ with vertex set $\mathbb{Z}_{n-r} \cup I_r$. Set $n' = n - r$, $k' = k - j$, $d' = \gcd(n', k')$, $n' = dn'_0$, and $k' = d'k'_0$. The achievable sizes of orbits of $\mathcal{E}_{n,r}^{(k)}$ with exactly j elements in I_r are the integers $s = en'_0$, where e is a positive divisor of d' , and the number $N_j(s)$ of such orbits is*

$$N_j(s) = \binom{r}{j} \frac{1}{s} \sum_{f|s_0} \mu(s_0/f) \binom{n'_0 f}{k'_0 f},$$

where $s_0 = s/n'_0$.

Example 5. Consider $n = 30$, $k = 24$, $r = 3$. If $j = 0$, we have $n' = 27$, $k' = 24$, $d' = 3$, $n'_0 = 9$, $k'_0 = 8$, and $s = 9e$, where $e \mid 3$. Thus s is 9 or 27, and

$$N_0(9) = \binom{3}{0} \frac{1}{9} \sum_{f \mid 1} \mu(1/f) \binom{9f}{8f} = 1,$$

$$N_0(27) = \binom{3}{0} \frac{1}{27} \sum_{f \mid 3} \mu(3/f) \binom{9f}{8f} = \frac{1}{27} \left((-1) \binom{9}{8} + \binom{27}{24} \right) = 108.$$

Note that n' does not change with j . If $j = 1$, we have $k' = 23$, $d' = 1$, $n'_0 = 27$, $k'_0 = 23$, and $s = 27e$, where $e \mid 1$. Thus $s = 27$, and

$$N_1(27) = \binom{3}{1} \frac{1}{27} \sum_{f \mid 1} \mu(1/f) \binom{27f}{23f} = 1950.$$

Likewise if $j = 2$, then $n' = 27$, $k' = 22$, $d' = 1$, $n'_0 = 27$, $k'_0 = 22$, and $s = 27e$, where $e \mid 1$. Thus $s = 27$, and

$$N_2(27) = \binom{3}{2} \frac{1}{27} \sum_{f \mid 1} \mu(1/f) \binom{27f}{22f} = \frac{1}{9} \binom{27}{22} = 8970.$$

Finally if $j = 3$, then $n' = 27$, $k' = 21$, $d' = 3$, $n'_0 = 9$, $k'_0 = 7$, and $s = 9e$, where $e \mid 3$. Thus s is 9 or 27, and

$$N_3(9) = \binom{3}{3} \frac{1}{9} \sum_{f \mid 1} \mu(1/f) \binom{9f}{7f} = \frac{1}{9} \binom{9}{7} = 4,$$

$$N_3(27) = \binom{3}{3} \frac{1}{27} \sum_{f \mid 3} \mu(3/f) \binom{9f}{7f} = \frac{1}{27} \left((-1) \binom{9}{7} + \binom{27}{21} \right) = 10,962.$$

Notice that there are $1 + 4 = 5$ orbits of size 9 and $108 + 1950 + 8970 + 10,962 = 21,990$ of size 27, for a total of 21,995 orbits, just as we computed in Example 4 using Theorem 7. These include $5 \cdot 9 + 21,990 \cdot 27 = 593,775$ edges, which is $\binom{30}{24}$.

6 Forcing Same-Sized Orbits

In order to decompose a complete uniform hypergraph $K_n^{(k)}$ into isomorphic subgraphs, it is convenient to use a group action that yields same-sized

orbits. As mentioned in the beginning of Section 4, if $\gcd(n, k) = 1$ the action described in Section 1 achieves same-sized orbits. If $\gcd(n, k) > 1$, it may be possible to find an r -pyramidal action that yields same-sized orbits. We call $K_n^{(k)}$ *balancing* if there exists an integer $r \in [0, k - 1]$, such that $\mathcal{E}_{n,r}^{(k)}$ has all orbits of size $n - r$.

Theorem 9. *The hypergraph $K_n^{(k)}$ is balancing if and only if $\gcd(n - r, k - j) = 1$ whenever $0 \leq j \leq r$.*

Proof. Suppose $\gcd(n - r, k - j) = 1$ whenever $0 \leq j \leq r$. For a fixed such j by Theorem 8, the achievable sizes of orbits of edges containing exactly j elements of I_r are the integers $s = en'_0$, where e is a positive divisor of $d' = \gcd(n - r, k - j) = 1$. Thus the only possible orbit size is $n'_0 = n'/d' = n - r$. Since this number does not depend on j , all orbits must be the same size.

Conversely, assume that for some j , $0 \leq j \leq r$, we have $\gcd(n - r, k - j) = d' > 1$. Achievable orbit sizes for orbits containing exactly j elements of I_r include n'_0 and $d'n'_0$ by Theorem 8. Thus there are orbits of at least two different sizes. \square

If $\gcd(n, k) = 1$, then there exists a graph G with $\binom{n}{k}/n$ edges such that $K_n^{(k)}$ admits a cyclic G -decomposition. Let G be a subgraph of $K_n^{(k)}$, where $V(K_n^{(k)}) = \mathbb{Z}_{n-r} \cup I_r$ and let Γ be a G -decomposition of $K_n^{(k)}$. Then Γ is said to be r -pyramidal if Γ is closed under clicking. Thus among decompositions, 0-pyramidal is equivalent to cyclic. If we partition $\mathcal{E}_{n,r}^{(k)}$ into m distinct \mathbb{Z}_{n-r} -orbits each of size $n - r$ and if G with $E(G) \subseteq \mathcal{E}_{n,r}^{(k)}$ is a subgraph of $K_n^{(k)}$ with edge-set containing exactly one edge from each of the m distinct \mathbb{Z}_{n-r} -orbits, then $\Gamma = \{G + i : i \in \mathbb{Z}_{n-r}\}$ is an r -pyramidal G -decomposition of $K_n^{(k)}$. For example, if G is the subgraph of $K_9^{(3)}$, with $E(K_9^{(3)}) = \mathcal{E}_{9,2}^{(3)}$, such that $E(G) = \{\{0, 1, 2\}, \{0, 1, 3\}, \{0, 1, 4\}, \{0, 1, 5\}, \{0, 2, 4\}, \{0, 1, \infty_1\}, \{0, 2, \infty_1\}, \{0, 3, \infty_1\}, \{0, 1, \infty_2\}, \{0, 2, \infty_2\}, \{0, 3, \infty_2\}, \{0, \infty_1, \infty_2\}\}$, then $\Gamma = \{G + i : i \in \mathbb{Z}_7\}$ constitutes a 2-pyramidal G -decomposition of $K_9^{(3)}$.

The requirement that the graph G in the previous paragraph contains exactly one edge from each of the m different \mathbb{Z}_{n-r} -orbits of size $n - r$ can again be viewed as an extension of the notion of a ρ -labeling of G . Suppose $K_n^{(k)}$ is balancing for some $r \in [0, k - 1]$. Let $K_n^{(k)}$ have edge set $\mathcal{E}_{n,r}^{(k)}$. Then a subgraph G of $K_n^{(k)}$ with $m = \binom{n}{k}/(n - r)$ edges is said to admit an r -pyramidal ρ -labeling if each of the m edges of G belongs to a different orbit under the action of \mathbb{Z}_{n-r} .

We call the integer $k > 1$ *completely balancing* if $K_n^{(k)}$ is balancing for all $n > k$.

Theorem 10. *Let ≥ 2 . If k is completely balancing, then for every integer $n > k$ there exists r with $0 \leq r \leq k - 1$ and a graph G with $\binom{n}{k}/(n - r)$ edges such that $K_n^{(k)}$ admits an r -pyramidal G -decomposition.*

The following lemma allows us to decide whether k is completely balancing by checking a finite number of cases.

Lemma 11. *Let π_k denote the product of the primes that are at most k . If $K_n^{(k)}$ is balancing for all $n \in [k + 1, k + \pi_k]$, then k is completely balancing*

Proof. Suppose $K_n^{(k)}$ is balancing for all $n \in [k + 1, k + \pi_k]$. Now let $n > k$. Then there exists $n' \in [k + 1, k + \pi_k]$ such that $n' \equiv n \pmod{\pi_k}$. That is, $n = s\pi_k + n'$ for some $s \geq 0$. By the assumption and Theorem 9, there exists $r \in [0, k - 1]$, such that $\gcd(n' - r, k - j) = 1$ for $0 \leq j \leq r$. That is, any prime that divides $k - j$ does not divide $n' - r$. Since any prime dividing $k - j$ also divides π_k , we have $\gcd(n - r, k - j) = \gcd(s\pi_k + n' - r, k - j) = \gcd(n' - r, k - j) = 1$ for $0 \leq j \leq r$. \square

One can easily verify that every k with $2 \leq k \leq 6$ is completely balancing. For example, $k = 2$ is completely balancing with $r = 0$ when n is odd and with $r = 1$ when n is even. Similarly, $k = 3$ is completely balancing with $r = 0$ when $n \equiv 1$ or $2 \pmod{3}$, and with $r = 1$ when $n \equiv 0 \pmod{6}$ and $r = 2$ when $n \equiv 3 \pmod{6}$.

Using Theorem 9 and Lemma 11 and a computer, one can easily verify that every $k \leq 14$ is completely balancing. Thus we have the following.

Theorem 12. *For each k with $2 \leq k \leq 14$ and each $n > k$, there exists an r with $0 \leq r \leq k - 1$ and a graph G with $\binom{n}{k}/(n - r)$ edges such that $K_n^{(k)}$ admits an r -pyramidal G -decomposition.*

It is simple to verify that $K_n^{(15)}$ is balancing for all $n < 2199$. To show that $K_{2199}^{(15)}$ is not balancing, it suffices to note that for $r \in \{0, 3, 4, 6, 9, 12, 14\}$, we have $\gcd(2199 - r, 15) > 1$; for $r \in \{1, 5, 7, 8, 11, 13\}$, we have $\gcd(2199 - r, 14) > 1$; for $r = 2$, we have $\gcd(2199 - r, 13) > 1$; and for $r = 10$, we have $\gcd(2199 - r, 11) > 1$. In fact, we have verified that $K_n^{(15)}$ is balancing if and only if $n \not\equiv b \pmod{30030}$ where $b \in \{2199, 2200, 5765, 5766, 9125, 9126, 9455, 9456, 9459, 9460, 13355, 13356, 20585, 20586, 20589, 20590, 20919, 20920, 27845, 27846\}$. We have also verified that no $k \in [15, 50]$ is completely balancing and conjecture that no $k > 14$ is completely balancing.

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